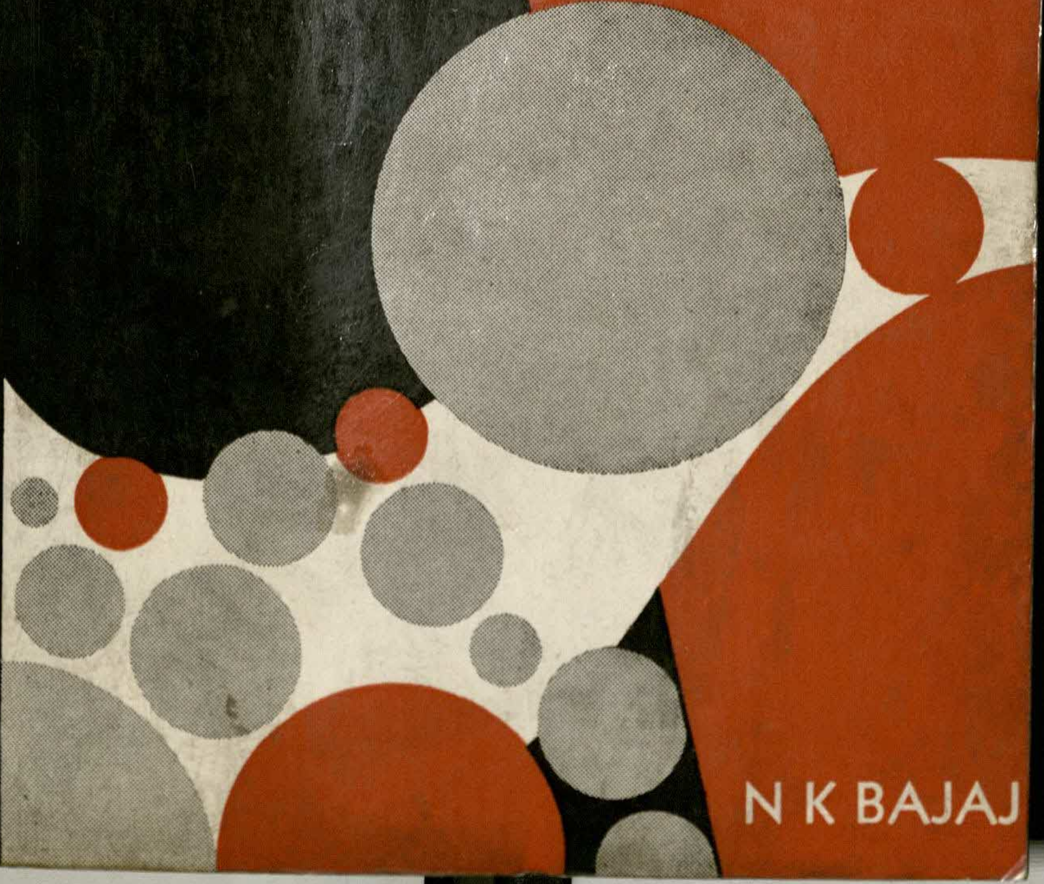
TATA McGRAW HILL'S

PHYSICS

FOR CLASS XI



N K BAJAJ

PHYSICS
FOR CLASS XI

PHYSICS
FOR CLASS XI

N K Bajaj
St Stephen's College
University of Delhi
Delhi

Tata McGraw-Hill Publishing Company Limited
NEW DELHI

McGraw-Hill Offices

New Delhi New York St Louis San Francisco Auckland Bogotá
Guatemala Hamburg Lisbon London Madrid Mexico
Milan Montreal Panama Paris San Juan São Paulo
Singapore Sydney Tokyo Toronto

Reviewed by:

Stalin Malhotra
Delhi Public School
R K Puram
New Delhi

©1989, TATA McGRAW-HILL PUBLISHING COMPANY LIMITED

No part of this publication can be reproduced in any form or by any means without the prior written permission of the publishers

This edition can be exported from India only by the publishers, Tata McGraw-Hill Publishing Company Limited

Assistant Sponsoring Editor: Amit Bhattacharjee
Assistant Production Editor: Kaushik Ghosh
Production Officer: B L Dogra

Published by Tata McGraw-Hill Publishing Company Limited,
4/12 Asaf Ali Road, New Delhi 110 002, and printed at
R N Printers, 1/2208 Ram Nagar, Shahdara, Delhi 110 032

Acc no -

Preface

This textbook is written strictly in accordance with the latest physics syllabus for Class XI prescribed by the Central Board of Secondary Education, New Delhi, under the 10+2+3 pattern of education.

The National Policy on Education, adopted in 1986, lays considerable emphasis on student-oriented rather than teacher-oriented books. According to this policy, books must be designed in such a way that a student is able to read and understand on his own without undue reliance on formal classroom instruction. The student must play an active role in the learning process and the teacher a passive one, serving only as a guide. Merely reading about the experiences and observations of others is not stimulating. If students are to understand and enjoy physics, they must have some of these experiences themselves. To encourage this kind of direct experiences with reality, I have recommended easily performed "home" experiments in the main body of the text. The student is advised to perform these experiments which require easily available material.

The objective of this book is to present physics in an interesting, understandable and enjoyable manner. An appreciation, rather than a working knowledge of physics is intended to be inculcated. Therefore, stress has been laid on the concepts of physics, rather than the mathematical aspects or techniques. Although calculus has been used, the mathematics involved is quite simple. The physical meaning of the mathematics used is clearly explained, thus enabling the student to appreciate the important part mathematics plays in the development of physics. Stress has been laid on practical applications throughout the book. The language is kept as simple as possible.

The book is divided into 13 chapters. Chapter 1 is essentially introductory; it deals with the scope, excitement and application of physics. Chapter 2 on measurement deals with units and dimensions of physical quantities and the accuracy in their measurements. Chapters 3 and 4 deal with kinematics of motion in one, two and three dimensions. Chapters 5 to 8 are devoted to dynamics of motion. Chapters 9 to 11 are concerned with the structure and properties of matter including heat and thermodynamics. Chapters 12 and 13 are on oscillations and waves. The coverage of topics is more or less traditional, but I have departed from the conventional approach in which general principles or laws are stated first and various results are deduced from them. To introduce a physical concept, I have begun with commonplace observations of our daily experience and cited concrete examples and then proceeded to suggest generalizations or laws that unify these observations. A study of the history of science reveals that this approach is more in keeping with how science is actually practised.

The introduction of each new concept is followed by a number of solved numerical examples. The book contains 181 solved examples. It is my earnest hope that these examples will enable students to master a concept or a technique before they move on to the next. The physical concepts are amply illustrated by means of diagrams and graphs wherever possible. The book contains over 295 illustrations.

I am conscious of the fact that some students can learn much more rapidly and can assimilate more difficult concepts as compared to the average students. These better and brighter students often do not find a subject challenging enough. To sustain and stimulate the interest of such students, some thought-provoking topics are discussed as supplements at the end of each chapter. The average student may omit these supplements at first reading; this will not affect his or her learning process.

The questions at the end of each chapter have been carefully designed. The numerical problems have been graded in order of the level of difficulty and the answers to all problems are given at the end of the book. The book contains over 200 numerical problems. For a quick recapitulation, a summary of the salient features is included at the end of each chapter.

No scientific theory is born in a vacuum. Each scientist builds on the work of his predecessors or contemporaries. Therefore, a brief historical development of ideas has been included in the body of the text wherever necessary.

I wish to express my deep sense of gratitude to my colleagues and students whose encouragement and cooperation have been a source of inspiration. It is a pleasure to thank Mr. Vineet Bajaj for going through the solutions of the numerical problems and making helpful suggestions and comments. Finally, I would like to thank my wife Mrs. Kamlesh Bajaj whose dedication and encouragement made this book possible.

Suggestions for the improvement of the book would be most welcome.

N K Bajaj

A Note to the Student

The main objective of this book is to present physics in an interesting, understandable and enjoyable manner so that you can understand the physical phenomena of the world around you. Each phenomenon is very carefully described by citing concrete examples from our everyday experience and a rational scientific explanation is provided for each phenomenon starting from fundamental principles. It is my earnest hope that a careful reading of the text in each chapter, and going through solved examples and working out the problems at the end of each chapter, will contribute immensely towards your understanding of the finer points of the subject of physics.

The somewhat bulky size of the book should not scare you. The book has assumed this size because of its following salient features:

1. The detailed steps of all mathematical derivations are given, starting from fundamental principles and bringing out clearly the mathematical logic and technique involved.

2. The introduction of each new concept is followed by a number of solved numerical examples. The book contains as many as 181 solved examples which will enable you to master a concept before you move on to the next. These examples have been carefully graded. We begin with simpler examples and gradually go on to more difficult ones involving the use of more than one concept or one formula.

3. The physical concepts are illustrated by means of a large number of diagrams and graphs. The book contains about 295 illustrations.

4. To sustain the interest of brighter students, supplements are appended at the end of a chapter which deal with more challenging and thought-provoking topics. The average student may omit these supplements at first reading.

The book contains about 400 questions and numerical problems. The numerical problems have been graded in order of the level of difficulty and the answers to all problems are given at the end of the book. You are strongly urged to solve as many problems as you can. Do not despair if you cannot solve all of them; some are quite difficult. If you are unable to solve a particular problem, read the relevant text more closely, especially the solved examples. You should develop the habit of working out at least 2 or 3 problems *regularly*. This is of the *utmost* importance. Only then will you be able to enjoy, appreciate and comprehend the subject of physics. Problem-solving not only sharpens your intellect and intuitions but can also be a source of immense fun and satisfaction. So keep solving problems!

N K Bajaj

Contents

Preface v

A Note to the Student vii

1 Introduction 1

1.1 What is Physics ? *1* 1.2 Scope and excitement of Physics *3* 1.3 Physics and technology *4* Summary *5* Exercises *5*

2 Measurement 7

2.1 Introduction: the need for measurement *7* 2.2 Fundamental physical quantities: the building blocks of Physics *8* 2.3 Length and its measurement *8* 2.4 Time and its measurement *11* 2.5 The SI system of units *12* 2.6 Dimensions of physical quantities *16* 2.7 Principle of homogeneity of dimensions *18* 2.8 Uses of dimensional equations *18* 2.9 Limitations of dimensional analysis *26* 2.10 Order of magnitude *26* 2.11 Accuracy and errors in measurement *28* Summary *33* Exercises *34*

3 Description of Motion in One Dimension 36

3.1 Introduction *36* 3.2 The idea of a point object *36* 3.3 Motion in one, two and three dimensions *37* 3.4 Motion in a straight line: groundwork *37* 3.5 Uniform motion *39* 3.6 Non-uniform motion with constant acceleration *45* 3.7 General relation between position and velocity *57* 3.8 Acceleration in a general one-dimensional motion *59* Supplement *60* Summary *61* Exercises *61* Problems *62*

4 Description of Motion in Two and Three Dimensions 65

4.1 Introduction: the need for vectors *65* 4.2 Position and displacement vectors in two dimensions *65* 4.3 Vector and scalar quantities *67* 4.4 Vector operations *67* 4.5 Zero vector and its properties *75* 4.6 Resolution and components of vectors in a plane *78* 4.7 Motion in two dimensions *83* 4.8 Projectile motion *87* 4.9 General position-velocity-acceleration relations for motion in two dimensions *96* 4.10 Uniform circular motion *99* 4.11 Motion of objects in three-dimensional space *106* Summary *107* Exercises *108* Problems *108*

5 Law of Motion **111**

5.1 Introduction *111* 5.2 Galileo's experiments on the motion of objects *112* 5.3 Newton's first law of motion: the law of inertia *114* 5.4 Newton's second law of motion *115* 5.5 Concurrent forces *118* 5.6 Newton's third law of motion *120* 5.7 The law of conservation of linear momentum *123* 5.8 Impulse *133* 5.9 Forces in nature *135* 5.10 Friction *135* 5.11 Inertial and non-inertial frames *144* Supplement *147* Summary *148* Exercises *149* Problems *149*

6 Work, Energy and Power **152**

6.1 Introduction *152* 6.2 Scalar product of two vectors *152* 6.3 Work *154* 6.4 Work done by a constant force *156* 6.5 Work done by a variable force *157* 6.6 Energy *160* 6.7 Power *163* 6.8 Transformation of energy *168* 6.9 Conservation of energy *169* 6.10 Conservative and non-conservative forces *172* 6.11 Collisions *176* Summary *191* Exercises *192* Problems *193*

7 Centre of Mass, Rigid Bodies and Rotational Motion **195**

7.1 Introduction *195* 7.2 Centre of mass of a two-particle system *195* 7.3 Rigid bodies *204* 7.4 Description of rotational motion in a plane *206* 7.5 Plane rotational motion of a single particle: the torque *208* 7.6 Angular momentum of a particle *212* 7.7 Some examples of circular motion of rigid bodies *219* 7.8 Torque and angular momentum for a system of particles *225* 7.9 Rotational motion and moment of inertia *227* 7.10 Some examples of two-dimensional rigid body motion *236* Supplement *245* Summary *248* Exercises *249* Problems *250*

8 Gravitation **253**

8.1 Introduction *253* 8.2 Acceleration due to gravity near earth's surface *254* 8.3 Newton's law of universal gravitation *254* 8.4 Inertial and gravitational mass *260* 8.5 Variation of g *262* 8.6 Gravitational potential energy *265* 8.7 Escape velocity *268* 8.8 Geostationary satellites *274* 8.9 Planetary motion *277* 8.10 Torque due to gravity: centre of gravity *282* Supplement *283* Summary *286* Exercises *287* Problems *287*

9 Molecules **289**

9.1 Introduction *289* 9.2 The atomic hypothesis: atoms and molecules *289* 9.3 Brownian motion *290* 9.4 Avogadro number *294* 9.5 Avogadro's hypothesis *297* 9.6 Interatomic and intermolecular forces *298* 9.7 States of matter *303* Summary *306* Exercises *306* Problems *307*

10 Properties of Matter **309**

10.1 Introduction *309*

Section A: Solids 309

10.2 Introduction *309* 10.3 Crystalline and amorphous solids *310* 10.4 Elasticity *312*

Section B: Fluids 329

10.5 Introduction *329* 10.6 Fluid pressure *329* 10.7 Pascal's law *332* 10.8 Atmospheric pressure *337* 10.9 Variation of atmospheric pressure with altitude *340* 10.10 Archimedes' principle *343* 10.11 Viscosity *348* 10.12 Fluid flow *357*

Section C: Liquids 370

10.13 Surface tension *370* 10.14 Surface energy *375* 10.15 Angle of contact *378* 10.16 Drops and bubbles: pressure difference across a film *382* 10.17 Capillarity *384* 10.18 Effect of temperature on surface tension *392* 10.19 Detergents and surface tension *392*

Section D: Gases 392

10.20 Introduction *392* 10.21 Kinetic theory of gases *393* 10.22 Gas laws *400* 10.23 Equation of state of an ideal gas *403* 10.24 Kinetic interpretation of temperature *403* Supplement: S.1 Derivation of Hooke's law *408* S.2 Derivation of Poiseiulle's formula *409* S.3 Van der Waals' equation of state *411* S.4 Distribution of molecular speeds *412* S.5 Collisions, mean free path and diffusion *413* Summary *414* Exercises *415* Problems *417*

11 **Heat and Thermodynamics** . **421**

Section A: *Heat* *421*

11.1 Introduction: heat and temperature *421* 11.2 Modern concept of heat *422* 11.3 Temperature *423* 11.4 Thermal expansion *428* 11.5 Specific heat *435* 11.6 Specific heats of a gas *437*

Section B: *Thermodynamics* *443*

11.7 Introduction *443* 11.8 Definition of some terms used in thermodynamics *444* 11.9 Zeroth law of thermodynamics: thermal equilibrium and concept of temperature *445* 11.10 Isothermal and adiabatic processes *447* 11.11 Internal energy *447* 11.12 First law of thermodynamics *448* 11.13 Applications of the first law of thermodynamics *449* 11.14 Phases and phase diagrams *451* 11.15 Heat engines and second law of thermodynamics *457*

Section C: *Transfer of Heat* *465*

11.16 Introduction *465* 11.17 Conduction *465* 11.18 Convection *469* 11.19 Radiation *470* Supplement: S.1 Pressure-volume relation in an adiabatic change *478* S.2 Evaporation and boiling *479* Summary *480* Exercises *480* Problems *482*

12 **Oscillations** **484**

12.1 Introduction *484* 12.2 What causes a system to oscillate *484* 12.3 Simple harmonic motion *485* 12.4 Periodic and harmonic functions *489* 12.5 Characteristics of an SHM *492* 12.6 Relation between linear SHM and uniform circular motion *495* 12.7 Velocity and acceleration of SHM *497* 12.8 Total energy in SHM *504* 12.9 Some examples of SHM *508* 12.10 Initial conditions *522* 12.11 Damped oscillations *523* 12.12 Forced oscillations and resonance *525* Supplement *527* Summary *529* Exercises *530* Problems *530*

13 **Waves** **533**

13.1 Introduction *533* 13.2 Waves in a continuous medium *534* 13.3 Types of waves *535* 13.4 Characteristics of a harmonic wave *536* 13.5 Relation between frequency, wavelength and wave velocity *540* 13.6 Mathematical description of harmonic wave: the wave equation *540* 13.7

The wave function *542* 13.8 Graphical representation of a harmonic wave *544* 13.9 Phase and phase difference *548* 13.10 Velocities in wave motion *553* 13.11 Newton's formula and Laplace's correction *557* 13.12 Effect of change of pressure, temperature and humidity on velocity of sound in a gas *556* 13.13 Boundary effects *563* 13.14 Superposition of waves *565* 13.15 Standing (or stationary) waves *568* 13.16 Beats *583* 13.17 The Doppler effect *588* 13.18 Musical sound and noise *595* 13.19 Musical scale *596* 13.20 Acoustics of buildings *597* 13.21 Dispersion *600* Supplement on 'Periodic waves' *600* Summary *602* Exercises *604* Problems *606*

Answers to Problems **609**

1

Introduction

1.1 WHAT IS PHYSICS?

Broadly speaking, physics is a branch of science which deals with the study of the phenomena of nature. The word 'physics' is derived from a Greek word meaning nature. The word 'science' comes from the Latin word *scientia* which means 'to know'.

Man has been observing various natural phenomena from time immemorial. He has always been curious about nature and the world around him. The motion of the moon and other heavenly bodies in the sky has aroused awe and amazement in him. The regular repetition of sunrise and sunset and the seasons of the year have fascinated him. Man has observed these and other natural phenomena and responded to them in an orderly manner. The experiences gained over a period of time were transmitted from generation to generation and began to be termed as 'knowledge'. Each generation added new facts to the knowledge obtained from the previous generation. The systematized knowledge thus gained was termed 'science'.

The Scientific Method

Every bit of knowledge is not called science—only such knowledge as is collected by what is called the *scientific method.* The study of the history of the progress of science reveals that it has evolved through a method or procedure. The so-called scientific method is the basis of scientific development. The method involves four steps:

1. Observation of the relevant facts,
2. proposal of a hypothesis or a theory based on these observations,
3. testing of the proposed theory to see if its consequences or predictions are actually observed in practice, and
4. modification of the theory, if necessary.

The first and the most important step in the study of a natural phenomenon is *observation* and *measurement* to arrive at a truth. This is the subject matter of the next chapter. Observation can be *subjective* or *objective.* An observation that varies from individual to individual is subjective. For example, different individuals observing the same thing, e.g. a painting or a flower, feel differently. Physics does not deal with such subjective observations. *Physics is a science of objective observation*, an observation which is the same for all individuals.

The next step is the proposal of a hypothesis or theory to explain the observations in terms of a limited number of laws. A good theory is one which not only successfully explains the observed facts but can also predict those likely to be brought to light in the future, and whose consequences can be tested by further observation. If a new measurement reveals a discrepancy between the existing theory and the observations, then the theory is not fully acceptable; it needs to be modified or refined. If the modified theory cannot explain the significant difference, however small, or if the predictions of the theory are not confirmed by measurements, then the theory is discarded and a new theory has to be developed involving new concepts. We come across many such instances in the history of the progress of science. Early thinkers believed that the earth was flat which was an obvious conclusion. The observations of distant ships in the sea (to elaborate, that at first only the mast of the ship was visible and later the rest of the ship) could not be explained on the basis of the concept of the earth being flat. This new observation gave a clue to the shape of the earth. So the earlier concept of the flat earth was abandoned and the spherical shape of the earth came to be accepted. Later, when precision measurements of the acceleration due to gravity were made at various places on the earth, it was found that the spherical shape could not account for the small variations in the value of the acceleration due to gravity. To account for these small discrepancies, the shape of the earth had to be considered as oblate spheroidal. With the advent of earth satellites, it is now possible to photograph (or see) the earth from far above and determine its exact shape.

To cite another example, for over two centuries Newton's equations of motion (which are dealt with in Chapter 5) were accepted as correct. But when the motion of bodies moving at very high speed (close to the speed of light) was studied, Newton's equations were found to be totally inadequate. To deal with objects moving at such high speeds, Albert Einstein developed a new theory called the Special Theory of Relativity using entirely new and revolutionary concepts regarding space and time. All predictions of his new theory have been experimentally confirmed. Similarly Einstein developed the General Theory of Relativity to explain some facts about gravitation which could not be understood in terms of Newton's law of gravitation (see Chapter 8).

To cite yet another example, Newton proposed the corpuscular theory of light to account for his observations of shadow formation. According to this theory, light is made up of corpuscles or particles. But later experimental observation on the diffraction (bending) of light by small obstacles could be explained only on the basis of the wave theory of light. When photoelectric effect was discovered, both these theories failed to explain it. So a new theory of light, involving entirely new concepts of the true nature of light was born, the quantum theory of light.

These examples show how a scientist practises physics and how physics has developed to its present stage. Thus the progress of physics (and indeed of all other branches of science) is directly related to the advances in experimental observation and to the development of new concepts.

1.2 SCOPE AND EXCITEMENT OF PHYSICS

The various sciences may be divided into two broad classes, *physical* and *biological*. Physical sciences deal with non-living matter and biological sciences with living matter. About a hundred years ago, it was possible for one man to master the knowledge of both sciences and many outstanding workers in physical sciences were also competent doctors and biologists. There was no clear-cut division between the several branches of physical science as we know them today. In fact, all were included in the term *natural philosophy*. Aristotle, Archimedes and even Galileo and Newton called themselves natural philosophers. But today the situation is different. The tremendous upsurge of scientific activity and the accumulation of knowledge in the last century have forced scientists to narrow down their field of activity. Not even a genius could hope to keep up with the developments in the various branches of physical sciences as we know them today, namely, physics, chemistry, astronomy, biology, medicine, geology, engineering, etc.

Those concerned mainly with the applications of science to the betterment of human life and environment are called engineers. The invention of the steam engine and the electric motor gave rise to engineering. Physicists and chemists are concerned more with the basic aspects of nonliving matter. Chemistry deals primarily with molecular changes and the rearrangement of the atoms which form molecules. Physics deals with the phenomena of the non-living world such as mechanics (motion), heat, sound, electricity, magnetism and light. The division of the subject of physics into these and other branches is largely a matter of convenience. These branches are inter-related as you will discover in the course of your study of the subject. *Physics may be defined as that branch of knowledge that deals with the phenomena of non-living matter*.

In physics, we deal with many physical phenomena and experiences. Merely reading about the experiences and observations of others is not enough. If students are to understand and enjoy physics, they must have some of these experiences themselves. These experiences are not only exciting but also very educative. The swinging hanging lamp in a Church led Galileo to a method of measuring time. The fall of an apple and the motion of the moon led Newton to his famous law of gravitation. The rattling (or dancing) of the lid of a kettle led to the invention of the steam engine. The blowing of a flute causes vibrations that produce sound. The light from stars tells us something about stars and their evolution. The study of electricity helps us to design motors and dynamos. The study of semiconductors helps us to design radios, televisions, calculators and even computers.

The physical world presents a very wide diversity. In Chapter 2 you will learn about the enormous ranges of length, mass and time in physical processes. The phenomena involving sub-atomic particles deal with masses as small as 10^{-30} kg whereas astronomical phenomena may involve masses as large as 10^{30} kg or more. Astronomical distances are as large as 10^{26} m and time scales as long as 10^{9} years. Subatomic processes involve distances as small as 10^{-15}m and time scales as short as 10^{-20} s. Thus physical phenomena have an enormous range of length, mass and time measurements. Apart from this, a physical phenomenon is never an isolated or independent happening; there are a variety of interactions one has to deal with. In spite of such a wide range and complexity of physical phenomena, it is amazing that these phenomena can be analysed and understood in terms of a few basic laws and principles. One such law is the law of conservation of energy. In the complex processes (e.g. interactions) there is something that remains constant. It is the energy. The law of conservation of energy is the one unifying principle that brings order to the jumble of facts concerning the natural processes in our universe.

Another important reason why it is possible to analyse a complex phenomenon is that we can (with ingenuity and intuition) identify a few important features in any phenomenon out of a large number of parameters; many of which are of less significance in relation to the phenomenon under study. We can then understand the complex phenomenon in terms of the important parameters and unfold the hidden simplicity behind the complexity. Nature is a very fine example of unity in diversity. In the later years of his life, Albert Einstein tried (unsuccessfully) to discover a unified theory of interactions as he strongly believed in the unity of natural processes.

1.3 PHYSICS AND TECHNOLOGY

Practical applications of physics and other branches of science have played a very important role in the development of industry and in improving the standard of living of man. The inventions of steam engines and electrical motors constitute the beginning of industrial revolution. The understanding of electromagnetic waves in the longer wavelength domain has found applications in radio, television and wireless communication. We can watch television programmes from distant countries 'live' via satellites called geostationary satellites. These satellites also help in forecasting weather and in geophysical survey such as exploration of oil wells. The electricity that we use in our homes and industry is generated at a power plant by the conversion of some other form of energy into electrical energy. In a thermal power plant, the heat energy of superheated steam is converted into electrical energy. In a hydro-electric power plant, the gravitational potential energy of water stored in a raised reservoir in a dam is converted into electrical energy. Nuclear energy released in a fission process is the source of energy in a nuclear reactor which produces electricity. These and a host of other applications of physics have contributed a great deal to the technical advancement of society. Thus physics plays a very significant role in technology and in our daily lives.

We will begin our journey through the world of physics with the measurements of various physical quantities in terms of internationally accepted units and their order of accuracy. We will then learn the motion of objects in one, two and three dimensions and deduce the equations governing the motion. The causes of motion will be dealt with separately. The motion of extended objects (rotational motion) and its special features will be discussed. A separate chapter is devoted to gravitation because of its relevance in the understanding of our universe. We will then focus our attention on the microscopic description of matter in terms of molecules and later discuss the bulk properties of matter. The subject of heat and thermodynamics has be been dealt with in some detail as it is important in the development of heat engines and in the understanding of chemical reactions. Finally we will study in detail some mechanical oscillations and their relevance in waves.

SUMMARY

Physics is a branch of science that deals with the study of the phenomena of nature. The scientific method used in the study of sciences involves observation, proposal of a theory, testing the consequences of the proposed theory and modification or refinement of the theory in the light of new facts. The applications of physics have played a very great role in technology and in our daily lives.

EXERCISES

1. What do you understand by the 'scientific method'? Explain with the help of two examples.
2. Describe five natural phenomena relevant to physics. How have these phenomena been studied by man?
3. Briefly describe the important contributions of the following scientists: Galileo Galilei, Issac Newton, Michael Faraday, Rutherford, Maxwell, Raman and Albert Einstein. (Read the biographies of these great men of physics from your school library. They are fascinating reading.)
4. 'A good student of science should doubt any scientific statement he hears or reads'. Comment.
5. No physicist has ever seen an electron or even an atom. Yet all physicists believe in their existence. How far are they justified in their belief?
6. 'Subjective observation has no place in physics'. Comment.
7. Comment on the inter-disciplinary nature of the different branches of science.
8. Describe the wide diversity of the physical phenomena in the universe. How does physics help unfold the underlying unity in diversity?

9. State one law of nature which holds good in all natural processes.
10. Write an essay on the role of physics in technology and in our daily lives.
11. 'A theory is not scientific if it cannot be proved wrong at any later point of time'. How far would you agree with this statement?
12. After you have read this book, look out for some equations and laws which strike you most. Give the reason for each choice.

2

Measurement

2.1 INTRODUCTION: THE NEED FOR MEASUREMENT

The study of the history of the progress of science reveals that it has evolved through a method or procedure. The so-called scientific method is the basis of scientific development. This method involves four steps:

1. Observation of the relevant facts,
2. proposal of a hypothesis or a theory based on these observations,
3. testing of the proposed theory to see if its consequences or predictions are actually observed in practice, and
4. modification of the theory, if necessary.

We shall discuss the first and the most important step, namely, *observation*. Observation can be *subjective* or *objective*. An observation that varies from individual to individual is subjective. For example, different individuals observing the same thing, e.g. a painting or a flower, feel differently. Physics does not deal with such subjective observations. Physics is a science of *objective* observation, an observation which is the *same* for all individuals. An individual observes through his sense of touch or sight, but these senses are not always reliable. To illustrate the inaccuracy of our sense of touch, we consider three pans containing cold, warm and hot water. If you put your finger first in cold water and then in warm water, your sense of touch will tell you that it is hot. But if you put your finger first in hot water and then in warm water, your sense tells you it is cold. This clearly suggests the necessity of making a measurement to arrive at the truth. It is necessary to measure the degree of hotness of water in each pan. In other words, it is not enough to describe a phenomenon in a general and qualitative way. A number must be tied to it. Thus, *physics is a science of measurement*. Lord Kelvin, a leading physicist of the 19th century, once said:

"When you can measure what you are talking about and express it in numbers, you know something about it; but when you cannot, your knowledge is of a meagre and unsatisfactory kind; it may be the beginnig of knowledge, but you have scarcely in your thoughts advanced to the stage of a science."

2.2 FUNDAMENTAL PHYSICAL QUANTITIES: THE BUILDING BLOCKS OF PHYSICS

The building blocks of physics are the physical quantities in terms of which the laws of physics can be expressed. *All quantities which can be measured either directly or indirectly are called physical quantities*; such as length, time, mass, temperature, speed, force, electric current, etc. Physical quantities are divided into two classes, *fundamental* or *derived*. Examples of quantities usually classified as fundamental are length, mass and time. Speed is an example of a derived quantity as it is derived from the measurement of length and time.

2.3 LENGTH AND ITS MEASUREMENT

Length is the extent of the separation (i.e. distance) between two points in space. We become aware of the separation when we look at two objects at different positions, such as two persons standing a certain distance apart on earth or when we look at two heavenly bodies. The measurement of length or distance is of obvious significance.

The measurement of a physical quantity involves the choice or selection of a suitable standard or *unit*. The size of the unit is arbitrary but it must be the same kind of quantity as the one to be measured. For example, for measuring a length, the unit itself must be a length. The unit chosen for the measurement of length is a standard length named *metre*. A physical quantity is completely specified by two characteristics:

1. the unit in terms of which the quantity is measured, and
2. the numerical value, i.e. the number that tells us how many times the unit occurs in the quantity.

Lengths or distances have a large range of variation. The particles inside a nucleus are separated by a distance several billion times smaller than the length of, say, a tiny insect, whereas the distance of the nearest star from the earth is several billion times the height of a man.

Measurement of Length

There are two methods of measuring the distance between two points: (i) direct method, and (ii) indirect method.

Direct Methods These methods involve a direct comparison of the length to be measured with a chosen standard, i.e. a metre stick. Suppose you wish to measure the length of a long iron rod. Lay the metre stick successively along the rod starting from one end and find out how many times this can be done until the other end of the rod is reached. If you find that the length of the rod is five times the length of a metre, then the length of the rod is 5 metres.

Indirect Methods If the distance to be measured is exceedingly small or much larger than the chosen standard, then a direct comparison with the standard is not possible. For example, the size of an atom or the distance of a star from the earth cannot be measured directly. An indirect method, involving more complicated procedures, is usually necessary.

Let us first consider the measurement of large distances. Suppose we wish to measure the distance of an inaccessible wall or a hill from a certain point on the surface of the earth. Standing at that point we fire a gun by directing it towards the wall and we measure that time interval between the instant the gun is fired and the instant the echo is heard. Now, we need a subsidiary experiment to determine the speed of sound at that temperature. The value of the speed of sound multiplied by half the time measured gives the required distance. Even the indirect method involves the use of the standard of length. In this case the standard of length is used in the determination of the speed of sound. Adopting the same procedure, the distance of the moon from a rocket launching station can be measured. A radar signal is sent from that station. The signal reflected from the surface of the moon is received by a receiver at the sending station. If the time between the sending and receiving of the signal is measured, and the speed of the radar signal is known, the required distance can be measured.

Only indirect methods can be used to measure astronomical distances. The distances of nearby stars can be measured by what is called the *triangulation* method. This method uses the geometrical properties of a triangle in a plane. The principle of this method is as follows: Consider a triangle SAB [Fig. 2.1 (a)]. If the length AB of the base and the angles θ_1 and θ_2 subtended by the point S at the base AB are known, then, using trigonometric relations. the vertical distance h of S from the base line AB can be determined. Suppose S is a satellite and A and B are two points on the surface of the earth. Then the height of the satellite can be determined by measuring angles θ_1 and θ_2 by pointing a telescope toward the satellite standing at points A and B respectively. Using this method, the height of the first Sputnik was found to be 500 km. This method has to be modified if the point S (e.g. a star) is too far away from the earth, for in that case, $\theta_1 = \theta_2 = 90°$ (and the height will come out to be infinity) unless the points A and B are very far away. The length of the base line can be increased by pointing a telescope at the star at a six-month interval when the earth has moved from one point of its orbit to a diametrically opposite point. In Fig. 2.1 (b), these points are A and B. The length of the base line is now increased and is equal to the diameter of the earth's orbit round the sun ($\sim 3 \times 10^8$ km). Let S be the nearby star whose distance is to be measured. We choose, now, a very distant star (say N), whose direction may be taken to be practically the same at all positions of the earth in its orbital motion. Suppose at some position of the earth (22 December), say A, we measure, with the help of a telescope, the angle between the direction of the distant star N and that of the star, S, i.e. the angle between AN and AS. Then we wait for six months and measure the angle again when the earth is, say, at position B (22 June).

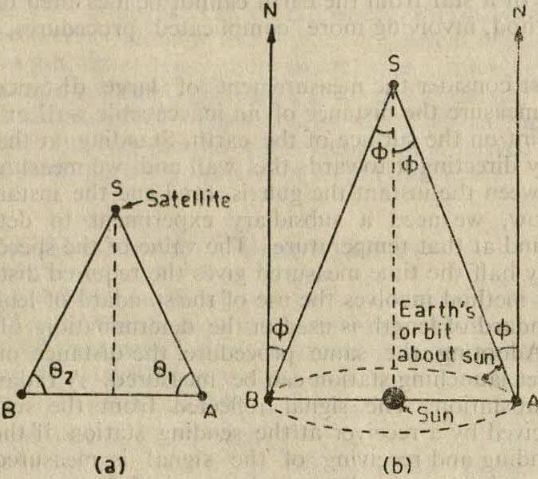


Fig. 2.1 The triangulation method

The sum of the two measured values of ϕ is equal to the angle the star subtends on the earth's orbital diameter (AB). It is easy to see from Fig. 2.1 (b) that

$$\frac{d}{h} = \tan\phi$$

$$h = \frac{d}{\tan\phi}$$

For very distant stars (such as N), the triangulation method cannot be used since then $\phi \simeq 0$ and cannot be measured accurately. Distances of such stars are measured by spectroscopic methods, about which you will learn in higher classes.

For the measurement of extremely small distances, such as the size of molecules and atoms, only indirect methods can be used. For example, the size of a molecule can be estimated by Avogadro's hypothesis. Subatomic distances can be measured by *alpha-particle scattering* experiments. Distances of the order of the wavelength of visible light are measured by interference experiments. Distances between the molecules of a crystal can be measured by *x-ray diffraction* experiments. The details of these methods are outside the scope of this book.

Table 2.1 gives the value of some measured distances. Notice the vast range over which length measurements can be made.

Table 2.1 Some measured lengths

	Metres
Distance of the most distant detected heavenly object	5×10^{25}
Radius of our galaxy	6×10^{19}
Distance of the nearest star (Alpha Centauri)	4×10^{16}
Mean orbit radius of the planet Pluto	6×10^{12}
Radius of the sun	7×10^{8}
Mean distance from Earth to Moon	2×10^{8}
Radius of the earth	6.4×10^{6}
Height of an average man	1.8×10^{0}
Thickness of a sheet of paper	1×10^{-4}
Radius of a hydrogen atom	5×10^{-11}
Radius of a proton	1.2×10^{-15}

2.4 TIME AND ITS MEASUREMENT

Physics is concerned not only with the measurement of length but also with the motion of objects. The idea of motion involves the concept of time. The separation between the occurrence of two events is called the time interval. Any phenomenon that repeats itself can be used as a measure of time interval. For example, a pendulum can be used to measure time interval. The rotation of the earth on its axis, which determines the length of the day, has been used as a time standard from the earliest times. The day is subdivided into hours, minutes and seconds.

There are direct and indirect methods for measuring time intervals. Direct methods make use of a watch or a clock. Indirect methods are used to measure very short and very long time intervals. The laws of radioactive decay have been used to measure very long as well as very short time intervals. Some radioactive substances, such as carbon 14, take millions of years to decay whereas others decay in a millionth fraction of a second. Carbon dating is used to estimate the age of rocks. The details of these methods are outside the scope of this book.

Table 2.2 gives the orders of magnitude of some measured time intervals. Notice the vast range over which the measurements of time intervals can be made.

The measurement of the third fundamental physical quantity, namely, mass, will be discussed in Chapter 3.

Table 2.2 Some measured time intervals

	Seconds
Age of the earth	1×10^{17}
Human life expectancy	2×10^{9}
Time of Earth's orbit around Sun (1 year)	3×10^{7}
Time of Earth's rotation about its axis (1 day)	8.6×10^{4}
Time interval between normal heartbeats	1×10^{0}
Time period of sound waves	1×10^{-3}
Time period of radio waves	1×10^{-7}
Time period of rotation of a molecule	1×10^{-12}
Half-life of neutral pion	2.2×10^{-16}
Period of nuclear vibrations	1×10^{-21}

2.5 THE SI SYSTEM OF UNITS

In 1960, the eleventh General Conference of Weights and Measures introduced a new logical and rationalized system of units called the International System of Units (abbreviated SI in all languages). In this system there are *six fundamental units*. There are the units of six fundamental quantities, namely, *length*, *mass*, *time*, *electric current*, *temperature and luminous intensity*. The names of the units respectively are *metre*, *kilogram*, *second*, *ampere*, *kelvin and candela*. In 1971, one more fundamental unit named *mole* was added for the amount of substance. In addition to these seven basic units, there are *two supplementary units – radian* for angle in a plane and *steradian* for solid angle. The internationally accepted definitions of these units are as follows:

Fundamental Units

Metre (Symbol m) Until recently, the international standard of length was a bar of platinum-iridium alloy called the *standard metre* kept at the International Bureau of Weights and Measures near Paris, in France. The distance between two fine lines engraved near the ends of the bar (when at 0.00 °C) was defined to be one metre. There are several objections to the metre bar as the primary standard of length. First, it can be destroyed by fire or stolen. Secondly, it is not easily accessible. Thirdly, the metal is likely to undergo minute changes in length with the passage of time. Nowadays, all countries have adopted an indestructible, reproducible and easily accessible standard of length which is defined in terms of the wavelength of light. According to this definition *one standard metre is equal to 1650763.73 wavelengths (in vacuum) of the orange-red radiation corresponding to the transition between the energy levels* $2p_{10}$ *and* $5d_5$ *of the* ^{86}Kr *atom.*

For expressing very small lengths, submultiples of a metre are used. They are

$$\text{centimetre (cm)} = \frac{1}{100}\text{ m} = 10^{-2}\text{ m}$$

$$\text{millimetre (mm)} = 0.1\text{ cm} = 10^{-3}\text{ m}$$

$$\text{micron} = 10^{-6}\text{ m}$$

$$\text{angstrom } (\text{Å}) = 10^{-10}\text{ m}$$

$$\text{nanometre (nm)} = 10^{-9}\text{ m}$$

$$\text{fermi} = 10^{-15}\text{ m}$$

For expressing larger lengths or distances, bigger units are used. They are

$$\text{kilometre} = 10^{3}\text{ m}$$

For astronomical distances, a unit called the *light year* is used. A light year is the distance which light, travelling at a speed of 3×10^8 ms^{-1}, covers in one year. Thus

$$\begin{aligned}1\text{ light year} &= (3 \times 10^8) \times 60 \times 60 \times 24 \times 365.25\\ &= 9.467 \times 10^{15}\text{ m approximately}\end{aligned}$$

The nearest star, Alpha Centauri, is 4.3 light years away from the earth.

Sometimes the *astronomical unit* (A.U.) is used to express very large distances. The mean distance of the sun from the earth is called the astronomical unit and is equal to 1.496×10^{11} m.

Second (Symbol s) One mean day[1] has been used as standard of time from earliest times. One mean solar second was defined to be 1/86400 of a mean solar day. Nowadays, a much more accurate standard of time is used. It is based on a particular type of atomic called the cesium clock. The internationally accepted second is defined as *the time interval occupied by 9 192 631 772 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of cesium 133 atom.*

Kilogram (Symbol kg) It is a fundamental unit of mass and is equal to *the mass of the platinum-iridium cylinder of diameter equal to its height* kept at the International Bureau of Weights and Measures.

Like length and time, mass is measured by direct and indirect methods. Table 2.3 gives the order of magnitude of some measured masses. Notice the wide range over which mass measurements have been made.

---

[1] Solar day is the time interval between one noon to the next noon and a mean solar day is the average of all solar days in one year.

Table 2.3 Some measured masses

	Kilogram
Our galaxy	2.2×10^{41}
Earth	6×10^{24}
Average man	7×10^{1}
A house fly	1×10^{-4}
Dust particle	1×10^{-13}
Helium atom	6.67×10^{-27}
Electron	9.1×10^{-31}

Ampere (Symbol A) It is the fundamental unit of electric current. An ampere is defined as *that constant current which, when flowing in two straight parallel conductors of infinite length and of negligible area of cross-section and placed one metre apart in vacuum, would produce between these conductors a force equal to* 2×10^{-7} *newton per metre of length.*

Kelvin (Symbol K) It is the unit of temperature and is defined as *the fraction 1/273.16 of the thermodynamic temperature of the triple point of water*[2].

Candela (Symbol cd) It is the unit of luminous intensity. A candela is defined as the *luminous intensity, in a perpendicular direction, of a surface of area 1/600000 square metre of a black body at the temperature of freezing platinum under a pressure of 101 325 newton per square metre.*

Mole (Symbol mol) It is *the amount of the substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.*

Supplementary Units

Radian (Symbol rad) It is *the plane angle between two radii of a circle which cut off on the circumference an arc equal to the length of the radius.* Note that

$$\text{Angle (in radian)} = \frac{\text{Arc}}{\text{Radius}}$$

Steradian (Symbol sr) It is *the solid angle which, with its vertex at the centre of a sphere, cuts off an area on the surface of the sphere equal to that of a square having sides of length equal to the radius of the sphere.* If S is the area cut off on the surface of a sphere of radius r, the solid angle at the centre of the sphere is given by

$$\phi \text{ (in steradian)} = \frac{S}{r^2}$$

---

[2] Triple point of water is the temperature at which water exists in all the three states, namely, solid, liquid and vapour (gas).

The student is not expected to memorize these definitions which are given here for the sake of completeness, interest and importance of devising a permanent, rational and reproducible internationally acceptable standard.

SI Derived Units

Any rationalized system of units must keep the number of units for physical quantities as small as possible, because otherwise one would have to remember a separate unit for each physical quantity. In the SI system, the nine physical quantities mentioned above are regarded as *fundamental quantities* which are measured in terms of *fundamental units*. Other physical quantities which can be defined in terms of these fundamental quantities are, therefore, called *derived quantities* and their units are called *derived units*.

The name and symbol used for the derived quantities which we shall come across in this book are given in Table 2.4.

Table 2.4 Some SI derived uuits

Derived Physical Quantity	*Name of the Unit*	*Symbol*
Area	square metre	m^2
Volume	cubic metre	m^3
Frequency	hertz	Hz (or s^{-1})
Mass density	kilogram per cubic metre	$kg\ m^{-3}$
Speed (or velocity)	metre per second	$m\ s^{-1}$
Angular velocity	radian per second	$rad\ s^{-1}$
Acceleration	metre per second squared	$m\ s^{-2}$
Angular acceleration	radian per second squared	$rad\ s^{-2}$
Force	newton	N
Work, energy	joule	J
Power	watt	W
Pressure	pascal	Pa
Viscosity	pascal second	Pa s
Surface tension	newton per metre	$N\ m^{-1}$
Potential difference	volt	V
Electric field strength	volt per metre	$V\ m^{-1}$
Electric resistance	ohm	Ω (or VA^{-1})
Magnetic flux	weber	Wb
Magnetic flux density	tesla	T (or $Wb\ m^{-2}$)
Specific heat	joule per kilogram per kelvin	$J\ kg^{-1}\ K^{-1}$

They will be defined as and when we encounter them in the following chapters.

Merits of the SI System

The merits of the SI system of units are:

1. It is a coherent system of units, i.e. it is a system in which all the derived units are obtained from the fundamental units by only multiplication or division without introducing numerical factors.
2. the units are easily reproducible.
3. they do not vary with time as they are based on the properties of an atom.
4. it is a rationalized system.

2.6 DIMENSIONS OF PHYSICAL QUANTITIES

In Table 2.4 the units of the derived physical quantities have been written in terms of those of the fundamental physical quantities. We shall now introduce the concept of the dimension of a physical quantity and show how this can be done. For simplicity, we shall consider quantities that we come across in mechanics.

In mechanics, there are three fundamental quantities, namely, mass, length and time. Let the symbols M, L, and T respectively denote the units of mass, length and time. It should be remembered that these symbols only indicate the nature of the unit and not its magnitude. Then the unit of velocity can be written as:

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{L}{T} = LT^{-1}$$

Similarly, the units of area and density are:

$$\text{Area} = \text{Length} \times \text{Breadth} = L \times L = L^2$$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3} = M\,L^{-3}$$

Thus we say that the unit for velocity has zero dimension in mass (since speed does not depend on mass), 1 dimension in length and -1 dimension in time. Area has 2 dimensions of length; it has zero dimension of mass and zero dimension of time. Density has one dimension of mass, -3 dimensions of length and zero dimension of time. Thus, *the dimension of the units of a derived physical quantity may be defined as the number of times the fundamental units of mass, length and time appear in the physical quantity.*

Dimensional Formulae and Equations

Since the area does not depend on mass and time, its dimensions in mass and time are said to be zero. We may, therefore, represent area by $M^0L^2T^0$ also. Similarly, velocity and density may respectively be represented by M^0LT^{-1} and $ML^{-3}T^0$. The unit of a quantity written in this form is called its *dimensional formula*. Thus the dimensional formula for velocity is M^0LT^{-1}. When this formula is written in the form of an equation, it is called the dimensional equation. Thus, the dimensional equation for velocity is[3]

$$\text{Velocity } (v) = M^0LT^{-1}$$

Thus, M^0LT^{-1} is the dimensional formula for velocity and $(v) = (M^0 LT^{-1}$ is the dimensional equation for velocity. Table 2.5 gives the dimensional formulae of some physical quantities.

Table 2.5 Dimensional formulae of some physical quantities

Physical quantity	*Relation with other physical quantities*	*Dimensional formula*
Speed (or velocity)	Distance/Time	$\frac{L}{T} = LT^{-1} = M^0LT^{-1}$
Acceleration	Change in velocity/Time	$LT^{-1}/T = LT^{-2} = M^0LT^{-2}$
Force	Mass × Acceleration	$M \times LT^{-2} = MLT^{-2}$
Momentum	Mass × Velocity	$M \times LT^{-1} = MLT^{-1}$
Work (or energy)	Force × Distance	$MLT^{-2} \times L = ML^2T^{-2}$
Power	Work/Time	$ML^2T^{-2}/T = ML^2T^{-3}$
Density	Mass/Volume	$M/L^3 = ML^{-3} = ML^{-3}T^0$
Pressure	Force/Area	$MLT^{-2}/L^2 = ML^{-1}T^{-2}$
Stress	Force/Area	$ML^{-1}T^2$
Strain	$\frac{\text{Change in length or volume}}{\text{Original length or volume}}$	$\frac{L}{L}$ or $\frac{L^3}{L^3}$ — No dimensions
Elasticity modulus	Stress/Strain	$ML^{-1}T^{-2}$
Angle	Arc/Radius	L/L — No dimensions
Angular velocity	Angle/Time	$\frac{1}{T} = T^{-1} = M^0L^0T^{-1}$
Angular acceleration	$\frac{\text{Change in angular velocity}}{\text{Time}}$	$M^0L^0T^{-1}/T = M^0L^0T^{-2}$
Torque	Force × Distance	ML^2T^{-2}
Frequency	No. of vibrations/Time	$\frac{1}{T} = T^{-1} = M^0L^0T^{-1}$
Surface tension	Force/Length	$MLT^{-2}/L = ML^0T^{-2}$
Viscosity	Stress/Velocity gradient	$ML^{-1}T^{-1}$

---

[3] Sometimes the quantity is put within square brackets to represent its dimensions and not its magnitude, e.g.

$$[\text{Velocity}] = [M^0LT^{-1}] \quad \text{or} \quad [v] = [M^0LT^{-1}]$$

Some authors use round brackets, e.g.

$$(v) = (M^0LT^{-1})$$

Dimensional Variables and Constants

In physics, we come across many variables and constants: some of them have dimensions (or units) while others are dimensionless (i.e. they have no units). For example, variables such as velocity, force, etc. have dimensions. But variables such as angle (= arc/radius) strain, specific gravity are dimensionless; they are defined as a ratio of two quantities of the same dimensions. The examples of dimensional constants are velocity of light in free space, universal gravitational constant, etc. These constants have dimensions; but constants such as numbers (1, 2, . . .) etc. are dimensionless.

2.7 PRINCIPLE OF HOMOGENEITY OF DIMENSIONS

Consider a simple equation

$$A + B = C$$

If this is an equation of physics, i.e. if A, B and C are physical quantities, then this equation says that one physical quantity A, when added to another physical quantity B, gives a third physical quantity C. This equation will have no meaning in physics if the nature (i.e. the dimensions) of the quantities on the left-hand side of the equation is not the same as the nature of the quantity on the right-hand side. For example, if A is a length, B must also be a length and the result of addition of A and B must express a length. In other words, the dimensions of both sides of a physical equation must be identical. This is called the *principle of homogeneity of dimensions.*

2.8 USES OF DIMENSIONAL EQUATIONS

Dimensional equations provide a very simple method of deriving relations between physical quantities involved in any physical phenomenon. The analysis of any phenomenon carried out by using the method of dimensions is called *dimensional analysis.* This analysis is based on the principle of homogeneity of dimensions explained above.

There are four important uses of dimensional equations:

1. Checking the correctness of an equation.
2. Derivation of the relationship between the physical quantities involved in any phenomenon.
3. Finding the dimensions of constants or variables in an equation.
4. Conversion of units from one system to another.

We shall illustrate each use by giving a suitable example.

Checking the Correctness of an Equation

Using the principle of homogeneity of dimensions, we can check whether or not any relation or equation obtained for any physical process or phenomenon is correct. The equation will be correct only if the dimensions of each term on both sides of the equation are the same.

EXAMPLE 2.1 In Chapter 3 we shall derive (from first principles) an expression for the distance travelled by a particle in uniform acceleration. The expression reads

$$S = ut + \tfrac{1}{2} at^2 \tag{2.1}$$

where u is the initial velocity, a is the constant acceleration and S is the distance travelled in time t.

We know that the dimensional formulae of these quantities are:

$$\text{Distance } (S) = (\text{L})$$
$$\text{Velocity } (u) = (\text{LT}^{-1})$$
$$\text{Time } (t) = (\text{T})$$
$$\text{Acceleration } (a) = (\text{LT}^{-2})$$

Using these in Eq. (2.1), we have

$$(\text{L}) = (\text{LT}^{-1})\,(\text{T}) + (\text{LT}^{-2})\,(\text{T}^2) \qquad (\because \text{ constant } \tfrac{1}{2} \text{ is dimensionless})$$
$$= (\text{L}) + (\text{L})$$

i.e. the dimensions of each term on the right-hand side are the same as those on the left-hand side. Hence Eq. (2.1) is dimensionally correct or consistent.

EXAMPLE 2.2 In Chapter 12 we shall study the periodic oscillations of a simple pendulum and derive an expression for its time period t, which reads

$$t = 2\,\pi \sqrt{\frac{l}{g}} \tag{2.2}$$

where l is the length of the pendulum and g is the acceleration due to gravity. To check whether this equation is correct or not, we find the dimensional formula for each quantity in this equation.

$$\text{Time period } (t) = (\text{T})$$
$$\text{Length } (l) = (\text{L})$$
$$\text{Acceleration due to gravity } (g) = (\text{LT}^{-2})$$

The constant (2π) is dimensionless

Therefore, the dimensions of $2\,\pi \sqrt{\frac{l}{g}}$ are $\sqrt{\frac{\text{L}}{\text{LT}^{-2}}} = \sqrt{\text{T}^{+2}} = \text{T}$, the same as that of $t = (\mathrm{T})$. Thus the dimensions of the quantities of both sides of Eq. (2.2) are the same and the equation is dimensionally correct.

Derivation of Relationship between Different Physical Quantities

The principle of homogeneity of dimensions can be used to obtain a relationship between the different physical quantities involved in a given physical phenomenon. In many cases, the exact dependence of a given physical quantity on other physical quantities can be found if we know the factors on which that quantity depends.

Let us consider the following examples.

EXAMPLE 2.3 Let us take the example of a simple pendulum mentioned above. The time period of its oscillation is given by Eq. (2.2). Let us now see how we can obtain it from dimensional considerations.

A simple pendulum consists of a string of length l, fixed at one end and having a bob of a small mass m at the other (Fig. 2.2). When the bob A of the pendulum is displaced a little to a position B and released, it begins to return to its original position under the effect of gravity and executes periodic oscillations. The time period t of these oscillations may possibly depend on some or all of the following physical quantities of the system, namely, m the mass of the bob, l the length of the string, and g the acceleration due to gravity.

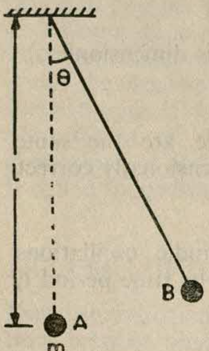


FIG. 2.2 A simple pendulum

The procedure to solve this problem by dimensional considerations is to assume that the time period

$$t \propto m^a$$

$$\propto l^b$$

$$\propto g^c$$

or

$$t \propto m^a l^b g^c$$

or

$$t = k\, m^a l^b g^c \tag{2.3}$$

where k is a dimensionless proportionality constant and a, b and c are constants to be determined.

Writing down the dimensional formula for each quantity in Eq. (2.3), we have (since k has no dimensions),

$$(\mathrm{T}) = (\mathrm{M})^a(\mathrm{L})^b(\mathrm{LT}^{-2})^c$$

or

$$(\mathrm{T}) = (\mathrm{M})^a(\mathrm{L})^{b+c}(\mathrm{T})^{-2c}$$

or

$$(\mathrm{M})^0(\mathrm{L})^0(\mathrm{T})^1 = (\mathrm{M})^a(\mathrm{L})^{b+c}(\mathrm{T})^{-2c}$$

According to the principle of homogeneity of dimensions, the dimensions of all the terms on either side of this equation must be the same.

Therefore, equating the exponents (i.e. powers) of M, L and T, we have

$$0 = a$$

$$0 = b + c$$

$$1 = -2c$$

Therefore $c = -\frac{1}{2}$, $b = +\frac{1}{2}$ and $a = 0$. Substituting these values in Eq. (2.3), we get,

$$t = k\, m^{\circ} l^{1/2}\, (g)^{-1/2}$$

or

$$t = k\sqrt{\frac{l}{g}}$$

It may be remarked that the value of the constant k cannot be determined by the method of dimensions because it is a number. Its value is determined either experimentally or from a theoretical derivation. It is found that $k = 2\pi$.

Therefore $$t = 2\pi\sqrt{l/g}$$

Although we are not able to determine the value of the proportionality constant by this method, yet it yields very useful information about the exact dependence of the time period on other quantities. We have, for example, found that time period t is independent of m, the mass of the bob. We have also discovered that the time period t will depend on l, the length of the pendulum and g, the acceleration due to gravity at that place. We have, in fact, gone a step further and found out how exactly t will depend on l and g, namely, t is directly proportional to the square root of l and inversely proportional to the square root of g. This is no small achievement. The merit of this method lies in the fact that, without using any law or hypothesis about pendulum oscillations, we have been able to show, how its time period will depend on its length and the acceleration due to gravity.

EXAMPLE 2.4 A uniform string of length l is stretched with a force F. If m is the mass per unit length of the string, show, using dimensional considerations, that the frequency of vibration of the string is

$$n = k\,\frac{1}{l}\sqrt{\frac{F}{m}}$$

where k is a dimensionless constant.

Solution Let us suppose that

$$n \propto l^{a} F^{b} m^{c}$$

or

$$n = k\, l^{a} F^{b} m^{c}$$

Writing down the dimensions on both sides (see Table 2.5), we have

$$(T^{-1}) = (L)^a(MLT^{-2})^b(ML^{-1})^c\left(\because\ m = \frac{\text{mass}}{\text{length}}\right)$$
$$= (M)^{b+c}(L)^{a+b-c}(T)^{-2b}$$

Equating the powers of M, L and T on either side, we get

$$0 = b + c$$
$$0 = a + b - c$$

and $$-1 = -2b$$

$\therefore$ $$b = \tfrac{1}{2} \quad \text{and} \quad c = -b = -\tfrac{1}{2}$$

But $a + b - c = 0,$

Therefore $a + \frac{1}{2} - (-\frac{1}{2}) = 0$

or $$a = -1$$

Hence $$n = k\, l^{-1}\, F^{1/2}\, m^{-1/2}$$

or $$n = k\frac{1}{l}\sqrt{\frac{F}{m}}$$

The value of k is found to be $\frac{1}{2}$ experimentally or from a detailed mathematical deduction.

EXAMPLE 2.5 Using dimensional consideration, show that the speed of sound in a material medium of modulus of elasticity E and density ρ is given by

$$v = k\sqrt{\frac{E}{\rho}}$$

where k is a dimensionless constant.

Solution Let

$$v = k\, E^a \rho^b$$

Writing down the dimensions, we get (see Table 2.5),

$$(LT^{-1}) = (ML^{-1}T^{-2})^a(ML^{-3})^b$$
$$= (M)^{a+b}(L)^{-a-3b}(T)^{-2a}$$

Equating the powers of M, L and T on both sides, we get,

$$0 = a + b$$
$$1 = -a - 3b$$

and $$-1 = -2a$$

which give $a = \frac{1}{2}$ and $b = -\frac{1}{2}$

$$v = k\,E^{1/2}\,\rho^{-1/2} = k\sqrt{\frac{E}{\rho}}$$

A complete mathematical investigation shows that $k = 1$.

EXAMPLE 2.6 Assuming that the mass m of the largest stone that can be moved by a flowing river depends upon the velocity v of water, its density ρ and acceleration due to gravity g, show that m varies as the sixth power of the velocity of water in the river.

Solution We assume that

$$m = k\,v^a\,\rho^b\,g^c$$

where k is a dimensionless constant and a, b and c are unknown powers to be determined. Substituting the dimensions of all the quantities involved, we have

$$(\mathrm{M}) = (\mathrm{LT}^{-1})^a\,(\mathrm{M\,L}^{-3})^b\,(\mathrm{LT}^{-2})^c$$

Equating powers of M, L and T on both sides, we have

$$1 = b$$

$$0 = a - 3b + c$$

$$0 = -a - 2c$$

$\therefore$ $$c = -a/2, \text{ so that}$$

$$0 = a - 3 \times 1 - a/2$$

or $$0 = a/2 - 3$$

$\therefore$ $$a = 6 \text{ and } c = -3$$

Thus $$m = k\,v^6\,\rho\,g^{-3} = \frac{k\,\rho\,v^6}{g^3}$$

or $m \propto v^6$, i.e. the mass of the largest stone moved is proportional to the sixth power of the velocity of water in the river.

EXAMPLE 2.7 Given that the time period T of oscillation of a gas bubble from an explosion under water depends upon p, d and E where p is the static pressure, d the density of water and E is the total energy of explosion, find dimensionally a relation for T.

Solution Assuming that $T = k\,p^a\,d^b\,E^c$ and substituting dimensions of all the quantities involved, we have

$$(T) = (\mathrm{ML}^{-1}\mathrm{T}^{-2})^a\,(\mathrm{ML}^{-3})^b\,(\mathrm{ML}^2\mathrm{T}^{-2})^c$$

Equating powers of M, L and T on both sides, we have

$$a + b + c = 0$$

$$-a - 3b + 2c = 0$$

$$-2a - 2c = 1$$

Solving these equations, we get

$$a = -5/6,\ b = 1/2 \text{ and } c = 1/3$$

$$\therefore \quad T = k\, P^{-5/6}\, d^{1/2}\, E^{1/3}$$

or
$$T = k\, \frac{d^{1/2}\, E^{1/3}}{P^{5/6}}$$

This is the required relation for T.

EXAMPLE 2.8 Experiments show that the frequency (n) of a tuning fork depends upon the length (l) of the prong, the density (d) and the Young's modulus (Y) of its material. From dimensional considerations, find a possible relation for the frequency of a tuning fork.

Solution Assuming that

$$n = k\, l^a\, d^b\, Y^c$$

and substituting dimensions of all the quantities involved, we have

$$(\mathrm{T}^{-1}) = (\mathrm{L})^a\, (\mathrm{M\,L}^{-3})^b\, (\mathrm{M\,L}^{-1}\,\mathrm{T}^{-2})^c$$

Equating powers of M, L and T on both sides, we have

$$b + c = 0$$

$$a - 3b - c = 0$$

$$-2c = -1$$

These give $c = 1/2$, $b = -1/2$ and $a = -1$.

$$\therefore \quad n = k\, l^{-1}\, d^{-1/2}\, Y^{1/2}$$

or
$$n = \frac{k}{l}\sqrt{\frac{Y}{d}}$$

This is the required relation for the frequency of a tuning fork.

Finding the Units of Constants in an Equation

In Chapter 10 we shall obtain the equation of state of an ideal gas, namely, $PV = RT$, where P is the pressure exerted by the gas, V its volume, T its absolute temperature and R is a constant. This equation is not applicable to real gases such as hydrogen, oxygen etc. Van der

Waals modified this equation and obtained a new equation of state for real gases. His equation, called the Van der Waals' equation, is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

where a and b are constants for a given gas. Now the question is: In what unit must a and b be expressed? The principle of homogeneity of dimensions answers this question. It tells that the dimensions of $\frac{a}{V^2}$ must be the same as those of P, (since it is added to P) and the dimensions of b must be the same as those of V. Hence,

$$(b) = \mathrm{L}^3$$

and $$\frac{(a)}{\mathrm{L}^6} = \mathrm{ML}^{-1}\,\mathrm{T}^{-2}$$

or $$(a) = \mathrm{ML}^5\,\mathrm{T}^{-2}$$

Thus in the SI system, the unit of constant b must be m^3 and that of constant a must be kg $\mathrm{m}^5/\mathrm{s}^{-2}$.

Conversion of Units from One System to Another Before 1960, scientists were using the so-called cgs (c for centimetre, g for gram and s for second) system of units. In this system (to take a specific example) the unit of force is called the *dyne*. *A dyne is that force which produces an acceleration of 1 cm* s^{-2} *in a mass of 1 g.*

When the new SI system was adopted, the conversion of units from the cgs to the SI system became necessary. In the SI system, the unit of force is called the *newton*. *A newton* is that force which produces an acceleration of 1 m s^{-2} in a mass of 1 kg.

Dimensional considerations can be used to find the relationship between the units of the same quantity in two different systems. We shall use this method to convert newton into dyne or vice versa. The important point to remember is that the dimensions of a physical quantity cannot be different for different systems of units because the nature of the quantities is the same. In other words, corresponding to a length in the cgs system (cm) we have a length in the SI system (m); only the magnitudes of the length are different (1 m = 100 cm). Thus, the dimensions of a force must be the same in the cgs system as well as in the SI system, i.e.

$$(\text{Force}) = (\text{Mass} \times (\text{Acceleration})$$

$$= (\mathrm{M}) \times (\mathrm{LT}^{-2}) = (\mathrm{ML\,T}^{-2})$$

If M = 1 kg. L = 1 m and T = 1 s, the force is said to be 1 newton. But when M = 1 g, L = 1 cm, and T = 1 s, the force is said to be 1 dyne. Hence, from the dimensions of force, we have

$$\begin{aligned} 1 \text{ newton} &= (1 \text{ kg}) \times (1 \text{ m}) \times (1 \text{ s})^{-2} \\ &= (1000 \text{ g}) \times (100 \text{ cm}) \times (1 \text{ s})^{-2} \\ &= 100{,}000 \times (1 \text{ g}) \times (1 \text{ cm}) \times (1 \text{ s})^{-2} \\ &= 100{,}000 \times 1 \text{ dyne (by definition of dyne)} \\ &= 10^5 \text{ dyne} \end{aligned}$$

Thus $\quad 1 \text{ newton} = 10^5 \text{ dyne}$

or $\quad 1 \text{ dyne} = 10^{-5} \text{ newton}$

2.9 LIMITATIONS OF DIMENSIONAL ANALYSIS

Though the dimensional method is a simple and a very convenient way of finding the dependence of a physical quantity on other quantities of a given system, it has its own limitations, some of which are listed as follows.

1. In more complicated situations, it is often not easy to find out the factors on which a physical quantity will depend. In such cases, one has to make a guess which may or may not work.

2. This method gives no information about the dimensionless constant which has to be determined either by experiment or by a complete mathematical derivation.

3. This method is used only if a physical quantity varies as the product of other physical quantities. It fails if a physical quantity depends on the sum or difference of two quantities. Try, for instance, to obtain the relation $S = ut + \frac{1}{2} at^2$ using the method of dimensions.

4. This method will not work if a quantity depends on another quantity as sin or cos of an angle, i.e. if the dependence is by a trigonometric function. The method works only if the dependence is by power functions only.

5. This method does not give a complete information in cases where a physical quantity depends on more than three quantities, because by equating the powers of M, L and T, we can obtain only three equations for the exponents. In our example of a simple pendulum (Fig. 2.2) we could assume that the time period t depends on angle θ of the swing, besides the factors l, m and g and write down Eq. (2.3) as

$$t = k\, m^a\, l^b\, g^c\, \theta^d$$

If we do this, we would find that a, b and c can be determined, but d remains undetermined.

2.10 ORDER OF MAGNITUDE

The concept of *order of magnitude* is very useful in estimating how big or how small a given physical quantity is. In order to determine the order of magnitude of a physical quantity, we first express it in the *nearest* power of 10. The power of 10 so obtained is called the order of magnitude of that physical quantity. Table 2.6 below gives a few examples.

Table 2.6 Order of Magnitude

Quantity	*Expressed in nearest power of 10*	*Order of magnitude*
8	0.8×10^1	1
13	1.3×10^1	1
49	4.9×10^1	1
51	0.51×10^2	2
92	0.92×10^2	2
118	1.18×10^2	2
195	1.95×10^2	2
485	4.85×10^2	2
995	0.995×10^3	3
1050	1.050×10^3	3

A careful study of the above table reveals the method for determining the order of magnitude of a quantity. Note that the order of magnitude of 49 is not the same as that of 51. The number 51 is one order of magnitude higher than the number 49. This can be understood if we express these numbers as 4.9×10^1 and 5.1×10^1. Since 5.1 is greater than 5, the convention is to regard it closer to 10 and hence it contributes one more power of 10. Hence the number 51 should be expressed as 0.51×10^2 and not as 5.1×10^1 while determining its order of magnitude. The number 4.9, being less than 5, is not taken to be nearer 10 and hence it should be expressed as it is. Hence the order of magnitude of 49 is 1 but that of 51 is 2.

Referring to Table 2.1, we find that the order of magnitude of the radius of our galaxy ($= 0.6 \times 10^{20}$ m) is 20 while that of the radius of a proton ($= 1.2 \times 10^{-15}$ m) is—15. Similarly, from Table 2.2, we find that the order of magnitude of the age of the earth ($= 1 \times 10^{17}$ s) is 17 while that of the period of nuclear vibrations ($= 1 \times 10^{-21}$ s) is $-$ 21.

Abbreviations in Powers of 10

We will consider measurement of length as an example. The unit of length is the metre. Sometimes it is inconvenient to express large and small lengths in metres. For example, the distance between Delhi and Agra is about 200,000 metres. It is more convenient to say that this distance is 200 kilometres. 100 metres is abbreviated as 1 kilometre, i.e. kilo is an abbreviation for 1000. This 20 kilograms is 20,000 grams. Milli is an abbreviation for 1/1000 or 10^{-3}. Thus

$$10 \text{ milliseconds} = \frac{10}{1000} = \frac{1}{100} = 10^{-2} \text{ s}$$

Apart from kilo and milli, several other prefixes are used for powers of 10. The following table gives some prefixes.

Table 2.7 Prefixes for powers of 10

Power of 10	*Prefix*	*Symbol*
10^{12}	tera	T
10^{6}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deca	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

Using this table we can compactly express many large and small quantities. For example, the mean orbital radius of planet pluto is 6 terametres = 6×10^{12} m, period of rotation of a molecule is 1 picosecond = 10^{-12} s and a 10 megaton bomb has a destructive power of 10 million tons of TNT.

2.11 ACCURACY AND ERRORS IN MEASUREMENT

In many experiments in the physics laboratory, the aim is to determine the value of a physical constant. To determine a physical constant, we have to measure the various quantities which are connected with that physical constant by a formula.

For example, to a determine the density (ρ) of a metal block, we have to measure its mass (m) and its volume (V) which are related to ρ by the formula

$$\rho = \frac{m}{V}$$

The accuracy in the value of ρ obviously depends upon the accuracy in the measurements of m and V. Measurements of the quantities in the formula involve errors which are of two types: (a) *Random errors* and (b) *Systematic errors*.

Random Errors

Random errors may be due to (i) small changes in the conditions of a measurement and (ii) the incorrect judgement of the observer in making a measurement.

Examples (1) Suppose you are asked to locate the null point in an electrical experiment using a metre bridge. If you determine the null point 3 or 4 times you will notice that it shows small variations. The small changes in the null point are an example of random errors. This error is caused by small changes in the condition of the experiment, such as the heating effect of current and other relevent causes.

(2) Suppose you are determining the weight of a body with the help of a spring balance. You will usually make an error in estimating the coincidence of the pointer with the scale reading or in assessing the correct position of the pointer when it lies between two consecutive graduations of the scale. This error, which is due to incorrect judgement of the observer, is also an example of random errors.

Random errors cannot be traced to any systematic or constant cause of error. They do not obey any well-defined law of action. Their character can be understood or appreciated from the illustration of firing shots at a target using a rifle. The target is usually a bull's eye with concentric rings round it. The result of firing a large number of shots at the target is well known. The target will be marked by a well grouped arrangements of shots. A large number of shots will be nearer a certain point and other shots will be grouped around it on all sides. These shots which are grouped on both sides of the correct point obey the law of probability which means that large random errors are less probable to occur than small ones. A study of the target will show that the random shots lie with as many to one side of the centre as to the other. They will also show that small deviations from the centre are more numerous than large deviations and that a large deviation is very rare.

Method of Minimising Random Errors

If we make a large number of measurements of the same quantity then it is very likely that the majority of these measurements will have small errors which might be positive or negative. The error will be positive or negative depending on whether the observed measurement is above or below the correct value. Thus *random errors can be minimized by taking the arithmetic mean of a large number of measurements of the same quantity*. This arithmetic mean will be very close to the correct result. If one or two measurements differ widely from the rest, they should be rejected while finding the mean.

Systematic Errors

During the course of some measurements, certain sources of error operate constantly or systematically making the measurements systematically greater or smaller than the correct reading. These errors, whose cause can be traced, are called *systematic errors*. All instrumental errors belong to this category, such as the zero error in vernier calipers and micrometer screw, the index error in an optical bench, the end error in a metre bridge, faulty graduations of a measuring scale, etc.

Elimination of Systematic Errors To eliminate systematic errors, different methods are used in different cases.

1. In some cases, the errors are determined previously and the measurements corrected accordingly. For example, the zero error in an instrument is determined before a measurement is made and each measurement is corrected accordingly.

2. In some cases the error is allowed to occur and finally eliminated with the data obtained from the measurements. The heat loss due to radiation is taken into account and corrected for from the record of the temperature at different times.

Order of Accuracy

Even after random and systematic errors are minimized, the measurement has a certain order of accuracy which is determined by the least count of the measuring instruments used in that measurement.

Suppose we determine the value of a physical quantity u by measuring three quantities x, y and z whose true values are related to u by the equation

$$u = x^{\alpha} y^{\beta} z^{-\gamma} \tag{2.4}$$

Let the expected small errors in the measurement of quantities x, y and z be respectively $\pm \delta x$, $\pm \delta y$ and $\pm \delta z$ so that the error in u by using these observed quantities is $\pm \delta u$. The numerical values of δx, δy and δz are given by the least count of the instruments used to measure them.

Taking logarithm of both sides of Eq. (2.4), we have

$$\log u = \alpha \log x + \beta \log y - \gamma \log z$$

Partial differentiation of the above equation gives

$$\frac{\delta u}{u} = \alpha \frac{\delta x}{x} + \beta \frac{\delta y}{y} - \gamma \frac{\delta z}{z} \tag{2.5}$$

The proportional or relative error in u is $\delta u/u$. The values of δx, δy and δz may be positive or negative and in some case the terms on the right hand side of Eq. (2.5) may counteract each other. This effect cannot be relied upon and it is necessary to consider the worst case which is the case when all errors *add* up giving an error δu given by the equation:

$$\left(\frac{\delta u}{u}\right)_{\max} = \alpha \frac{\delta x}{x} + \beta \frac{\delta y}{y} + \gamma \frac{\delta z}{z} \tag{2.6}$$

Thus to find the maximum proportional error in u, multiply the proportional errors in each factor (x, y and z) by the numerical value of the power to which each factor is raised and then add all the terms so obtained.

The sum thus obtained will give the maximum proportional error in the result of u. When the proportional error of a quantity is multiplied by 100, we get the percentage error of that quantity. It is evident from Eq. (2.6) that a small error in the measurement of the quantity having the highest power will contribute maximum percentage error in the value of u. Hence the quantity having the highest power should be measured with as great a precision as possible. This is illustrated in the following example.

EXAMPLE 2.9 In an experiment for determining the density (ρ) of a rectangular block of a metal, the dimensions of the block are measured with callipers having a least count of 0.01 cm and its mass is measured with beam balance of least count 0.1 g. The measured values are:

Mass of block (m) = 39.3 g
Length of block (x) = 5.12 cm
Breadth of block (y) = 2.56 cm
Thickness of block (z) = 0.37 cm
Error in $m = \delta m = \pm 0.1$ g
Error in $x = \delta x = \pm 0.01$ cm
Error in $y = \delta y = \pm 0.01$ cm
Error in $z = \delta z = \pm 0.01$ cm

Find the maximum proportional error in the determination of ρ.

Solution The density of the block is given by

$$\rho = \frac{m}{xyz}$$

Calculating ρ omitting errors, $\rho = \dfrac{39.3}{5.12 \times 2.56 \times 0.37}$

$$= 8.1037 \text{ g cm}^{-3}$$

Proportional error in ρ is given by the equation.

$$\frac{\delta\rho}{\rho} = \frac{\delta m}{m} - \frac{\delta x}{x} - \frac{\delta y}{y} - \frac{\delta z}{z}$$

The maximum proportional error in ρ is given by

$$\left(\frac{\delta\rho}{\rho}\right)_{\max} = \frac{\delta m}{m} + \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$= \frac{0.1}{39.3} + \frac{0.01}{5.12} + \frac{0.01}{2.56} + \frac{0.01}{0.37}$$

$$= 0.0025 + 0.0019 + 0.0039 + 0.027$$

$$= 0.0353$$

$\therefore$ Maximum percentage error $= 0.0353 \times 100 = 3.53\%$

Hence the error is 3.53% of 8.1037. Clearly it is absurd to give the result for ρ to five significant figures. The error in ρ is given by

$$\delta\rho = 0.0353 \times \rho = 0.0353 \times 8.1037$$

$$= 0.286$$

$$\simeq 0.3$$

Hence the value of $\rho = 8.1037$ is not accurate up to the fourth decimal place. In fact, it is accurate only up to the first decimal place. Hence the value of ρ must be rounded off as 8.1 and the result of measurements is written as

$$\rho = (8.1 \pm 0.3) \text{ g cm}^{-3}$$

It is clear that such a large error in the measurement of ρ is due to a large error ($= 0.027$) in the measurement of z, the smallest of the quantities measured. Hence the order of accuracy of ρ should be increased by measuring z with an instrument having a least count which is smaller than 0.01 cm. Thus a micrometre screw (least count $= 0.001$ cm), rather than a vernier calliper should be used to measure z.

EXAMPLE 2.10 The coefficient of viscosity (η) of a liquid by the method of flow through a capillary tube is given by the formula

$$\eta = \frac{\pi}{8} \frac{R^4}{l} \frac{P}{Q}$$

where R = radius of the capillary tube,

l = length of the tube,

P = pressure difference between its ends and

Q = volume of liquid flowing per second. Which measurement needs to be made most accurately and why?

Solution The maximum proportionate error in η is given by the relation.

$$\frac{\delta\eta}{\eta}=4\frac{\delta R}{R}+\frac{\delta l}{l}+\frac{\delta P}{P}+\frac{\delta Q}{Q}$$

It is clear that the error in the measurement of R is magnified four times on account of the occurrence of R^4 in the formula. Hence the radius (R) of the capillary tube must be measured most accurately. Thus the quantity which is raised to the highest power needs the most accurate measurement.

SUMMARY

Physics is a science of measurement. All quantities which can be measured either indirectly or directly such as length, mass, time, force, temperature, light intensity, electric current, etc. are called physical quantities. Of numerous such quantities, length, mass and time are regarded as fundamental quantities. The measurement of these quantities involves the choice of a unit. The internationally accepted units of length, mass and time respectively are metre, kilogram and second. The units of all other mechanical quantities are derived from these three basic units. These quantities are measured by direct and indirect methods. The measured values of these quantities show a very wide range variation.

Different physical quantities have different dimensions. The dimensions of a physical quantity are the number of times the fundamental units of mass, length and time appear in that quantity. Thus, if the units of a quantity are $M^xL^yT^z$, then the quantity has x dimensions of mass, y dimensions of length and z dimensions of time.

The method of dimensions finds use in (i) checking the accuracy of an equation, (ii) obtaining the dependence of a physical quantity on other quantities involved in a given phenomenon and (ii) converting the units from one system into another. This method has its own limitations.

The measurements of physical quantities involve two types of errors—random and systematic errors which must be minimized in order to increase the accuracy of that measurement. The order of accuracy of a measurement depends on the value of the count least of the instruments used.

EXERCISES

1. Name the basic units in the SI system.
2. What is a physical standard? What characteristics should it have?
3. What are the merits of the SI system of units?
4. Why is the standard metre measured in terms of the wavelength of light?
5. Suggest a method for measuring the height of the Qutab Minar.
6. What is a fundamental physical quantity? Give three examples.
7. What is a derived physical quantity? Give three examples.
8. What do you understand by the dimensions of a physical quantity? Explain the principle of homogeneity of dimensions.
9. Explain, with the help of an example, what you mean by dimensional formula and dimensional equation.
10. Write down the dimensional formulae of the following quantities: (i) acceleration, (ii) force, (iii) energy, (iv) power, (v) pressure (vi) stress, (vii) viscosity, (viii) surface tension, (ix) torque, and (x) frequency.
11. Using dimensional considerations, check the correctness of the following equations.
    (i) $v = u + at$, $S = ut + \frac{1}{2}at^2$ and $v^2 - u^2 = 2aS$, where S is the distance travelled by a particle in uniform acceleration a in time t: u and v being the initial and the final velocities respectively.
    (ii) $E_k = \frac{1}{2}mv^2$, where E_k is the kinetic energy of a body of mass m moving with a velocity v.
    (iii) $E_p = mgh$ where E_p is the gravitational potential energy of a body of mass m held at a height h above the surface of the earth.
    (iv) $\eta = \frac{\pi P r^4}{8lV}$, where η is the coefficient of viscosity of the liquid flowing in a capillary tube of radius r and length l. P is the pressure difference at the ends of the tube and V is the volume of liquid flowing per second.
    (v) $h = \frac{2T\cos\theta}{r\rho g}$, where h is the height of a liquid column in a capillary tube of radius, r, T the surface tension of the liquid, θ the angle of contact, ρ the liquid density and g the acceleration due to gravity.
12. When white light travels through glass, the refractive index μ (= velocity of light in air/velocity of light in glass) is found to vary with the wavelength as

$$\mu = A + \frac{B}{\lambda^2}$$

where A and B are constants. Using the principle of homogeneity of dimesions, determine the SI units in which A and B must be expressed.

(Ans. A is a number; the SI units of B is m^2)

13. A stone of mass m is tied to a string and swung around in a circular path of radius r with a constant speed v. The stone exerts an inward (centripetal) force F on your hand. From dimensional considerations, show that

$$F = k\,\frac{mv^2}{r}$$

where k is a dimensionless constant.

14. Transverse waves are produced in a long uniform string stretched with a tension F. If the mass of unit length of the string is m, show, using the method of dimensions, that the velocity of the waves is given by

$$v = k\sqrt{\frac{F}{m}}$$

where k is a dimensionless constant.

15. A liquid column in a U-tube is executing vertical periodic oscillations. The total length of the liquid column is l and ρ is the density of the liquid. Let r be the radius of the U-tube and g be the acceleration due to gravity. The time period T of oscillations may depend on all the above four quantities, namely, l, ρ, r and g.
    (i) Assuming that T depends only on l, ρ, and g, show that the time period will be independent of the density of the liquid.
    (ii) Assuming now that T depends on r, ρ and g, show that the time period will also be independent of the radius of the U-tube.
    (iii) Hence show that T is given by

$$T = k\sqrt{\frac{l}{g}}$$

where k is a dimensionless constant.

16. What are random and systematic errors? Give two examples of each. How are they minimized?

3

Description of Motion in One Dimension

3.1 INTRODUCTION

Everything in this world is in motion; from the tiniest particles within atoms to the largest galaxies of stars. The motion of objects, living and non-living, on or near the surface of the earth is an obvious phenomenon. It was the first aspect of nature to be studied by man. The subject of motion was placed on a sound scientific footing by two great men, Galileo Galilei (1564-1642) and Sir Isaac Newton (1642-1727). Galileo laid the foundation of the modern scientific method. He advocated the idea that every theory of nature should be put to an experimental test. He studied the simplest examples of the natural phenomenon of motion and set up simple experiments to examine them. Only after Galileo had clarified the idea of motion was the world of physics ready for the great work of Isaac Newton.

The study of motion is divided into two branches called *kinematics* and *dynamics*. The word 'kinematics' comes from a Greek word *kinema* which means motion. In kinematics we learn how to describe the motion of objects without worrying about the *causes* of motion. The word 'dynamics' comes from a Greek word *dynamis* which means 'power'. In dynamics we study the *causes* that influence motion of objects.

In this chapter and the next we will study the kinematics of motion and learn how to *describe* motion in one and two dimensions. In later chapters we will study the dynamics of motion and learn how to *explain* motion. Galileo studied the kinematics of the motion of objects while Newton analysed the dynamics of the motion and introduced the concept of force, as discussed in Chapter 5.

3.2 THE IDEA OF A 'POINT' OBJECT

Before discussing how the motion of an object can be described, we shall make a very important assumption. We will assume that the object is so small that it can be treated as a particle or a point object, for then we need not, at this stage, worry about the internal structure of the object or any rotational motion it might have. So, in this chapter, whenever we speak of an object, we will mean a point object. In Chapter 7 we shall discuss the motion of finite objects.

In general, if the distance moved by an object in a reasonable amount of time is much larger than its size , then we can, to a good approximation, neglect its size and treat it as a point object. For example, a cricket ball can be regarded as a point object for describing its motion in a cricket field. Similarly a bus (which may be 5 m × 3 m × 2 m) may be regarded as a point object while describing its motion over a distance of a few kilometres. Even the earth and the sun may be treated as point objects while describing the motion of the earth round the sun; their sizes are negligibly small compared to the distance between them. In Chapter 7 we shall analyse this again and justify this assumption.

3.3 MOTION IN ONE, TWO AND THREE DIMENSIONS

The motion of a car speeding on a long and straight road is an example of motion in one dimension. To describe the motion of the car, we neglect the width and sideways motion of the car. But if the car goes along a zig-zag path through the traffic on a busy road, its motion is two dimensional. The motion of a carrom coin or a billiard ball on a horizontal plane surface are examples of motion in two dimensions. The motion in a circle or a curve in a plane (such as that of a planet round the sun) is also two-dimensional. The most general motion one can consider is the motion of objects in space which involves all the three dimensions. For example, if a carrom coin 'hops' up after rebound, its motion will be three-dimensional. Similarly the motion of a kite flying on a windy day is three-dimensional.

In this chapter we will learn how to describe the simplest of motions, namely, one-dimensional motion, i.e. motion in a straight line. In the next chapter we will deal with motion along a curved path which is two-dimensional. *Rectilinear motion* (i.e. motion in a straight line) is physically different from *curvilinear* motion (i.e. motion in a curve) as we shall see in Chapter 5 when we study Newton's laws of motion.

3.4 MOTION IN A STRAIGHT LINE: GROUNDWORK

If an object changes its *position* with *time* it is said to be in motion. For example, a car moving on a straight level road or a stone dropped from the top of a building are in motion in a straight line. The first step in the description of motion in all such cases is to know the position of the object at different instants of time during the motion. The motion can then be represented on a graph by plotting position against time.

Origin Unit and Sense for Time Measurement

In any phenomenon involving a change with time, we must be able to assign times to the various events that are relevant to the study of that phenomenon. In order to fix the time of each event, we have to choose a *reference* or *origin* of time which is given a value zero. We must also choose a unit of time, say the second. The *origin* and the *unit* are arbitrarily chosen for convenience. Sometimes it is more convenient to use minute, hour or even year as the unit of time. Having chosen a convenient unit of time and the origin, then the events that occurred *before* the origin (the zero point of time) are assigned *negative* numbers of unit of time and those that occur *after* the origin are assigned *positive* numbers of unit of time.

As stated earlier, the choice of the origin and the unit of time are a matter of convenience. To describe historical events the origin of time is taken to be the birth of Jesus Christ and the unit of time is taken to be the year. For example, we say that Emperor Ashoka ruled form 273 B.C. to 236 B.C. while Emperor Akbar ruled from 1556 A.D. to 1605 A.D. Instead of using B.C. and A.D. after the number of years, we could also use negative and positive values to convey the sense of time and say that Ashok ruled from − 273 years to − 236 years and Akbar ruled from + 1556 years to + 1605 years. The formulae and algebraic expressions which will describe the phenomenon of motion (or any other phenomena) can then be easily written in terms of time with the use of positive and negative time values.

Origin, Unit and Direction for Position Measurement

Suppose an object is moving in a straight line which we shall denote by the letter L. At each instant of time, we wish to describe the position of the object on that line L. For this purpose, some point O on L is chosen as the *reference point* or the *origin*. It does not matter where on L the origin is chosen. For example, to describe the position of a car moving on a road, a kilometre stone on the side of the road may be chosen as the origin. The position can now be described with respect to the origin by chosing a unit, which may be kilometre, metre or centimetre whichever is convenient. The position to the *rigth* of origin O is given by a *positive* number of units and the position to the *left* of O by a *negative* number of units.

At each time t, the position of the object is given by a number written as $x(t)$ which is positive for positions to the right of O, negative for position to the left of O and zero at the origin O. The notation $x(t)$ means that the position $x(t)$ *is a function of time t*, i.e. the value of x (t) depends on the value of t. Function $x(t)$ is called the *position coordinate* of the object. Figure 3.1 (a) shows how we indicate positions at two different values of t, say t_1 and t_2. we show $x(t_1) = 2.5$ units and $x(t_2) = -3$ units at times t_1 and t_2 respectively.

For motion in a vertical line, positions upwards from O are taken as positive while those downwards from O are taken as negative (see Fig. 3.1 (b)).

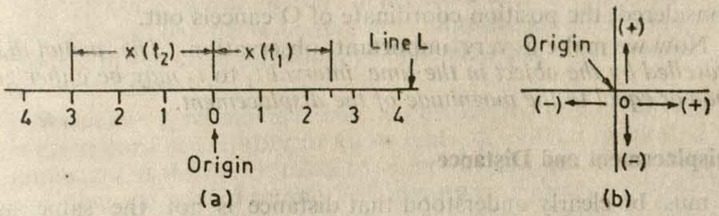


Fig. 3.1 Position coordinate on a line

Stationary Objects

If the position of an object does not change with time, it is said to be stationary or at rest. The position coordinate $x(t)$ is a constant, say x_0 which gives the position of the stationary object with respect to the origin. Since $x(t)$ does not change with time, the graph showing $x(t)$ against t is a straight line parallel to the time axis. This graph is shown in Fig. 3.2. This graph describes a train standing at a station.

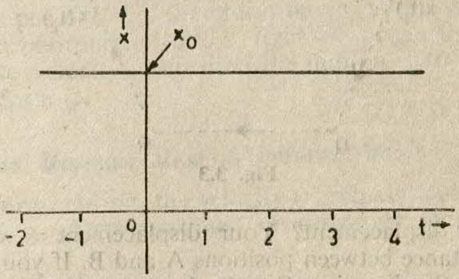


Fig. 3.2 Position-time graph of an object at rest

3.5 UNIFORM MOTION

Let us now consider the case where the object is actually moving in which case $x(t)$ is not a constant; as it changes with time. Let t_1 be a certain instant of time and t_2 a later instant of time. Let $x(t_1)$ and $x(t_2)$ be the position coordinates of the moving object at t_1 and t_2 respectively. Then $x(t_2) - x(t_1)$ is called the displacement of the object in a time interval from t_1 to t_2, i.e.

$$\text{Displacement in a time interval from } t_1 \text{ to } t_2 = x(t_2) - x(t_1) \quad (3.1)$$

Like position coordinate $x(t)$, displacement is also a length and is measured in length units. It follows from Eq. (3.1) that the displacement can be positive, negative or zero depending on whether $x(t_1)$ is less than, greater than or equal to $x(t_2)$. It also follows from Eq. (3.1) that the value of the displacement does not depend upon where on the line L we choose the origin O because both $x(t_2)$ and $x(t_1)$ are measured with respect to that origin O and when the difference $x(t_2) - x(t_1)$ is considered, the position coordinate of O cancels out.

Now we make a very important observation. *The actual distance travelled by the object in the time interval t_1 to t_2 may be either greater than or equal to the magnitude of the displacement.*

Displacement and Distance

It must be clearly understood that distance is not the same as displacement. Suppose You walk from an initial position A to a final position B (Fig. 3.3). You could do so by taking different routes (or paths) such as APB or AQRB as shown in the figure. The distance you travel along these two paths are (AP + PB) and (AQ + QR + RB) respectively.

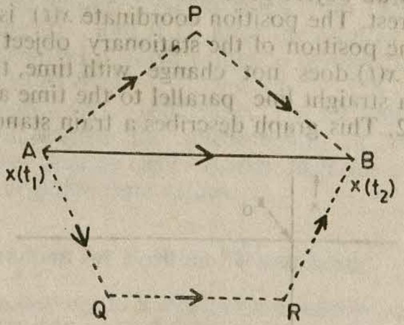


Fig. 3.3

What is your displacement? Your displacement $= x(t_2) - x(t_1)$, is the shortest distance between positions A and B. If you had travelled from A to B in the direction A to B, the distance you had travelled would have been equal to your displacement. If you travel along any other route, the distance you travel is greater than the displacement. Thus, if we know the displacement from time t_1 to time t_2, it does not tell us how exactly the object moves from position $x(t_1)$ to position $x(t_2)$, i.e where it was at times between t_1 and t_2.

Thus, distance is the total path travelled by a body in a certain interval of time and the displacement is the shortest distance (in a specified direction) between the initial and final positions of the body.

This distinction between distance and displacement is true not only for a uniform motion but also for a non-uniform motion. If a ball is thrown vertically upward from a position, say A, above the ground and after some time, it comes back to the same point A, the displacement of the ball is zero but the distance travelled by it is not zero—it is twice the distance between the point A and the highest point upto which it rises.

Velocity in Uniform Motion

Uniform motion is the simplest of all motions. The motion of an object is said to be *uniform* if it moves equal distances in equal intervals of time, *always moving in the same direction*. It is clear that the displacement can never be zero in a uniform motion because an object, moving in the same direction, cannot be at the same position at two different times.

When an object moves from one position to another, its *displacement* is an important quantity to know when describing its motion. Equally important is the *time* it takes to suffer that displacement. Displacement and time are related to another important property of motion called *velocity*. The *velocity of an object in uniform motion is defined as the displacement divided by the corresppoding time interval*, i.e.

$$\text{Velocity } v = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \tag{3.2}$$

It may be mentioned that, in the case of uniform motion, the actual distance covered between times t_1 and t_2 is the *magnitude* of the displacement. Also, the velocity can be calculated by any choices of time interval $(t_2 - t_1)$; it will always be the same for any time interval, howsoever small.

It follows from definition (3.2) that velocity is positive if displacement is positive, i.e. the object moves in the direction of increasing position coordinate (to the right of origin O or upwards from O). Negative velocity means that the motion is to the left of O or downwards from O.

Graphical Representation of Uniform Motion

In uniform motion, the velocity (v) remains constant. Hence the graph of velocity (v) against time (t) is a straight line parallel to the time

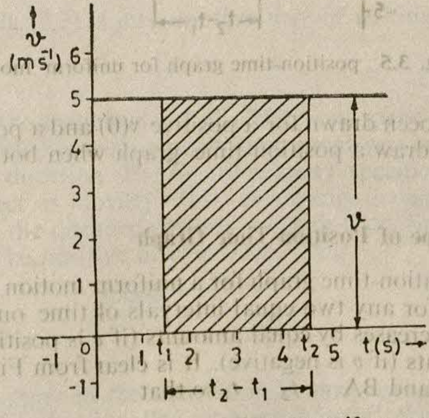


Fig. 3.4 Velocity-time graph for uniform motion

axis, just as the position-time graph of Fig. 3.2 for a stationary object. Figure 3.4 shows the velocity-time graph of a car having a constant velocity of $+5 \text{ m s}^{-1}$. If the velocity were negative, the graph would be below the time axis.

To draw the position-time graph of the motion of the car, let $x(0)$ be its position at time $t = 0$. Let us suppose that $x(0) = 4$ m from the origin. The position-time table can now be prepared for a few selected values of time t, as follows.

Time t (s)	−1	0	1	2	3	4
Position $x(t)$ (m)	−1	4	9	14	19	24

When $x(t)$ is plotted against t we obtain the position-time graph of the motion of the car. This is shown in Fig. 3.5.

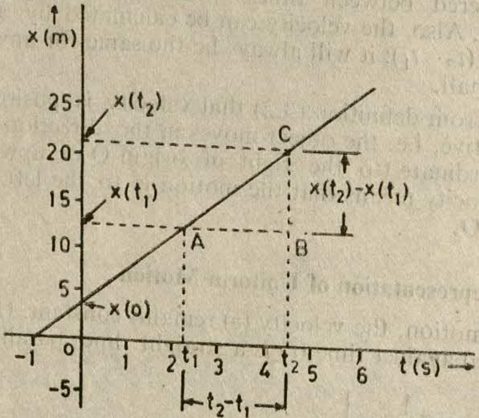


Fig. 3.5 position-time graph for uniform motion

This graph has been drawn for a positive $x(0)$ and a positive v. Students are advised to draw a position-time graph when both $x(0)$ and v are negative.

Velocity as Slope of Position-Time Graph

Because the position-time graph for a uniform motion is a straight line, it follows that for any two equal intervals of time on the t-axis, the coordinate x increases by equal amounts (if v is positive), or decreases by equal amounts (if v is negative). It is clear from Fig. 3.5 that CB = $x(t_2-) - x(t_1)$ and BA = $t_2 - t_1$ so that

$$\text{Slope} = \frac{\text{CB}}{\text{BA}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

which from Eq. (3.2) is the velocity v. Hence

$$v = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\text{CB}}{\text{BA}}$$

Thus the slope of the position-time graph for uniform motion determines the velocity of the motion.

Distance as Area under the Velocity-Time Graph

The velocity-time graph for uniform motion shown in Fig. 3.4 gives us a geometrical method of calculating the distance covered in a time interval from t_1 to t_2. Since

$$\begin{aligned} \text{Distance} &= \text{velocity} \times \text{time interval} \\ &= v \times (t_2 - t_1) \end{aligned}$$

it follows from Fig. 3.4 that this is the area under the velocity-time graph between times t_1 and t_2. This area is shown shaded in Fig. 3.4. Thus *the distance covered by an object in uniform motion between times t_1 and t_2 is given by the area under the velocity-time graph of the motions between times t_1 and t_2*. This method of finding distance covered is applicable even if v is negative.

Formulae for Uniform Motion

The graphical representation of position as a function of time (shown in Fig. 3.5) can be expressed as a mathematical formula also. It follows from this graph that (for v = constant)

$$x(t_2) = x(0) + vt_2$$

$$x(t_1) = x(0) + vt_1$$

so that

$$x(t_2) = x(t_1) + v(t_2 - t_1) \tag{3.3}$$

Notice that formula (3.3) is just another way of writing Eq. (3.2).

Speed

Just as distance is the magnitude of displacement, speed is the magnitude of velocity. Velocity can be positive or negative, i.e. it has both a magnitude and a direction. The sign of velocity specifies the direction in which the object is moving. Just as velocity is displacement per unit time, speed is the distance covered per unit time. Speed can never be negative, it can be positive or even zero. Thus

$$\text{Speed} = |v| \geqslant 0$$

Relative Velocity

When two objects move in the same straight line, we compare their motions in terms of what is called '*relative velocity*'. Suppose you are travelling in a train (labelled 1) and another train (labelled 2) moving in the same direction, on a parallel track, crosses your train. Train 2 must be moving faster than your train 1. If you look through your window at train 2, it seems to move slower than when someone standing on the ground watches both the trains. If train 2 is moving with the same speed as your train, then it will appear to you as though it were not moving at all. On the other hand, if train 2 is moving in the opposite direction and crosses your train, it will seem to you to move much faster. In such cases we speak of the velocity relative to the observer.

Consider two objects labelled 1 and 2 moving with uniform speeds v_1 and v_2 in the same straight line, i.e. in one dimension. Their positions $x_1(t)$ and $x_2(t)$ are given by [see Eq. (3.3)]

$$x_1(t) = x_1(0) + v_1 t$$

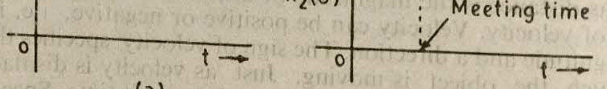
$$x_2(t) = x_2(0) + v_2 t$$

At any time t, the relative displacement (i.e. distance separating them) is $x_2(t) - x_1(t)$ which is obtained by subtracting the two equations:

$$x_2(t) - x_1(t) = x_2(0) - x_1(0) + (v_2 - v_1)\, t$$

This equation tells us that the displacement from 1 to 2 changes by an amount $(v_2 - v_1)$ in each unit of time. This means that, as seen from object 1, object 2 has a velocity $(v_2 - v_1)$, i.e. the relative velocity of object 2 with respect to object 1 is $(v_2 - v_1)$.

If $v_2 - v_1 = 0$, we find that $x_2(t) - x_1(t) = x_2(0) - x_1(0)$ which is constant independent of time. This means that two objects moving in the same direction with equal speeds, stay a constant distance apart. Their position-time graphs will be parallel straight lines as shown in Fig. 3.6 (a). Remember that parallel lines have equal slopes.



(a) (b)

Fig. 3.6 (a) $v_2 = v_1$ (b) $v_2 > v_1$

If $v_2 - v_1 \neq 0$, one graph will be steeper than the other. Since the graphs are no longer parallel, they must meet at a common point as shown in Fig. 3.6 (b) where $v_2 > v_1$. Notice that before the meeting time $x_2(t) - x_1(t)$ is negative but after the meeting time $x_2(t) - x_1(t)$ is positive.

If the two objects are moving in opposite directions, then v_1 and v_2 will have opposite signs as when two trains cross each other moving in opposite directions. In this case, the relative velocity $(v_2 - v_1)$ is greater in magnitude than either $|v_1|$ or $|v_2|$.

EXAMPLE 3.1 Two parallel rail tracks run north-south. Train A is moving northwards with a speed of 72 km h^{-1} and train B is moving southwards with a speed of 108 km h^{-1}. Find (a) the relative velocity of A with respect to B, (b) the relative velocity of the ground with respect to A, (c) relative velocity of B with respect to the ground, and (d) the velocity of a monkey running southwards on the roof of train B (with a velocity of 18 km h^{-1} with respect to train B) as observed by a person standing on the ground.

Solution Let us choose the positive direction of x-axis to be from south to north. Then

$$v_A = 72 \text{ km h}^{-1} = 20 \text{ m s}^{-1}$$

$$v_B = -108 \text{ km h}^{-1} = -30 \text{ m s}^{-1}$$

(a) Relative velocity of A with respect to B $= v_A - v_B = 20 - (-30) = +50$ ms^{-1}, i.e, the train A appears to B to move with a speed of 50 ms^{-1} in the northward direction.

(b) Relative velocity of ground with respect to A $= v_G - v_A = 0 - 20 = -20$ ms^{-1}, i.e. the ground appears to A to move with a speed of 20 m s^{-1} in the southward direction.

(c) Relative velocity of B with respect to the ground $= v_B - v_G = -30 - 0 = -30$ ms^{-1}, i.e. the train B is moving south at a speed of 30 ms^{-1}.

(d) Let v_M be the velocity of the monkey with respect to the ground. Now velocity of the monkey with respect to B $= v_M - v_B = -18$ km $h^{-1} = -5$ ms^{-1}.

$$\therefore \qquad v_M = -5 + v_B = -5 + (-30) = -35 \text{ ms}^{-1}$$

The relative velocity of the monkey with respect to the ground is 35 ms^{-1} in the south-ward direction.

3.6 NON-UNIFORM MOTION WITH CONSTANT ACCELERATION

Another important aspect of motion is *acceleration* which refers to a change in velocity with time. This change may be due to two reasons —either the magnitude of velocity (i.e. speed) is changing with time or the direction of the velocity is changing with time. The acceleration due to a change in the direction of the velocity will be discussed in the next chapter. In one-dimensional motion (i.e. motion in a straight line) the velocity can change only due to a change in speed.

Just as velocity is defined as the rate of change of displacement [Eq. (3.2)] *acceleration is defined as the rate of change of velocity.* The term acceleration refers to 'speeding up' or 'slowing down' of the motion. Suppose a car is moving on a straight road with at velocity $v(t_1)$ at a time t_1 when it is accelerated (or speeded up) at a constant rate and its velocity increases to $v(t_2)$ at a later time t_2. Then the constant acceleration is given by

$$\text{Acceleration } a = \frac{v(t_2) - v(t_1)}{t_2 - t_1} \tag{3.4}$$

The dimensions of acceleration are

$$\frac{\text{Dimensions of velocity}}{\text{Dimension of time}}$$

$$= \frac{\mathrm{LT^{-1}}}{\mathrm{T}} = \mathrm{L\,T^{-2}}$$

The SI unit of acceleration is, therefore, $\mathrm{ms^{-2}}$. Notice that the velocity of the car is changing at a constant rate in the time interval $(t_2 - t_1)$. During this interval the car has a velocity at every instant. This is called the *instantaneous velocity*. The speedometer of the car indicates its instantaneous velocity.

Velocity—Time Graph of Uniformly Acceleration Motion

Suppose a car is moving on a straight road at a speed of 5 $\mathrm{ms^{-1}}$. Let us say that time $t = 0$ it is accelerated at a constant rate of 3.5 $\mathrm{ms^{-2}}$. This means that the speed of the car increases by 3.5 $\mathrm{ms^{-1}}$ in every second. The velocity-time table of the motion of the car can now be prepared as follows.

Time (s)	**0**	**1**	**2**	**3**	**4**	**5**
Velocity ($\mathrm{ms^{-1}}$)	**5**	**8.5**	**12**	**15.5**	**19**	**22.5**

When the velocity is plotted against time we get the velocity-time graph of the motion of the car. This is shown in Fig. 3.7.

Acceleration as Slope of Velocity-time Graph

$$\text{Slope of the graph} = \frac{AC}{CB} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

which is the acceleration as defined by Eq. (3.4). Thus the slope of the velocity-time graph of a uniformly accelerated motion measures the acceleration.

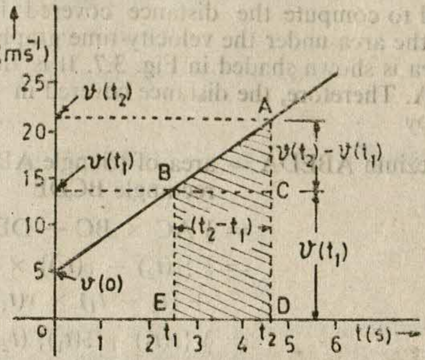


Fig. 3.7 Velocity-time graph for uniformly accelerated motion

Formulae for Uniformly Accelerated Motion

Distance-time relation Let us suppose that $v(t_1)$ and $v(t_2)$ are the velocities of the object at times t_1 and t_2 respectively (see Fig. 3.7). Since the velocity increases from $v(t_1)$ to $v(t_2)$ at a uniform rate, it is reasonable to expect that the average velocity during the interval $(t_2 - t_1)$ is

$$v_{av} = \tfrac{1}{2}\,[v(t_1) + v(t_2)] \tag{3.5}$$

From Eq. (3.4) we have

$$v(t_2) = v(t_1) + a\,(t_2 - t_1) \tag{3.6}$$

Using Eq. (3.6) in Eq. (3.5) we get

$$v_{av} = v(t_1) + \tfrac{1}{2}\,a\,(t_2 - t_1) \tag{3.7}$$

Now we know that

$$v_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \tag{3.8}$$

Equations (3.7) and (3.8) give

$$x(t_2) - x(t_1) = v(t_1)\,(t_2 - t_1) + \tfrac{1}{2}\,a\,(t_2 - t_1)^2 \tag{3.9}$$

This is the distance-time relation for a uniformly accelerated motion. The left-hand side of this relation is the distance covered by the object in a time interval $(t_2 - t_1)$.

Distance as area under velocity – time graph The relation (3.9) can also be obtained graphically. From the velocity-time graph (Fig. 3.4) of a uniform motion, we found a very useful result that the distance covered during a time interval from t_1 to t_2 is equal to the area under the velocity-time graph between times t_1 and t_2. This result (which we obtained for a uniform motion; v = constant) is actually applicable to *all motions*, including that with a constant acceleration. We will use this method to compute the distance covered in a time interval $(t_2 - t_1)$ from the area under the velocity-time graph between times t_1 and t_2. This area is shown shaded in Fig. 3.7. It is the area of the trapezium ABEDA. Therefore, the distance covered in time interval $(t_2 - t_1)$ is given by

$$\begin{aligned} \text{Area of trapezium ABEDA} &= \text{area of triangle ABC} + \text{area of rectangle BCDE} \\ &= \tfrac{1}{2}\,\text{AC} \times \text{BC} + \text{DE} \times \text{CD} \\ &= \tfrac{1}{2}\{v(t_2) - v(t_1)\} \times (t_2 - t_1) \\ &\quad + (t_2 - t_1) \times v(t_1) \\ &= \tfrac{1}{2}\{v(t_2) + v(t_1)\}(t_2 - t_1) \qquad (3.10) \\ &= v_{\text{av}} \times (t_2 - t_1) \qquad (3.11) \end{aligned}$$

Notice that Eq. (3.11) is the same as Eq. (3.8) we have used above to find the distance $x(t_2) - x(t_1)$. Thus our expectation that distance = average velocity × time is indeed correct. Now Eq. (3.10) is

$$x(t_2) - x(t_1) = \tfrac{1}{2}\{v(t_2) + v(t_1)\} \times (t_2 - t_1)$$

Using Eq. (3.6) in this equation gives

$$x(t_2) - x(t_1) = v(t_1) \times (t_2 - t_1) + \tfrac{1}{2}\, a \times (t_2 - t_1)^2$$

which is Eq. (3.9) obtained earlier.

Distance in terms of velocities We can obtain another equation in which time t does not appear explicitly. From Eq. (3.10) we have

$$v(t_2) + v(t_1) = \frac{2}{(t_2 - t_1)} \times \{x(t_2) - x(t_1)\} \qquad (3.12)$$

Also from Eq. (3.6) we have

$$v(t_2) - v(t_1) = a\,(t_2 - t_1) \qquad (3.13)$$

Multiplying Eq. (3.12) and Eq. (3.13) we get

$$v^2(t_2) - v^2(t_1) = 2a\,\{x(t_2) - x(t_1)\} \qquad (3.14)$$

We will discuss a physical interpretation of this result in terms of energy in Chapter 6.

We will collect our results [Eqs. (3.6), (3.9) and (3.14)] at one place. These equations can be written in alternative form if we replace t_1 by zero and t_2 by t.

$$v(t) = v(0) + at \qquad (3.15)$$

$$x(t) - x(0) = v(0)\,t + \tfrac{1}{2}\,at^2 \qquad (3.16)$$

$$v^2(t) - v^2(0) = 2\,a\,\{x(t) - x(0)\} \qquad (3.17)$$

We will now apply these equations to solve a few numerical problems. For this purpose, it is convenient to rewrite these equations in terms of a simpler (and more conventional) notation which is as follows.

u = initial velocity at time 0 = $v(0)$

v = final velocity at time $t = v(t)$

S = displacement (i.e. distance covered) in time t

$= x(t) - x(0)$.

In terms of this notation, Eqs (3.15), (3.16) and (3.17) read as follows:

$$v = u + at \tag{3.18}$$

$$S = ut + \tfrac{1}{2} at^2 \tag{3.19}$$

and

$$v^2 - u^2 = 2\,aS \tag{3.20}$$

A body moving with a velocity u (which we call the initial velocity) in a straight line is given a constant acceleration a at time $t = 0$. As a result of the acceleration, its velocity increases to v in time t (which we call the final velocity) during which the body suffers a displacement S.

While solving numerical problems, we will consider only the magnitudes of u, v, a and S and take care of their direction by assigning a positive or negative sign to the quantity. For example $+\,a$ will mean an acceleration and $-\,a$ will mean *retardation* or *deceleration.*

In the case of a body falling freely under gravity, the acceleration is due to gravity and is denoted by g. The value of g varies slightly from place to place on earth. The average value based on experimental measurements is found to be (see Ch. 8).

$$a = g = 9.8 \text{ m s}^{-2}$$

If a body is thrown vertically upwards, it is retarded. Thus, in this case, the acceleration is (according to our convention)

$$a = -\,9.8 \text{ ms}^{-2}$$

EXAMPLE 3.2 A train is moving due east with a velocity of 72 km h^{-1}. The brakes are applied and the train is brought to rest in 40 s. (i) What is the value and the direction of the acceleration, assuming it to be constant? (ii) How much distance does the train travel during this time?

Solution Let us assume that the east is in the x-direction. Let us take the velocity and displacement to be positive along $+\,x$-direction. Thus

$$\text{Initial velocity } u = +\,72 \text{ km s}^{-1} = \frac{72 \times 1000}{60 \times 60}$$

$$= 20 \text{ m s}^{-1} \text{ along } +\,x\text{-direction}$$

Final velocity $v = 0$ ($\because$ the train is brought to rest)

Time $t = 40$ s

(i) The acceleration can now be obtained from Eq. (3.18), i.e.

$$a = \frac{v - u}{t} = \frac{0 - 20}{40} = -\,0.5 \text{ ms}^{-2}$$

Since a is negative, it means that the acceleration is directed along— x-direction (opposite to that of u), The magnitude (value) of acceleration is 0.5 ms^{-2}. It is a case of deceleration or retardation.

(ii) The displacement S can then be found from either Eq. (3.19) or Eq. (3.20). Using Eq. (3.20), we have

$$S = \frac{v^2 - u^2}{2a} = \frac{(0)^2 - (20)^2}{2 \times (-\,0.5)} = 400 \text{ m}$$

Check Using Eq. (3.19), we have

$$S = ut + \tfrac{1}{2} at^2 = 20 \times 40 + \tfrac{1}{2}(-\,0.5)(40)^2$$
$$= 800 - 400 = 400 \text{ m}$$

S is positive. Therefore, a displacement of 400 m takes place in the $+\,x$-direction, i.e. the distance moved in the direction of the velocity is 400 m.

EXAMPLE 3.3 A stone is thrown vertically upwards with an initial velocity of 14 ms^{-1}. After how long will the stone strike the ground? The acceleration due to gravity is 9.8 ms^{-2}.

Solution Let us assume that all quantities acting along the upward direction ($+\,y$-direction) are positive and those in the downward direction negative. Since the velocity is directed upwards and the acceleration due to gravity acts downwards, we have,

$$u = +\,14 \text{ ms}^{-1}$$
$$a = -\,9.8 \text{ ms}^{-2}$$

Let us say that t is the time taken by the stone to strike the ground. The stone starts from the surface of the earth at time, say, $t = 0$ with a velocity of 14 ms^{-1}. It rises to a certain maximum height and after that returns to earth striking it at a time t, so that t is the total time for its entire journey. It is clear that the displacement of the stone is zero. It has come back from where it started; although the distance travelled is not zero. Thus,

$$S = 0$$

Now t can be calculated from the equation

$$S = ut + \tfrac{1}{2} at^2$$

or
$$0 = 14t - 4.9t^2$$
$$= 7t\,(2 - 0.7t)$$

Thus
$$t = 0 \text{ or } t = \frac{2}{2.7} = 2.86 \text{ s}$$

Now, $t = 0$ is not possible, and hence the stone will strike the ground after 2.86 s.

EXAMPLE 3.4 From the top of a multi-storeyed building 40 m tall, a boy projects a stone vertically upwards with an initial velocity of 10 ms^{-1} such that it eventually falls to the ground. (i) After how long will the stone strike the ground? (ii) After how long will it pass through the point from where it was projected? (iii) What will be its velocity when it strikes the ground? Take $g = 10 \text{ ms}^{-2}$.

Solution (i) Let us assign a positive sign to quantities directed in the upward direction and a negative sign to those acting downwards. Referring to Fig. 3.8, let us say that the stone is projected upwards from a point. A at time $t = 0$. It rises to its maximum height B, returns, passes through the the point C from where it was projected and eventually strikes the ground at D.

It is clear from the diagram that, although the total distance travelled by the stone is AB + BC + CD, the net displacement is only CD directed downwards, i.e.

$$S = -\text{CD} = -40 \text{ m}.$$

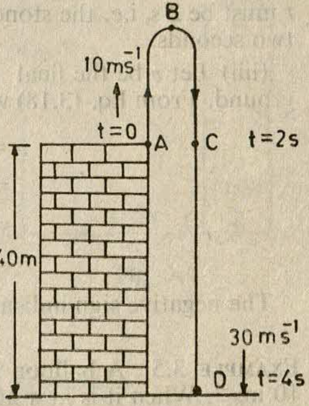


Fig. 3.8

Since the initial velocity u is directed upwards and the acceleration due to gravity acts downwards we have,

$$u = +10 \text{ ms}^{-1}$$
$$a = -10 \text{ ms}^{-2}$$

Now using the equation
$$S = ut + \tfrac{1}{2}at^2$$

we have
$$-40 = 10t + \tfrac{1}{2}(-10)\,t^2 = 10t - 5t^2$$

or
$$5t^2 - 10t - 40 = 0$$

or
$$t^2 - 2t - 8 = 0$$

or
$$t^2 - 4t + 2t - 8 = 0$$

or
$$t(t - 4) + 2(t - 4) = 0$$

or $$(t+2)(t-4)=0$$

i.e. $$t=-2\ \text{s} \quad \text{or} \quad 4\ \text{s}$$

The negative value of t is not possible. Hence, the stone will strike the ground 4 s after it is projected.

(ii) It is clear that the net displacement suffered by the stone in going from A to B and back to A (the point from where it was projected) is zero. Thus

$$S=0$$

Using Eq. (3.19) we have

$$\begin{aligned} 0 &= 10t + \tfrac{1}{2}(-10)t^2 \\ &= 10t - 5t^2 \\ &= 5t(2-t) \end{aligned}$$

$$\therefore \quad t = 0 \quad \text{or} \quad 2\ \text{s}$$

Since $t=0$ is not possible because at $t=0$ the stone was projected, t must be 2 s, i.e. the stone passes through the point of projection after two seconds.

(iiii) Let v be the final velocity just before the stone strikes the ground. From Eq. (3.18) we have

$$\begin{aligned} v &= u + at \\ &= 10 + (-10) \times 4 \\ &= 10 - 40 \\ &= -30\ \text{m s}^{-1} \end{aligned}$$

The negative sign indicates that the velocity v is directed downwards.

EXAMPLE 3.5 A balloon is rising vertically upwards at a velocity of 10 ms^{-1}. When it is at a height of 45 m from the ground, a parachutist bails out from it. After 3 s he opens the parachute and decelerates at a constant rate of 5 ms^{-2}. (i) What was the height of the parachutist above the ground when he opened his parachute? (ii) How far is he from the balloon at this instant of time? (iii) With what velocity does he hit the ground? (iv) After how long does he strike the ground after his exit from the balloon? Take $g = 10$ ms^{-2}.

Solution (i) Let us say that the parachutist bails out at time $t=0$ (see Fig. 3.9). He shares the velocity of the balloon and has an upward velocity of 10 ms^{-1}.

$$u = +10\ \text{m s}^{-1}$$

$$a = -10\ \text{m s}^{-2}\ \text{(acting downwards)}$$

$$t = 3\ \text{s}$$

The displacement in 3 s is obtined from the equation

$$\begin{aligned} S &= ut + \tfrac{1}{2}\,at^2 \\ &= 10 \times 3 + \tfrac{1}{2}(-\,10) \times (3)^2 \\ &= -\,15 \text{ m} \end{aligned}$$

Since the displacement is negative, it is directed downwards. The parachutist will be at B at time $t = 3$ s, the displacement from A to B being 15 m downwards. So the height from the ground when he opened his parachute is 30 m.

(ii) In time 3 s, the balloon has risen through 30 m (as the velocity of the balloon is 10 ms^{-1} upwards). Hence, the parachutist is now 30 + 15 = 45 m away from the balloon at time $t = 3$ s.

(iii) Let us now calculate his velocity at B. We use

$$\begin{aligned} v &= u + at \\ &= 10 + (-\,10) \times 3 \\ &= -\,20 \text{ ms}^{-1} \text{ (directed downwards)} \end{aligned}$$

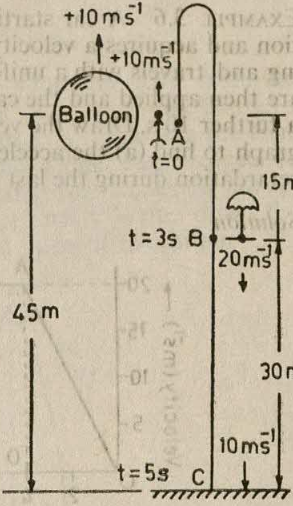


Fig. 3.9

To calculate the time taken for him to fall from B to C, the velocity of $-\,20$ m s^{-1} is to be treated as the initial velocity at B, i.e.

$$u = -\,20 \text{ ms}^{-1}$$

$$a = +\,5 \text{ ms}^{-2} \text{ (directed upwards)}$$

$$S = -\,30 \text{ m} \quad \text{(from B to C directed downwards)}$$

∴ Using $S = ut + \frac{1}{2}\,at^2$,

we have $-\,30 = -\,20t + \frac{1}{2} \times 5 \times t^2$

or $t^2 - 8t + 12 = 0$

or $(t - 6)\,(t - 2) = 0$

i.e. $t = 2$ or 6 s

Let us take $t = 6$ s. Then the velocity with which he hits the ground should be

$$\begin{aligned} v &= u + at \\ &= -\,20 + (5) \times 6 = +\,10 \text{ ms}^{-1} \end{aligned}$$

This is positive, i.e. v is directed upwards, which is not possible. Hence $t = 2$s, and the velocity with which he hits the ground is

$$v = -20 + 5 \times 2 = -10 \text{ ms}^{-1}$$

which is negative, as it should be.

(iv) The total time he takes (after his exist) to strike the ground is 3 s + 2 s = 5 s.

EXAMPLE 3.6 A car starting from rest is given a constant acceleration and acquires a velocity of 20 ms^{-1} in 4 s. The car stops accelerating and travels with a uniform velocity of 20 ms^{-1} for 6 s. The brakes are then applied and the car comes to rest with uniform retardation in a further 10 s. Draw the velocity-time graph of the motion. Use the graph to find (a) the acceleration of the car during the first 4 s (b) its retardation during the last 10 s and (c) the total distance the car travels.

Solution

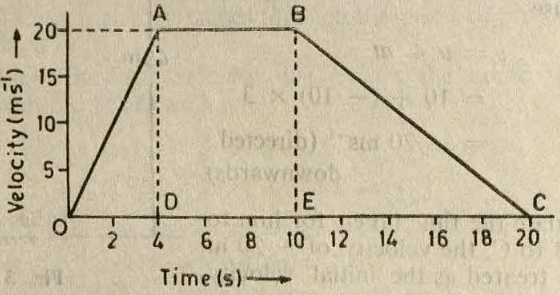


Fig. 3.10

Figure 3.10 is the velocity-time graph of the motion:

(a) Acceleration during the first 4 s = slope of OA.

$$= \frac{AD}{OD} = \frac{(20-0)\text{ ms}^{-1}}{(4-0)\text{ s}}$$

$$= \frac{20 \text{ ms}^{-1}}{4\text{s}} = 5 \text{ ms}^{-2}$$

(b) Acceleration during the last 10 s = slope of BC

$$= \frac{BE}{EC}$$

$$= \frac{(20-0)\text{ ms}^{-1}}{(10-20)\text{ s}} = \frac{20 \text{ ms}^{-1}}{-10 \text{ s}} = -2 \text{ ms}^{-2}$$

Therefore, the retardation of the car is 2 ms^{-2}

(c) Total distance travelled = distance travelled between O and 4 s + distance travelled between 4 s and 10 s + distance travelled between 10 s and 20 s.

$$= \text{area of triangle OAD} + \text{area of rectangle ADEB} + \text{area of triangle EBC}$$

$$= \tfrac{1}{2}\,\text{OD} \times \text{AD} + \text{DE} \times \text{AD} \times \tfrac{1}{2}\,\text{EC} \times \text{BE}$$

$$= \tfrac{1}{2} \times 4 \times 20 + 6 \times 20 + \tfrac{1}{2} \times 10 \times 20$$

$$= 40 + 120 + 100 = 260 \text{ m}$$

EXAMPLE 3.7 Two stones are thrown up simultaneously from the edge of a cliff 200 m high with initial speeds of 15 ms^{-1} and 30 ms^{-1}. Verify that the graph shown in Fig. 3.11 correctly represents the time variation of the relative position of the second stone with respect to the first. Neglect air resistance and assume that the stones do not rebound after hitting the ground. Take $g = 10\ \text{ms}^{-2}$. Write down the equations representing the linear and curved parts of the graph.

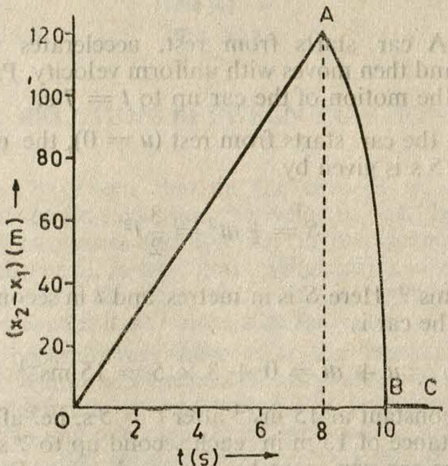


Fig. 3.11

Solution Let us suppose that the position coordinate of the ground is $x = 0$ and that of the edge of the cliff is $x_0 = 200$ m. The stone 1 is thrown up with an initial velocity $u_1 = 15\ \text{ms}^{-1}$ and the stone 2 with initial velocity $u_2 = 30\ \text{ms}^{-1}$. As in Example 3.4, we find that the stone 1 hits the ground after 8 s and stone 2 after 10 s. Up to the first 8 seconds, the positions of the two stones are given by

$$x_1 = x_0 + u_1 t + \tfrac{1}{2} g t^2$$

and

$$x_2 = x_0 + u_2 t + \tfrac{1}{2} g t^2$$

Subtracting them we get the separation between the stones:

$$x_2 - x_1 = (u_2 - u_1)t = (30 - 15)t$$

or $$x_2 - x_1 = 15t$$

This is the equation for separation for the first 8 seconds. Notice that $(x_2 - x_1)$ is a linear function of t having a slope 15 ms^{-1} which represents the linear part OA of the graph.

At $t = 8$ s, stone 1 hits the ground and stops there. Hence after 8 s, $x_1 = 0$ so that, we have, for time between 8 s and 10 s

$$x_2 - x_1 = x_0 + u_2t + \tfrac{1}{2} gt^2$$

$$= 200 + 15t - 5t^2 \quad (\because \; g = -10 \text{ ms}^{-2})$$

This is the equation for the graph BC between $t = 8$ s and 10 s. At $t = 10$ s, stone 2 also hits the ground and stops. So after 10 s both the stones are at rest on the ground i.e. their separation is zero after 10 s which explains the part BC of the graph.

Example 3.8 A car starts from rest, accelerates uniformly with 3 ms^{-2} for 5 s and then moves with uniform velocity. Plot the distance-time graph of the motion of the car up to $t = 7$ s.

Solution Since the car starts from rest ($u = 0$), the distance covered during the first 5 s is given by

$$S = \tfrac{1}{2} at^2 = \frac{3}{2} t^2 \qquad \text{(i)}$$

because $a = 3$ ms^{-2}. Here S is in metres and t in seconds. At $t = 5$ s, the velocity of the car is

$$v = u + at = 0 + 3 \times 5 = 15 \text{ ms}^{-1}$$

which remains constant at 15 ms^{-1} after $t = 5$ s, i.e. after $t = 5$ s, the car covers a distance of 15 m in each second up to 7 s. The distance-time table of the car can now be prepared using Eq. (i) for t up to 5 s and Eq. (ii) for t after 5 s.

Time (s)	0	1	2	3	4	5	6	7
Distance (m)	0	1.5	6.0	13.5	24	37.5	52.5	67.5

Figure 3.12 shows the distance-time graph of the motion of the car up to 7 s. Notice that during the accelerated motion (0 to 5 s) the graph

is a parabola (part OA) and during uniform motion (after 5 s) the graph is linear (part AB).

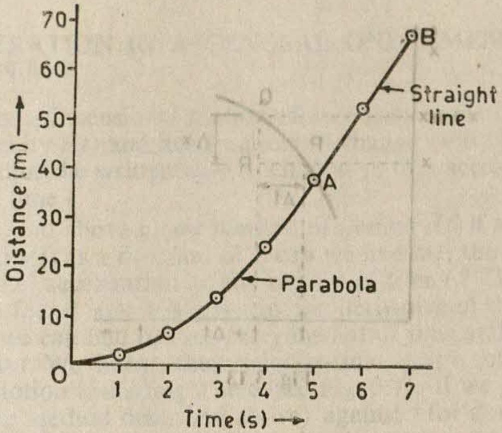


Fig. 3.12

3.7 GENERAL RELATION BETWEEN POSITION AND VELOCITY

In Section 3.5 we have seen that in the case of a uniform motion (v=constant), we could easily get the velocity just by knowing the positions at any two times and dividing the displacement by the time interval. Since the position-time graph (Fig. 3.5) is a straight line and the velocity is given by the slope of the line, it does not matter at what time the slope is found; it will always be the same.

The situation becomes very different in the case of a non-uniform motion with a constant acceleration. In this case, as discussed in Section 3.6, we can define an *average velocity* which refers to an interval of time and not to any one time. The position-time graph of such a motion is a curve called *parabola* as shown in Example 3.8. The slope of a parabola changes from point to point on the curve, as the velocity is changing with time. In such cases we can define what is called the *instantaneous velocity*, i.e. velocity at each instant of time. If we know the position $x(t)$ of a moving object at each time t, is it possible to calculate its velocity at each time? The answer is yes. The method of calculating the velocity is explained below.

Let us consider a general motion of an object whose acceleration may not even be constant. The position-time graph of such a motion is not even a parabola; it is some kind of a curve, a part of which is shown in Fig. 3.13 near the instant of time t, at which we wish to find the velocity $v(t)$.

We consider a *very small* time interval between t and $t + \Delta t$, so that Δt is very small. During this small interval Δt, the part PQ of the curve is very nearly straight, i.e. the motion is nearly uniform during this interval. The smaller the interval Δt, the better is our approximation.

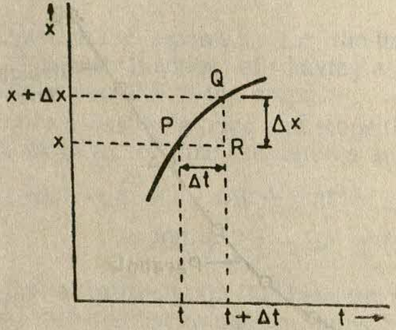


Fig. 3.13

We can therefore, calculate the velocity $v(t)$ at time t by using the relation for uniform motion, i.e. from the slope QR/PR as shown in the figure. Thus

$$v(t) \simeq \frac{\Delta x}{\Delta t} \tag{3.21}$$

where Δx is a *small* change in position in a small time interval Δt. Notice that we have used a sign $\simeq$ which stands for 'approximately equal to' which means that there is a slight error in our relation between velocity and position given by Eq. (3.21). The error will become smaller and smaller if the interval Δt is made shorter and shorter, i.e. Q becomes closer and closer to P. Our result will become exact 'in the limit' when Δt tends to zero, i.e. Q coincides with P. Thus

$$\begin{aligned} v(t) &= \text{velocity at P} \\ &= \operatorname*{Limit}_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t}\right) \end{aligned}$$

(i.e. $q \rightarrow p$)

In the language of differential calculus it is written as

$$v(t) = \frac{dx}{dt} \tag{3.22}$$

i.e. *velocity is the differential coefficient* (*or derivative*) *of the position with respect to time*. In fact, $\frac{dx}{dt}$ is the slope of the x-t curve at point P, that is, it is the velocity at P.

In the supplement at the end of this chapter, we have shown how the formulae for uniformly accelerated motion can be obtained by using this new method of finding $v(t)$ if $x(t)$ is known.

3.8 ACCELERATION IN A GENERAL ONE-DIMENSIONAL MOTION

The general one-dimensional motion of an object is one in which position $x(t)$, velocity $v(t)$ and acceleration all change with time so that acceleration must be written as $a(t)$ which says that acceleration $a(t)$ is a function of time t.

We have found above a new method of finding $v(t)$ if $x(t)$ is known. Having found $v(t)$ as a function of t, can we find $a(t)$ the *instantaneous acceleration*, i.e. acceleration at any instant of time t ? The answer is yes. We have found $v(t)$; it is given by the derivative of the position at time t. Thus we can find $v(t)$ for every instant of time and plot a graph of $v(t)$ against t. We know that velocity-time graph of a uniformly accelerated motion is a straight line (see Fig. 3.7). If we plot $v(t)$ (calculated by the method described above) against t for a non-uniformly accelerated motion, the graph will obviously not be a straight line; it will be some sort of a curve, a part of which around a time t is shown in Fig. 3.14.

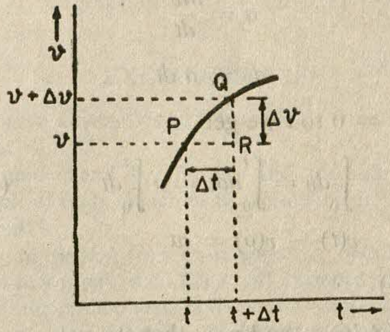


Fig. 3.14

In exactly the same manner as discussed in Section 3.7 above, the acceleration $a(t)$ at time t is given by

$$a(t) = \underset{\Delta t \to 0}{\text{Limit}} \left(\frac{\Delta v}{\Delta t} \right)$$

or

$$a(t) = \frac{dv}{dt} \tag{3.23}$$

i.e. *acceleration is the derivative of the velocity with respect to time.* In fact, $\frac{dv}{dt}$ is the slope of the v-t curve at time t, i.e. it is the acceleration at time t.

We can now express acceleration $a(t)$ directly in terms of position $x(t)$. Using Eq. (3.22) in Eq. (3.23), we get

$$a(t) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad (3.24)$$

In the language of calculus we say that *acceleration is the second derivative of position with respect to time.*

For an application of this new (calculus) method of finding $v(t)$ and $a(t)$, the student is advised to refer to the following Supplement where we have derived Eqs (3.15) and (3.16) of a uniformly accelerated motion by using this method.

SUPPLEMENT

Application of Calculus Method

We will show how Eqs (3.15) and (3.16) of uniformly accelerated motion can be obtained using the calculus method.

Derivation of Eq. (3.15) We have seen above that the acceleration is given by the derivative of velocity with respect to time, i.e.

$$a = \frac{dv}{dt}$$

$$\therefore \qquad dv = a\,dt \qquad \text{(i)}$$

Integrating from $t = 0$ to t we get

$$\int_0^t dv = \int_0^t a dt = a\int_0^t dt \qquad (\because\ a = \text{constant})$$

or

$$v(t) - v(o) = at \qquad \text{(i)}$$

which is Eq. (3.15)

Derivation of Eq. (3.16) We know that the velocity is given by the derivative of position with respect to time, i.e.

$$v(t)\,\frac{dx}{dt}$$

$$\therefore \qquad dx = v(t)\,dt$$

Using Eq. (i) we get

$$dx = \{v(0) + at\}\,dt$$

$$dx = v(0)\,dt + at\,dt$$

Integrating from 0 to t we have ($\because$ a and $v(0)$ are both constant)

$$\int_0^t dx = v(0) \int_0^t dt + a \int_0^t t\,dt$$

or
$$x(t) - x(0) = v(0)t + \tfrac{1}{2}at^2 \qquad \text{(ii)}$$

which is Eq. (3.16). Equation (3.17) is obtained on eliminating t between (i) and (ii).

SUMMARY

The three parameters which describe the motion of a moving object are the position coordinate, velocity and acceleration. Velocity is the rate of change of position with time and acceleration is the rate of change of velocity. The velocity is the slope of the position-time graph and acceleration is the slope of the velocity-time graph of the motion. The distance covered by the moving object in a certain time interval is equal to the area under the velocity-time graph in that time interval. The equations of a uniformly accelerated motion in a straight line are

$$v(t) = v(0) + at$$

$$x(t) = x(0) + v(0)\,t + \tfrac{1}{2}at^2$$

$$v^2(t) = v^2(0) + 2a\,[x(t) - x(0)]$$

EXERCISES

1. Distinguish between kinematics and dynamics.
2. What is a 'point' object?
3. What do you understand by one-two and three-dimensional motion. Give one example of each. Which of the following motions is one-two or three-dimensional?
   (a) A bird flying in the sky, (b) a train speeding on a long and straight track, (c) a boat in a rough sea. (d) a ball dropped from a height and rebounding from the ground and (e) the moon revolving round the earth.
4. Figure 3.15 shows two x—t graphs. Which graph represents a real situation and why ?

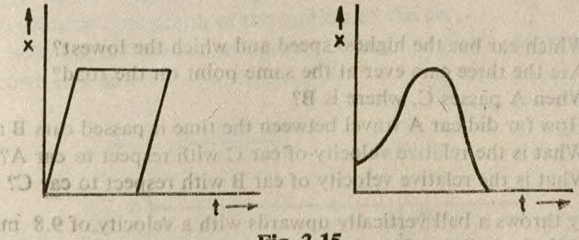


Fig. 3.15

5. Distinguish clearly between distance and displacement.
6. Draw the position-time graph for a uniform motion. How will you use the graph to find (a) the velocity of the moving object and (b), the distance covered in a certain time interval ?
7. Deduce the equations of a uniformly accelerated motion in a straight line from the velocity-time graph of the motion.
8, Obtain the relation between position and velocity for a general one-dimensional motion.
9 How can we find the acceleration at a time t for a general one-dimensional motion.
10. Deduce the equations of a uniformly accelerated motion in a straight line using the calculus method.

PROBLEMS

1. Figure 3.16 shows the position-time graphs of three cars A, B and C. On the basis of the graphs, answer the following questions:

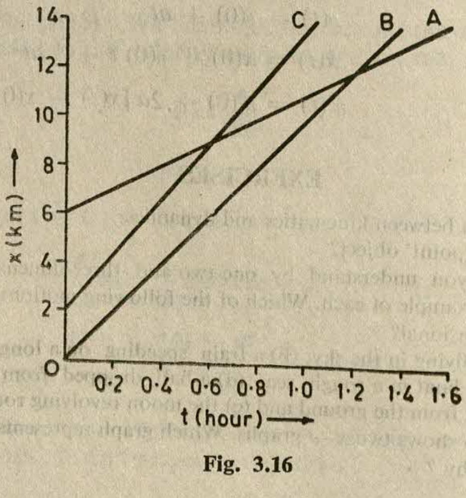


Fig. 3.16

(a) Which car has the highest speed and which the lowest?
(b) Are the three cars ever at the same point on the road?
(c) When A passes C, where is B?
(d) How far did car A travel between the time it passed cars B and C?
(e) What is the relative velocity of car C with respect to car A?
(f) What is the relative velocity of car B with respect to car C?

2. A boy throws a ball vertically upwards with a velocity of 9.8 ms^{-1} from the roof of a building 20 m high. How long will the ball take to reach the ground? What will be its velocity when it strikes the ground.

3. A boy standing on the roof of a building 50 m high throws a ball vertically downwards with an initial velocity of 5 ms^{-1} (i) What is the velocity of the ball 1.0 s after it was thrown? (ii) How high above the ground will the ball be after 2.0 s? (iii) With what velocity will the ball hit the ground?
4. Solve Problem 4 for the case when the ball is thrown vertically upwards with an initial velocity of 5 ms^{-1}.
5. A balloon is ascending at the rate of 14 ms^{-1} vertically upwards. A food packet dropped from the balloon is found to take $6\frac{1}{8}$ is to strike the ground.
   (i) What was the height of the balloon when the packet was dropped?
   (ii) With what velocity does it reach the ground?
6. A parachutist bails out from an aeroplane and after dropping through a distance of 19.6 m, he opens the parachute and decelerates at 2 ms^{-2}. If he reaches the ground with a velocity of 1.6 ms^{-1}; for how long was he in the air? What was the height of the aeroplane when he bailed out from it?
7. A man is running at a speed of 4 ms^{-1} towards a stationary bus. When he is 6 m behind the door (at $t = 0$), the bus suddenly starts moving at a constant acceleration of 1.2 ms^{-2}. How long does it take for the man to 'catch' the bus? If he where initially 10 m behind the bus, could he catch the bus?
8. A ball of mass 100 g is thrown vertically upward from the ground with a velocity of 49 ms^{-1}. At the same time, another identical ball is dropped from a height of 98 m to fall freely along the same path as followed by the first ball. After some time the two balls collide and stick together and finally fall together. Find the time of flight of the balls.
9. At time $t = 0$, the driver of a train moving at a speed of 108 km h^{-1} sights another train at a distance x ahead of him moving at a speed of 54 km h^{-1}. He immediately applies brakes which give his train a constant retardation of 5 ms^{-2}. What is the critical value of x that will just avert a collision?
10. A car starting at time $t = 0$ from rest, first moves for some time with a constant acceleration of 5 ms^{-2} and then, after moving at a constant velocity for some time, starts decelerating at the same rate to come to rest in a total time of 25 s. If the average velocity of the car over the entire journey is 20 ms^{-1}, for how long does it move with a uniform velocity? Draw the velocity-time graph of the motion of the car.
11. A body starts from rest at time $t = 0$ and undergoes an acceleration as shown in Fig. 3.17.

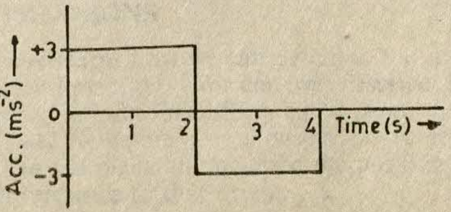


Fig. 3.17

(a) Draw the velocity-time graph of the motion during $t = 0$ to $t = 4$ s.

(b) Draw the displacement-time graph of the motion during $t = 0$ to $t = 4$ s.

12. A small ball is dropped vertically from the zero mark of a vertical scale. A 'snapshot' of the falling ball is taken and shows that while the shutter is open, the ball falls from the 4.9 m to the 5.1 m mark. How long is the shutter open?

13. If a, b and c be the distances moved by a particle travelling with a uniform acceleration during the xth, yth and zth, second of its motion respectively, show that

$$a\,(y - z) + b\,(z - x) + c\,(x - y) = 0$$

14. A particle traversed half the distance with a velocity V, the remaining part of the distance was covered with a velocity V' for half the time and with a velocity V'' for the other half of the time. Find the mean velocity of the particle averaged over the whole of its motion.

15. A circus artist maintains four balls in motion making each in turn rise to a height of 5 m from his hand. With what velocity does he project them and where will the other three balls be at the instant the fourth one leaves his hand? (Take $g = 10\ \text{m s}^{-2}$).

16. A steel ball is dropped from a height of 78 m on a perfectly elastic surface. Taking the speed of the ball at $t = 0$ as zero, draw a graph representing the velocity of the ball as a function of time from $t = 0$ to $t = 8$ s. What is the velocity of the ball at $t = 1$ s and $t = 7$ s?

17. The displacement of a particle varies with time according to the relation

$$x = -15\,t^2 + 20\,t + 30$$

Find the position, velocity and acceleration of the particle at $t = 0$. When and where will the velocity of the particle become zero? Can we call the motion of the particle as one with a uniform acceleration?

4

Description of Motion in Two and Three Dimensions

4.1 INTRODUCTION: THE NEED FOR VECTORS

In Chapter 3, we discussed motion in one dimension, i.e. motion in a straight line. We analysed the simplest kind of motion along a straight line, namely, the one with a constant acceleration. In such a motion, the displacement, velocity and acceleration are all in the same direction. In this chapter, we shall consider those motions in which velocity and acceleration are not in the same direction. The motion in this case follows a curved path, as we shall see below. Such a motion is called *curvilinear motion*.

In everyday life, we rarely come across rectilinear motion. Vehicles move along a curved path while negotiating a curve; athletes run on circular tracks, planets move in nearly circular orbits and electrons in an atom revolve around the nucleus.

We will use the concepts of position, velocity and acceleration developed in Chapter 3. In order to generalise these concepts to the case of two and three dimensions we need to use new mathematical quantities called *vectors*. We therefore, begin this chapter with vectors. We will learn the important properties of vectors and vector operations.

4.2 POSITION AND DISPLACEMENT VECTORS IN TWO DIMENSIONS

In Chapter 3 we have learnt how we can describe the position of an object moving in one dimension. How can we describe the position of an object moving in two dimensions, e.g. a plane surface such as the plane of this page? As before, we choose a convenient reference point or origin O in the plane and describe the position of an object on this plane with reference to that origin.

Suppose at a certain instant of time $t = t_1$, the object is at a point P in the plane. Then a straight line drawn from the origin O to the position P is called the *position vector* of the object at time t_1 (Fig. 4.1).

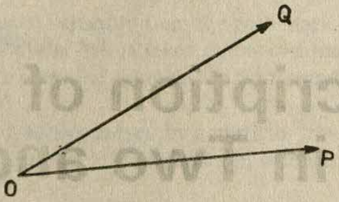


Fig. 4.1 Position vector in a plane

The position vector is written as $\overrightarrow{OP}$ will a tail at O and an arrowhead at the tip P. For every point in a plane. there is a position vector from O to that point.

The vector $\overrightarrow{OP}$ contains *two* items of information: (i) it tells us how far the point P is from the origin O, i.e. the straight-line distance from O to P and (ii) it tells us in what direction the point P is with respect to O or as seen from O. Thus it has a *magnitude* (i.e. length) as well as a *direction*. Figure 4.1 also shows another position vector $\overrightarrow{OQ}$ whose length is the same as that of vector $\overrightarrow{OP}$ but its direction is different. The magnitude of a vector is written as $|\overrightarrow{OP}|$ or simply as OP without an arrow above it. Note that $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$ but $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are different vectors because their directions are different.

The Displacement Vector

Suppose the moving object is at point P_1 at time t_1 and at point P_2 at a later time t_2, then the vector $\overrightarrow{P_1P_2}$ with the tail at P_1 and the arrowhead (or tip) at P_2 is the displacement vector representing the motion during the interval from t_1 to t_2.

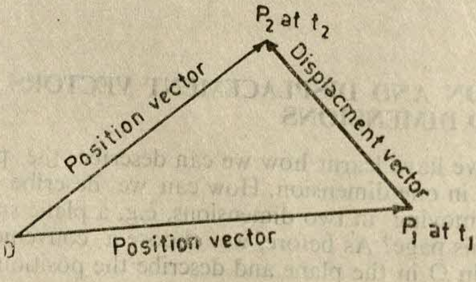


Fig. 4.2 The displacement vector

It must be clearly understood that the displacement vector is the shortest distance (straight-line distance) in the prescribed direction between the points P_1 and P_2. In actual motion, the object may not have travelled along the straight line, as discussed in Sec. 3.5.

4.3 VECTOR AND SCALAR QUANTITIES

All physical quantities can be classified into two broad categories—*scalar quantities* and *vector quantities*. Quantities *that have only a magnitude and no direction are called scalar quantities* or simply *scalars*. A few examples of scalars are: mass, time, area, volume, distance, speed, work, energy, power, heat, temperature, charge and potential. They are completely specified by merely stating a number (i.e. magnitude).

On the other hand, some physical quantities are specified not by merely stating a number but a direction as well. *Quantities that have a magnitude as well as a direction are called vector quantities* or simply *vectors*. A few examples of vectors are: position, displacement, velocity acceleration, force, momentum, torque, electric field and magnetic field.

4.4 VECTOR OPERATIONS

We will now learn a few mathematical operations with vectors useful in the study of physics. We will learn operations like addition and subtraction of vectors. It must be clearly understood that these operations are carried out with vectors representing the same physical quantity and in the same number of dimensions. For example, we cannot add a displacement vector to a velocity vector or a displacement in one dimension to a displacement in two dimensions. These points must be kept in mind while developing the mathematics of vectors.

Analytically, a vector may be represented by a single letter with an arrow on top of it as $\vec{A}$. Instead of putting an arrow, the normal practice is to write the letter in boldface to represent a vector (as in A).

Equality of Vectors

Two vectors $\vec{A}$ and $\vec{B}$ of the same physical quantity are equal, if and only if they have the same magnitude and the same direction. For example, vectors $\vec{A}$ and $\vec{B}$ shown in Fig. 4.3 (a) are equal. We can test the equality by shifting $\vec{B}$ parallel to itself until its tail coincides with the tail of $\vec{A}$. If the tips of the two vectors also coincide, the vectors are equal. (Remember, the vector shifted parallel to itself is equal to the original vector since the magnitude and the direction of the vector do not change when it is shifted parallel to itself). Equality of two vectors is indicated in the usual way:

$$\vec{A} = \vec{B}$$

The two vectors $\vec{C}$ and $\vec{D}$ shown in Fig. 4.3 (b) are equal in magnitude but opposite in direction. Therefore

$$\vec{C} = -\vec{D} \quad \text{or} \quad \vec{D} = -\vec{C}$$

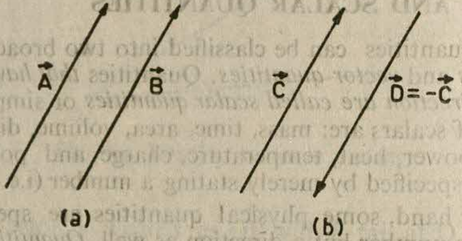


Fig. 4.3 (a) Two equal vectors. $\vec{A} = \vec{B}$
(b) Two vectors equal in magnitude but opposite in direction.
$\vec{C} = -\vec{D}$ or $\vec{D} = -\vec{C}$

Multiplication of Vectors by Real Numbers

When a vector is multiplied by a real number, the result is another vector. We will illustrate this by taking the example of position vectors. Let us first consider the multiplication of vectors by *positive* real numbers. Suppose two vectors $\vec{OP}$ and $\vec{OQ}$ are in the *same* direction (i.e. OPQ is a straight line) but the length of vector $\vec{OQ}$ is twice that of vector $\vec{OP}$ [Fig. 4.4 (a)]. We say that

$$\vec{OQ} = 2\,\vec{OP}$$

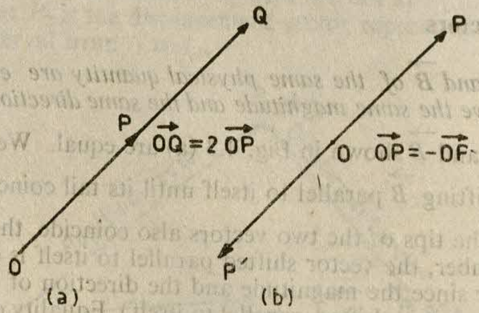


Fig. 4.4 (a) Multiplying a vector by 2
(b) Multiplying a vector by—1.

Thus th . multiplication of $\overrightarrow{OP}$ by 2 has dobuled its length but its direction remains unchanged. This means that if a vector $\overrightarrow{A}$ is multiplied by a real positive number λ, the result is a vector, which is written as $\lambda \overrightarrow{A}$ whose direction is the same as that of $\overrightarrow{A}$ but whose magnitude is changed by a factor λ:

$$|\lambda\overrightarrow{A}| = \lambda|\overrightarrow{A}|; \quad \text{if} \quad \lambda > 0$$

Let us now consider multiplication by *negative* numbers. Suppose two points P and P′ on a plane are at the same distance from the origin O, but in opposite directions as seen from O, i.e. P′ OP is a straight line (Fig. 4.4 b). It is clear that vector $\overrightarrow{OP'}$ is the *negative* of vector $\overrightarrow{OP}$, i.e. $OP' = (-1) \times \overrightarrow{OP}$. Thus multiplication of a vector by -1, merely reverses its direction without changing its magnitude. It follows that the multiplication of a vector by -2 reverses its direction and also doubles its magnitude. Thus the general rule is that the multiplication of a vector $\overrightarrow{A}$ by a negative number λ gives a vector $\lambda\overrightarrow{A}$ whose direction is opposite to that of $\overrightarrow{A}$ but whose magnitude is $(-\lambda)$ times $|\overrightarrow{A}|$:

$$|\lambda\overrightarrow{A}| = -\lambda\,|\overrightarrow{A}| = |\lambda|\,|\overrightarrow{A}|; \quad \text{if} \quad \lambda < 0$$

To summarise, *the multiplication of a vector by a real number either leaves the direction unchanged (if the number is positive) or reverses the direction (if the number is negative); while the magnitude of the vector is scaled by the magnitude of the number.*

The number λ need not be a pure number (suh as 2, 3, -1, -2, etc.) without any dimensions. In fact λ could be a scalar having its own dimensions and $\overrightarrow{A}$ could be a physical quantity having its own dimensions which could, be different from those of λ. Then the dimensions of the product $\lambda\overrightarrow{A}$ will be the product of the dimensions of λ and $\overrightarrow{A}$. For example, if we multiply a velocity vector (dimensions LT^{-1}) by a time interval (which is a scalar of dimension T), we get a displacement vector (dimension $LT^{-1} \times T = L$).

Addition and Subtraction of Vectors

If we have 4 kg of rice in a bag and add 3 kg of rice to it, we will have a total of $4 + 3 = 7$ kg of rice in the bag. This is how scalars add, but vectors do not add like this because their direction may be different.

Addition of Vectors We have already added two position vectors to define a displacement vector as shown in Fig. 4.2 (without knowing how vectors are added). We will now learn how this is done. We will illustrate the addition of vectors by taking the example of displacement vectors in a plane.

The addition of vectors in one dimension is straight forward. Suppose we give two displacements $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ one after another to the same object in the same direction. Figure 4.5 (a) shows a displacement $\overrightarrow{PQ}$ of magnitude, say 4 cm to the 'right' applied to an object originally at the point P, which brings it to a position Q. Now a second displacement $\overrightarrow{QR}$ of magnitude, say, 3 cm is applied to it in the same direction, which brings the object to the position *R*. Obviously the the same result could be obtained if a single displacement $\overrightarrow{PR}$ of magnitude 4 + 3 = 7 cm were given to the object in the same direction in which the individual displacements are given. In the language of vectors, the result is written as

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

or

$$\vec{A} + \vec{B} = \vec{C}$$

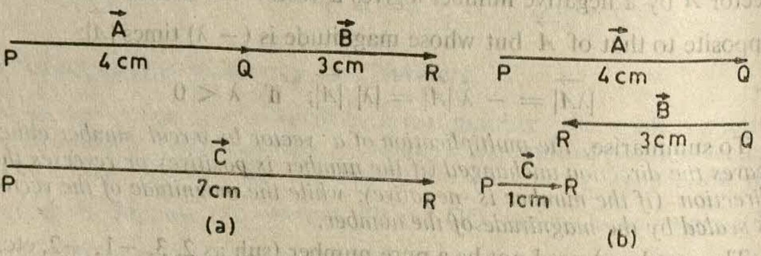


Fig. 4.5 Addition of vectors in one dimension

Figure 4.5 (b) shows how vectors in opposite directions are added. The result is a vector $\vec{C} = \overrightarrow{PR} = 1$ cm in the direction of the bigger of the two vectors $\vec{A}$ and $\vec{B}$. This result is also written as

$$\vec{A} + \vec{B} = \vec{C}$$

Since the vector $\vec{C}$ alone produces the same result as the vectors $\vec{A}$ and $\vec{B}$ together do, we say that $\vec{C}$ is the *resultant* (or sum) of vectors $\vec{A}$ and $\vec{B}$. Thus it is clear that vectors do not add as the scalars do.

We will now consider addition of vectors in two dimensions. We begin with the simplest case of two vectors at right angles to each other.

Case I—*Addition of Two Vectors at Right Angles to Each Other*: Suppose a man walks a distance of 4 m from P to Q in the east and then walks a distance of 3 m from Q to R in the north as shown in

Fig. 4.6. These two perpendicular displacement $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are respectively represented by two vectors $\vec{A}$ and $\vec{B}$. For convenience, 1 cm can be taken to represent a distance of 1 m. If we join P and R and measure the length of the vector $\vec{C}$ (or $\overrightarrow{PR}$), we will find it to be 5 cm which represents a displacement of magnitude 5 m. If instead of walking a distance of 4 m from P to Q and then walking a distance of 3 m from Q to R (a total distance of 4 + 3 = 7 m), the man walks a distance of 5 m from P to R, he would reach the same distination (which is R). Thus, the displacement of 5 m along PR produces the same effect as the displacement of 4 m along PQ and 3 m along QR together do. We say that the vector $\overrightarrow{PR}$ (or $\vec{C}$) is the resultant (or sum) of two vectors $\overrightarrow{PQ}$ (or $\vec{A}$) and $\overrightarrow{QR}$ (or $\vec{Q}$). In the language of vectors, we may write

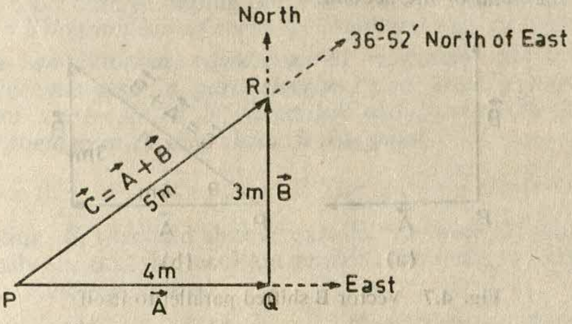


Fig. 4.6 Addition of two perpendicular vectors

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

or
$$\vec{A} + \vec{B} = \vec{C} \tag{4.1}$$

It is easy to see that the same resultant effect is obtained if the two displacements are applied simultaneously instead of one being applied after the other. This can be understood as follows: Suppose a displacement $\vec{A}$ of magnitude 4 m is applied to a particle originally at P in the direction of the east and simultaneously another displacement $\vec{B}$ of magnitude 3 m is applied to it in the direction of the north as shown in Fig. 4.7 (a). The tails of the arrows of vectors $\vec{A}$ and $\vec{B}$ are at P. Let us shift the vector $\vec{B}$ parallel to itself until the tail of vector $\vec{B}$ is at the head of vector $\vec{A}$ as shown in Fig. 4.7 (b). Figure 4.7 (b) is the same as Fig. 4.6 which represents the case when the displacements were applied one after the other. Hence the resultant is the same as before. The resultant displacement has a magnitude of 5 m and its direction is 36° 52′ north of east.

Thus the method (or rule) for vector addition is: *shift* (*if necessary*) *one vector parallel to itself until its tail is at the head of the other vector. The resultant is then given by the vector drawn from the tail of the first vector to the head of the second.*

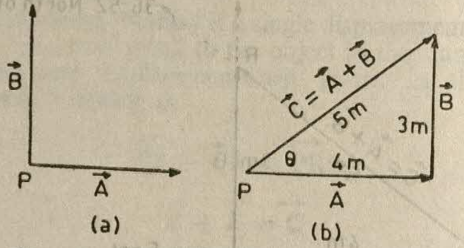


Fig. 4.7 Vector B shifted parallel to itself

Case II—*Addition of two Vectors Inclined to Each Other*: Consider two vectors $\vec{A}$ and $\vec{B}$ represented by $\overrightarrow{PQ}$ and $\overrightarrow{OR}$ respectively and inclined to each other at a certain angle as shown in Fig. 4.8 (a). Shift $\vec{B}$ parallel to itself till the tail O of $\vec{B}$ touches the head Q of $\vec{A}$ as shown in Fig. 4.8 (b). Then the resultant vector is given by $\overrightarrow{PR}$ which joins the tail of $\vec{A}$ to the head of $\vec{B}$. Now complete the parallelogram PQRS. Since $\overrightarrow{PR}$ is the resultant of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, we have

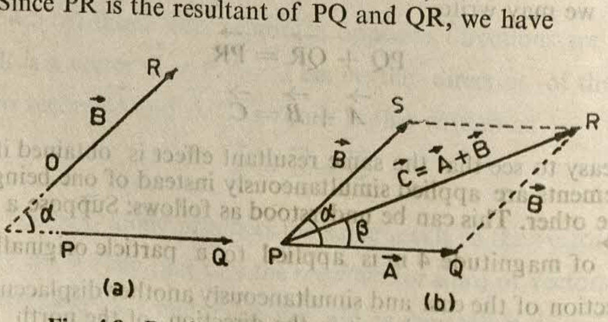


Fig. 4.8 Parallelogram law of vector addition

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

But $$\overrightarrow{PS} = \overrightarrow{QR} = \vec{B} \qquad (\because\ \overrightarrow{PS} \text{ and } \overrightarrow{QR} \text{ are equal and parallel})$$

$$\therefore \quad \overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PR}$$

or $$\vec{A} + \vec{B} = \vec{C}$$

Now $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ (which represent the two vectors $\vec{A}$ and $\vec{B}$ respectively) constitute the two adjacent sides of the parallelogram, meeting at point P and the resultant vector $\overrightarrow{PR}$ is the diagonal of the parallelogram passing through the same point P.

This procedure of finding the resultant of two vectors is known as the *parallelogram law of vector addition* and may be stated as follows.

If the two vectors are represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.

Notice that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The reason is the following. Instead of shifting $\vec{B}$, we could shift $\vec{A}$ parallel to itself. If you do that, you can easily see that the resultant remains the same, no matter which of the two vectors is shifted.

Case III—*Addition of More than Two Vectors*; *Polygen Law of Vector Addition*: Suppose we wish to determine the resultant of four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ as shown in Fig. 4.9 (a). These vectors may be the four displacements or forces applied to a body. The procedure to find the resultant is that we shift vectors $\vec{B}$, $\vec{C}$ and $\vec{D}$ parallel to themselves so that the tail of $\vec{B}$ touches the head of $\vec{A}$, the tail of $\vec{C}$ is at the head of $\vec{B}$ and the tail of $\vec{D}$ is at the head of $\vec{C}$. The result of this shifting is shown in Fig. 4.9 (b).

The resultant of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ is given by vector $\vec{E}$ as

$$\vec{E} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

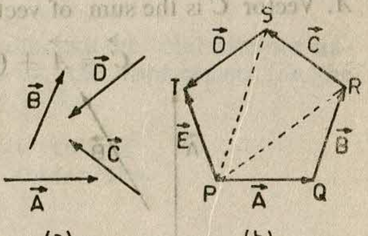
$$= \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}$$

$$= \overrightarrow{PR} + \overrightarrow{RS} + \overrightarrow{ST}$$

$$(\because \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR})$$

or $$\vec{E} = \overrightarrow{PS} + \overrightarrow{ST} = \overrightarrow{PT}$$

$$(\because \overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS})$$



(a) (b)

Fig. 4.9 The polygon law of vector addition

Thus, *if a number of vectors are represented in magnitude and direction by the sides of a polygon, taken in order, then the resultant is repre-*

sented in magnitude and direction by the closing side of the polygon taken in the opposite order. This is called the polygon law of vector addition.

Note: Equation $\vec{PQ} + \vec{QR} = \vec{PR}$ or $\vec{A} + \vec{B} = \vec{C}$ is a vector equation. It is important to understand that this equation does not say that the sum of the magnitude of $\vec{A}$ and $\vec{B}$ is equal to the magnitude of $\vec{C}$. This kind of simple addition holds only for scalars. In fact, a vector equation prescribes an operation. The equation $\vec{A} + \vec{B} = \vec{C}$ simply says [see Fig. 4.8(b)] that the operation of moving a body from P to Q and then from Q to R is equivalent to a simple operation of moving it from P to R. It may be mentioned that in writing $\vec{A} + \vec{B} = \vec{C}$ it is assumed that all vectors lie in the same plane which, in our case, is the plane of the paper.

Subtraction of Vectors Suppose we wish to subtract a vector $\vec{B}$ from a vector $\vec{A}$. Since

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

the subtraction of vector $\vec{B}$ from vector $\vec{A}$ is equivalent to the addition of vector $-\vec{B}$ to vector $\vec{A}$. Hence the procedure to find $(\vec{A} - \vec{B})$ is as follows:

Choose a convenient scale and draw the vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. 4.10(a). If $\vec{B}$ is to be subtracted from $\vec{A}$, draw the vector negative of $\vec{B}$, i.e. draw the vector $-\vec{B}$ [see Fig. 4.10(b)]. Now shift the vector $-\vec{B}$ parallel to itself so that the tail of $-\vec{B}$ is at the head of $\vec{A}$. Vector $\vec{C}$ is the sum of vectors $\vec{A}$ and $-\vec{B}$, i.e. [see Fig. 4.10(c)].

$$\vec{C} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

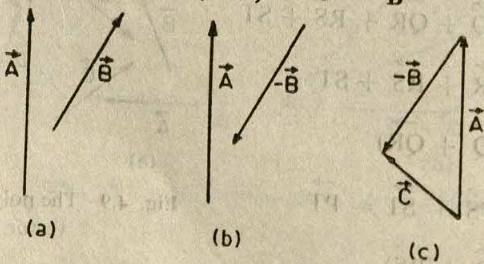


Fig. 4.10 Subtraction of vector $\vec{\mathbf{B}}$ from vector $\vec{A}$
$\vec{C} = \vec{A} - \vec{\mathbf{B}}$

4.5 ZERO VECTOR AND ITS PROPERTIES

Just as the real number zero is significant and necessary in real number operations, the zero vector is a very useful concept in vector operations.

If two vectors $\vec{A}$ and $\vec{B}$ are equal, their difference $(\vec{A} - \vec{B})$ is defined as the *zero* or *null vector* and is represented by the symbol $\vec{0}$. *It has zero magnitude and* no specific direction. A vector which is not null is called a *proper vector*. Thus if

$$\vec{A} = \vec{B}$$

then

$$\vec{A} - \vec{B} = \vec{0}$$

The main properties of the zero vector are:

$$\vec{A} + \vec{0} = \vec{A}$$

and

$$\lambda \vec{0} = 0 \vec{A} = \vec{0}$$

Physical Meaning of $\vec{0}$

Let us go back to the definition of a position vector as illustrated in Fig. 4.1. What is the position of the origin 0; it is the zero vector. Similarly, what is the displacement vector of a stationary object (see Fig. 4.2) from time t_1 to time t_2; it is the zero vector. Similarly the acceleration vector of uniform motion is the zero vector. Thus we see that it is useful and necessary to have the zero vector.

EXAMPLE 4.1 A passenger arriving in a new town wishes to go from the station to a hotel 10 km away on a straight road from the station. A dishonest taxi-driver takes him along a circuitous route 25 km long and reaches the hotel in 30 min. What is (a) the average speed of the taxi and (b) the magnitude of the average velocity?

Solution From Fig. 4.11 we find that, although the total distance travelled (along the circuitous route) is 25 km, the displacement (i.e. the shortest or straight line distance) is only 10 km.

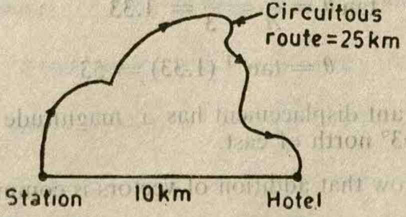


Fig. 4.11

Now, Total time taken = 30 min = 0.5 h

(a) Average speed of taxi $= \frac{\text{Total distance}}{\text{Time}} = \frac{25 \text{ km}}{0.5 \text{ h}} = 50 \text{ km h}^{-1}$

(b) Magnitude of average velocity $= \frac{\text{Displacement}}{\text{Time}} = \frac{10 \text{ km}}{0.5 \text{ h}}$

$= 20 \text{ kmh}^{-1}$

Notice that the average speed is not equal to the average velocity.

EXAMPLE 4.2 A car travels due east on a level road for 30 km. It then turns due north at an intersection and travels 40 km before stopping. Find the resultant displacement of the car.

Solution Referring to Fig. 4.12 vector $\vec{A}$ represents a displacement of 30 km due east and vector $\vec{B}$ represents a displacement of 40 km due north. The resultant displacement of the car is $\vec{C}$.

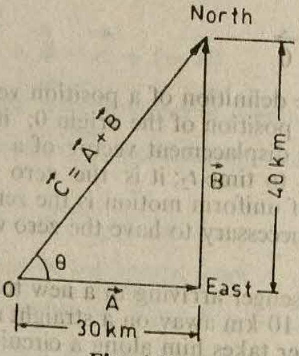


Fig. 4.12

Now, Magnitude of $\vec{C} = \sqrt{A^2 + B^2} = \sqrt{(30)^2 + (40)^2} = 50$ km

The direction of the resultant displacement is given by

$$\tan \theta = \frac{B}{A} = \frac{4}{3} = 1.33$$

$$\therefore \quad \theta = \tan^{-1} (1.33) = 53°$$

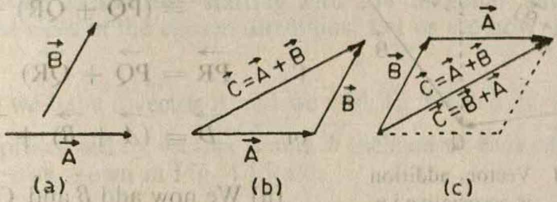
Thus the resultant displacement has a magnitude of 50 km and makes an angle 53° north of east.

EXAMPLE 4.3 Show that addition of vectors is commutative, i.e.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Solution Suppose we wish to add the two vectors $\vec{A}$ and $\vec{B}$ shown in Fig. 4.13 (a). Let us first add $\vec{B}$ to $\vec{A}$ by the method of vector addition discussed above. Shift $\vec{B}$ parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$ [Fig. 4.13 (b)]. The sum (or resultant) $\vec{C}$ is given by

$$\vec{C} = \vec{A} + \vec{B}$$



$\vec{B}$
$\vec{A}$
(a)
$\vec{C}=\vec{A}+\vec{B}$
$\vec{B}$
$\vec{A}$
(b)
$\vec{A}$
$\vec{B}$
$\vec{C}=\vec{A}+\vec{B}$
$\vec{C}=\vec{B}+\vec{A}$
(c)

Fig. 4.13 Vector addition is commutative $(\vec{A} + \vec{B} = \vec{B} + \vec{A})$

Now let us reverse the order of addition, i.e. let us now add $\vec{A}$ to $\vec{B}$. Shift $\vec{A}$ parallel to itself until the tail of $\vec{A}$ touches the head of $\vec{B}$ [Fig. 4.13(c)]. The resultant vector $\vec{C'}$ is given by

$$\vec{C'} = \vec{B} + \vec{A}$$

Comparing Figs 4.13(a) and 4.13(b) it is clear that the magnitudes as well as the directions of the resultant vectors $\vec{C}$ and $\vec{C'}$ are the same, i.e.

$$\vec{C} = \vec{C'}$$

Hence

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

This is the commutative law of vector addition which states that *the sum of vectors remains the same in whatever order they may be added.*

EXAMPLE 4.4 Show that vector addition is associative, i.e.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Solution Let us add three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represented respectively by $\vec{PQ}$, $\vec{QR}$ and $\vec{RS}$ as shown in Fig. 4.14. The resultant of these vectors can be found in two ways.

(i) We first add $\vec{A}$ and $\vec{B}$. The resultant (i.e. sum) is $(\vec{A}+\vec{B})$ represented by $\overrightarrow{PR}$. We then add this resultant to $\vec{C}$, the result is the final resultant vector $\vec{D}$. Thus

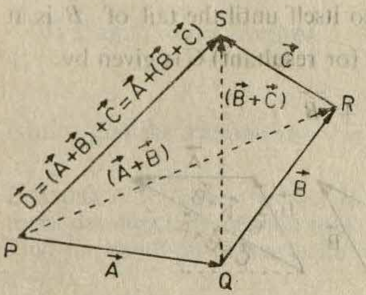


Fig. 4.14 Vector addition is associative i.e. $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$.

$$\vec{D}=\overrightarrow{PR}+\overrightarrow{RS}$$

$$=(\overrightarrow{PQ}+\overrightarrow{QR})+\overrightarrow{RS}$$

$$(\because \quad \overrightarrow{PR}=\overrightarrow{PQ}+\overrightarrow{QR})$$

or
$$\vec{D}=(\vec{A}+\vec{B})+\vec{C} \qquad \text{(i)}$$

(ii) We now add $\vec{B}$ and $\vec{C}$ to get the resultant $(\vec{B}+\vec{C})$ represented by $\overrightarrow{QS}$.

We then add $\vec{A}$ to this resultant, the result is again the same resultant vector $\vec{D}$. Thus

$$\vec{D}=\overrightarrow{PQ}+\overrightarrow{QS}$$

$$=\overrightarrow{PQ}+(\overrightarrow{QR}+\overrightarrow{RS}) \qquad (\because \quad \overrightarrow{QS}=\overrightarrow{QR}+\overrightarrow{RS})$$

$$=\vec{A}+(\vec{B}+\vec{C}) \qquad \text{(ii)}$$

From Eqs (i) and (ii) we have

$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$$

This is the associative law of vector addition.

Thus the order of addition of vectors is immaterial.

4.6 RESOLUTION AND COMPONENTS OF VECTORS IN A PLANE

So far our treatment of vectors has been largely *geometrical* (graphical). However, in many situations it is more useful to use the *algebraic* method of describing and using vectors. This is possible if we learn how to resolve (or split) a vector into its components. We shall first discuss the general case and then consider the very special and useful case of rectangular components.

Resolution of a Vector along Two Arbitrary Directions

From the parallelogram law (Fig. 4.8) we have seen that the resultant of two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle α is given by vector $\vec{C}$, the diagonal of the parallelogram. In other words, vector $\vec{C}$ is equivalent to vectors $\vec{A}$ and $\vec{B}$ together. Thus we can resolve any given vector into two other vectors in any two given directions by working the parallelogram rule backwards—starting with the diagonal and drawing the adjacent sides in the chosen directions. Let us see how this can be done.

Suppose we have a vector $\vec{A}$ and we wish to resolve it along two directions prescribed by vectors $\vec{a}$ and $\vec{b}$ inclined to each other at an angle, say, α as shown in Fig. 4.15(a).

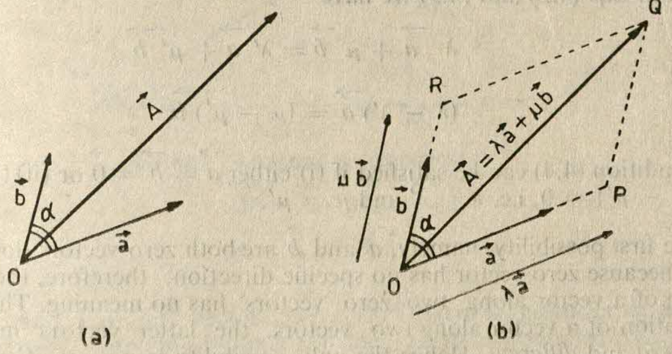


Fig. 4.15 Resolution of vector $\vec{A}$ along the directions of vectors $\vec{a}$ and $\vec{b}$

With vector $\vec{A}$ as diagonal we complete the parallelogram with adjacent sides in the direction of vectors $\vec{a}$ and $\vec{b}$. This construction is shown in Fig. 4.15(b). Vector $\vec{A} = \overrightarrow{OQ}$ is then the sum (or resultant) of vectors $\overrightarrow{OP}$ and $\overrightarrow{OR}$. Notice that the magnitudes of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ may not be equal to those of $\vec{a}$ and $\vec{b}$ respectively, but their directions must be the same. Since $\overrightarrow{OP}$ differs from $\vec{a}$ only in magnitude, it follows that $\overrightarrow{OP} = \lambda\,\vec{a}$ where λ is a real number by which $\vec{a}$ must be multiplied to get $\overrightarrow{OP}$. Similarly $\overrightarrow{OR} = \mu\,\vec{b}$ where μ is also a real number. The numbers λ and μ are different from zero. It follows from Fig. 4.15(b) that

$$\vec{A} = \lambda\,\vec{a} + \mu\,\vec{b} \tag{4.2}$$

and we say that *$\vec{A}$ has been resolved along $\vec{a}$ and $\vec{b}$*. Vectors. $\lambda\,\vec{a}$ and $\mu\,\vec{b}$ are called the *resolved parts* or *components* of $\vec{A}$ along $\vec{a}$ and $\vec{b}$.

Uniqueness of resolution If vector $\vec{A}$ is given then there is only one pair of numbers (λ, μ) which satisfy Eq. (4.2). In order to prove this uniqueness, let us say there are two equivalent ways of resolving $\vec{A}$ along $\vec{a}$ and $\vec{b}$ so that

$$\vec{A} = \lambda'\,\vec{a} + \mu'\,\vec{b} \tag{4.3}$$

From Eqs (4.2) and (4.3) we have

$$\lambda\,\vec{a} + \mu\,\vec{b} = \lambda'\,\vec{a} + \mu'\,\vec{b}$$

or

$$(\lambda - \lambda')\,\vec{a} = (\mu - \mu')\,\vec{b} \tag{4.4}$$

Condition (4.4) can be satisfied if (i) either $\vec{a} = \vec{b} = \vec{0}$ or (ii) $(\lambda - \lambda') = (\mu - \mu') = 0$, i.e. $\lambda = \lambda'$ and $\mu = \mu'$.

The first possibility, namely, $\vec{a}$ and $\vec{b}$ are both zero vectors does not exist because zero vector has no specific direction, therefore, the resolution of a vector along two zero vectors has no meaning. Thus for resolution of a vector along two vectors, the latter vectors must be *non-zero* and *different*. Hence the only possibility is

$$\lambda' = \lambda \qquad \mu' = \mu$$

This shows that there is one and only one way in which a given vector $\vec{A}$ can be resolved along vectors $\vec{a}$ and $\vec{b}$ inclined to each other at any arbitrary angle.

Resolution of a Vector into Two Rectangular Components

From the practical point of view, the most important case of resolution of a vector is one in which $\vec{a}$ and $\vec{b}$ are at right angles to each other and are unit vectors. The components of a vectors resolved along two unit vectors at right angles to each other are called the *rectangular components* of the vector.

Conventionally two perpendicular unit vectors in a plane are written as $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Since they are unit perpendicular vectors, they obey the properties:

Direction of $\hat{\mathbf{i}}$ = perpendicular to direction of $\hat{\mathbf{j}}$

and $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}|$

So we write $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in place of $\vec{a}$ and $\vec{b}$. The usual practice is to take $\hat{\mathbf{i}}$ along the x-axis and $\hat{\mathbf{j}}$ along the y-axis of a cartesian rectangular coordinate system. Figure 4.16 shows how a vector $\vec{A}$ is resolved along x and y axes. It is also usual to write A_x and A_y instead of λ and μ. So that we have

$$\vec{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$$

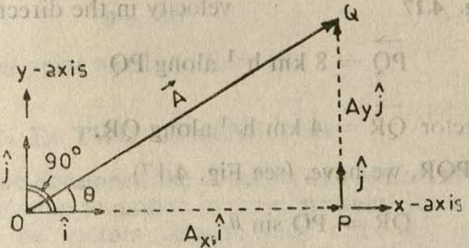


Fig. 4.16 Rectangular components:
$\vec{OP} = \vec{A_x}\,\hat{\mathbf{i}}$, $\vec{PQ} = \vec{A_y}\,\hat{\mathbf{j}}$,
$\vec{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$

Since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors, $\vec{OP}$ and $\vec{PQ}$ have magnitudes A_x and A_y respectively. Applying Pythagoras' theorem to right-angled triangle OPQ we can relate the magnitude of vector A to A_x and A_y as

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2} \tag{4.5}$$

We can also relate A_x and A_y to magnitude A of $\vec{A}$ and angle θ in triangle OPQ as

$$A_x = A\cos\theta \tag{4.6}$$

and $$A_y = A\sin\theta$$

Squaring these equations and adding we get (since $\sin^2\theta + \cos^2\theta = 1$)

$$A^2 = A_x^2 + A_y^2 \tag{4.7}$$

which is Eq. (4.5) obtained above. Also from Eqs (4.6) and (4.7) we find that

$$\tan\theta = \frac{A_y}{A_x} \tag{4.8}$$

or $$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

EXAMPLE 4.5 Assume that water is flowing at a constant velocity of 4 km h^{-1} as shown in Fig. 4.17. A person wants to cross the river by a boat at point P to a point R directly opposite on the other side of the river. In which direction should the person row the boat so as to ultimately arrive at R? Assume that he can row at a constant velocity of 8 km h^{-1} in still water.

Solution Let PQ be the direction along which he must row the boat. The velocity of the boat along PQ together with the velocity of water along QR, should yield a resultant, velocity in the direction PR. Thus

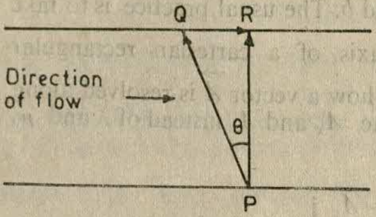


Fig. 4.17

Vector $\overrightarrow{PQ} = 8$ km h^{-1} along PQ

and vector $\overrightarrow{QR} = 4$ km h^{-1} along QR.

From triangle PQR, we have, (see Fig. 4.17)

$$QR = PQ \sin\theta$$

$$\therefore \quad \sin\theta = \frac{QR}{PQ} = \frac{4}{8} = \frac{1}{2}$$

$$\therefore \quad \theta = 30^\circ$$

Thus, the person must row the boat upstream in a direction PQ making an angle 30° with the direction PR.

EXAMPLE 4.6 In a harbour, the wind is blowing at a speed of 72 km h^{-1} and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km h^{-1} to the north, what is the direction of the flag on the mast of the boat?

Solution Refer to Fig. 4.18.

Wind velocity $v_w = 72$ km $h^{-1} = \overrightarrow{OP}$

Boat velocity $v_b = 51$ km $h^{-1} = \overrightarrow{OQ}$

The flag will point in the direction QP making an angle θ.

In triangle OPQ we have

$$\frac{OP}{\sin\theta} = \frac{OQ}{\sin\alpha}$$

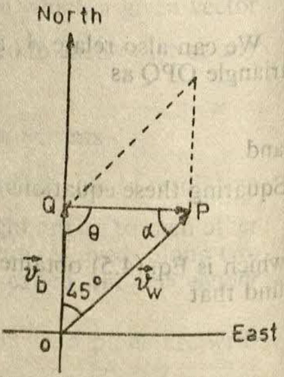


Fig. 4.18

Now $\alpha + \theta + 45° = 180°$, therefore $\alpha = 135° - \theta$

$$\therefore \qquad \frac{\sin(135° - \theta)}{\sin\theta} = \frac{OQ}{OP} = \frac{51}{72}$$

$$\text{or} \qquad \frac{1}{\sin\theta}(\sin 135° \cos\theta - \cos 135° \sin\theta) = \frac{51}{72}$$

$$\text{or} \qquad \frac{1}{\sin\theta}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) = \frac{51}{72}$$

$$\text{or} \qquad \cot\theta = \sqrt{2} \times \frac{51}{72} - 1 = 0.002$$

$$\therefore \qquad \theta \simeq 90°$$

i.e. the flag will point nearly to the east.

4.6 MOTION IN TWO DIMENSIONS

In Chapter 3 we developed the concepts of position, velocity and acceleration in relation to motion in one dimension. These quantities were not taken to be vectors because they acted only in one direction, namely, the direction of motion of the object. Quantities in the direction of motion were taken as positive while those against the direction of motion were taken as negative. In relation to two (and three) dimensional motion, position, velocity and acceleration are vectors. Now that we have learnt the necessary vector operations, we can describe motion in two (and later) in three dimensions. We shall denote the position, velocity and acceleration vectors by $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$ respectively. Their rectangular components in a plane will be represented as $\{x(t), y(t)\}$, for position vector, $\{v_x(t), v_y(t)\}$ for velocity vector and $\{a_x(t), a_y(t)\}$ for acceleration vector.

Let $\vec{r}(t_1)$ be the position vector at time t_1 of an object moving in a plane. Let $\vec{r}(t_2)$ be its position vector at a later time t_2. Then the displacement vector from t_1 to t_2 is

$$\text{Displacement vector from } t_1 \text{ to } t_2 = \vec{r}(t_2) - \vec{r}(t_1)$$

The rectangular components of the displacement vector are the differences of the rectangular components of $\vec{r}(t_2)$ and $\vec{r}(t_1)$. Thus

$$\begin{aligned}\vec{r}(t_2) - \vec{r}(t_1) &= \{x(t_2)\,\hat{\mathbf{i}} + y(t_2)\,\hat{\mathbf{j}}\} - \{x(t_1)\,\hat{\mathbf{i}} + y(t_1)\,\hat{\mathbf{j}}\}\\ &= \{x(t_2) - x(t_1)\}\,\hat{\mathbf{i}} + \{y(t_2) - y(t_1)\}\,\hat{\mathbf{j}}\end{aligned}$$

We can simplify this if we choose $t_1 = 0$ and $t_2 = t$. Then

$$\vec{r}(t) - \vec{r}(0) = \{x(t) - x(0)\}\,\hat{\mathbf{i}} + \{y(t) - y(0)\}\,\hat{\mathbf{j}} \qquad (4.9)$$

We will now discuss the following two special cases of motion in two dimensions, i.e. motion in a plane.

Case I—*Motion with Uniform Velocity* The simplest case is one in which the magnitude as well as the direction of the velocity of the object moving in a plane do not change with time. The object is moving in a straight line in the plane with a constant speed, i.e. the velocity is a constant vector. For such a motion, the velocity is simply the rate of change of displacement, i.e.

$$\vec{v} = \frac{\vec{r}\,(t) - \vec{r}\,(0)}{t}$$

or

$$\vec{r}\,(t) = \vec{r}\,(0) + \vec{v}\,t \tag{4.10}$$

Compare this formula with Eq. (3.3) of Chapter 3. We find that Eq. (4.10) is a generalisation in two dimensions of Eq. (3.3) of Chapter 3 for one dimension.

It must be clearly understood that the object is moving with a constant velocity in a plane. The velocity ($\vec{v}$) can be expressed in terms of its rectangular components v_x and v_y as:

$$\vec{v} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}}$$

The speed v of the object, is the magnitude of its velocity $\vec{v}$ and is given by [see Eq. (4.5)]

$$\text{Speed } (v) = |\vec{v}| = (v_x^2 + v_y^2)^{1/2} \tag{4.11}$$

We could also write Eq. (4.10) in terms of rectangular components (x, y) of $\vec{r}$ as follows:

$$x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = x(0)\,\hat{\mathbf{i}} + y(0)\,\hat{\mathbf{j}} + (v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}})\,t$$

which may be rewritten as

$$[x(t) - x(0) - v_x t]\,\hat{\mathbf{i}} + [y(t) - y(0) - v_y t]\,\hat{\mathbf{j}} = 0$$

Since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are non-zero unit vectors, this equation can be satisfied only if the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are separately zero, i.e. if

$$x(t) = x(0) + v_x t$$

and

$$y(t) = y(0) + v_y t \tag{4.12}$$

If we compare these equations with Eq. (3.3) we find that each rectangular position coordinate of the motion in two dimensions varies with time exactly as the position coordinate in one-dimensional motion with constant velocity.

Graphical Representation of Motion: To represent the motion on a plane graph paper, we plot the position coordinates $x(t)$ and $y(t)$ along x and y axes respectively for each instant of time between 0 and t. At each value of time, the position of the object is a point on the graph. As time passes, this point moves in the plane and we watch the motion of the point. This graph is shown in Fig. 4.19.

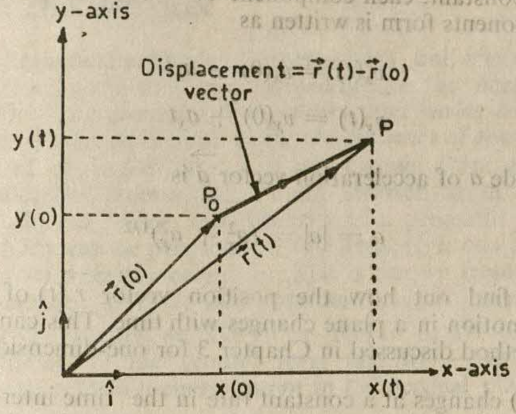


Fig. 4.19 Graphical representation of uniform motion in a plane

The points P_0 and P on the graph correspond to the positions of the object at times $t = 0$ and $t = t$. As time passes the object moves along the line P_0P. $\overrightarrow{P_0P}$ is the displacement in the plane.

Case II—Motion with Uniform Acceleration We will now study uniformly accelerated motion in a plane. The velocity vector in Eq. (4.10) is no longer constant; but changes with time and we denote it by $\vec{v}(t)$. Since the acceleration is assumed constant, the velocity vector changes at a constant rate from 0 to t. We can then define an average velocity vector during the time from 0 to t as

$$\vec{v}_{av} = \frac{\vec{r}(t) - \vec{r}(0)}{t} \tag{4.13}$$

The acceleration vector $\vec{a}$ (assumed constant) is defined as the rate of change of velocity with time, i.e.

$$\vec{a} = \frac{\vec{v}(t) - \vec{v}(0)}{t} \tag{4.14}$$

or
$$\vec{v}(t) = \vec{v}(0) + \vec{a}\,t \qquad (4.15)$$

In terms of the rectangular components a_x and a_y of vector $\vec{a}$, we have, as before

$$\vec{a} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$$

Since $\vec{a}$ is constant, each component of $\vec{a}$ is constant and Eq. (4.15) in the components form is written as

$$v_x(t) = v_x(0) + a_x t$$

and
$$v_y(t) = v_y(0) + a_y t \qquad (4.16)$$

The magnitude a of acceleration vector $\vec{a}$ is

$$a = |\vec{a}| = (a_x^2 + a_y^2)^{1/2} \qquad (4.17)$$

We will now find out how the position vector $\vec{r}(t)$ of a uniformly accelerated motion in a plane changes with time. This can be done by using the method discussed in Chapter 3 for one-dimensional motion.

Since $\vec{v}(t)$ changes at a constant rate in the time interval 0 to t, it is reasonable to except that the average velocity during the time interval is the average of the velocities at $t = 0$ and at t, so that

$$\vec{v}_{\text{av}} = \tfrac{1}{2}\,[\vec{v}(0) + \vec{v}(t)]$$

Using Eq. (4.15) in this equation, we get

$$\vec{v}_{\text{av}} = \vec{v}(0) + \tfrac{1}{2}\,\vec{a}\,t \qquad (4.18)$$

Now, Displacement in time interval (t)

= average velocity in time interval (t) × time interval (t)

i.e.
$$\vec{r}(t) - \vec{r}(0) = \vec{v}_{\text{av}} \times t$$

Using Eq. (4.18) we then get

$$\vec{r}(t) - \vec{r}(0) = \vec{v}(0)\,t + \tfrac{1}{2}\,\vec{a}\,t^2 \qquad (4.19)$$

The two rectangular components of the displacement are:

$$x(t) - x(0) = v_x(0)\,t + \tfrac{1}{2}\,a_x t^2$$

and
$$y(t) - y(0) = v_y(0)\,t + \tfrac{1}{2}\,a_y t^2 \qquad (4.20)$$

Notice that each of these equations is just like uniformly accelerated motion in one dimension; the first equation represents displacement along the x-direction and the second along the y-direction.

The path of the object moving in a plane with a constant acceleration can be found by plotting $y(t)$ and $x(t)$ at each value of time in the time internal 0 to t and watching how the point moves on the plane. It is clear from Eqs (4.20) that the path will not be a straight line because the displacements no longer vary linearly with time t (notice the presence of t^2 term in these equations). The path of the object, in general, is a *parabola.*

4.8 PROJECTILE MOTION

An important practical example of motion with uniform acceleration in two dimensions is the motion of projectiles on the earth's surface. *Projectile is the name given to a body which, after having been given an initial velocity, is allowed to move under the influence of gravity alone.* A few examples of projectiles are: a stone thrown from the top of a building, a bullet fired from a rifle, a bomb released from a plane, a javelin thrown by an athlete, etc. The path of a projectile is called its *trajectory.* A body can be projected in two ways: (i) It can be projected horizontally from a certain height, or (ii) it is thrown from the ground in a direction inclined to it. We shall consider these two cases separately.

In relation to projectiles, we take $y(t) - y(0)$ to be the vertical height of the projectile above the ground level and $x(t) - x(0)$ to be the horizontal distance from a chosen point in the vertical plane in which the projectile flies. In the case of projectiles, the acceleration $\vec{a}$ has only one component $a_y = -g$, which is directed downwards; here g is the acceleration due to gravity. Thus.

$$\vec{a} = (0, -g)$$

i.e. $$a_x = 0 \quad \text{and} \quad a_y = -g$$

Thus, for a projectile, Eqs (4.20) simplify to

$$x(t) - x(0) = v_x(0)t \tag{4.21}$$

and $$y(t) - y(0) = v_y(0)t - \tfrac{1}{2}gt^2 \tag{4.22}$$

Body Projected Horizontally from a Certain Height

Suppose a body is projected horizontally with a velocity of 0.1 ms^{-1} from the top of a tower as shown in Fig. 4.20. The body acquires the following two motions simultaneously:

1. Uniform velocity with which it is projected (i.e. 0.1 ms^{-1}) in the horizontal direction OX.
2. Uniform acceleration vertically downwards along OY.

The x, y coordinates at the point O of projection are taken, for convenience to be (0, 0), i.e, $x(0) = 0$ and $y(0) = 0$. Since the body is projected horizontally, $v_y(0) = 0$ so that

$$x(t) = v_x(0)\,t \quad \text{and} \quad y(t) = -\tfrac{1}{2}gt^2$$

These two motions are independent of each other, i.e. the horizontal velocity of the body does not depend on whether it is falling or not

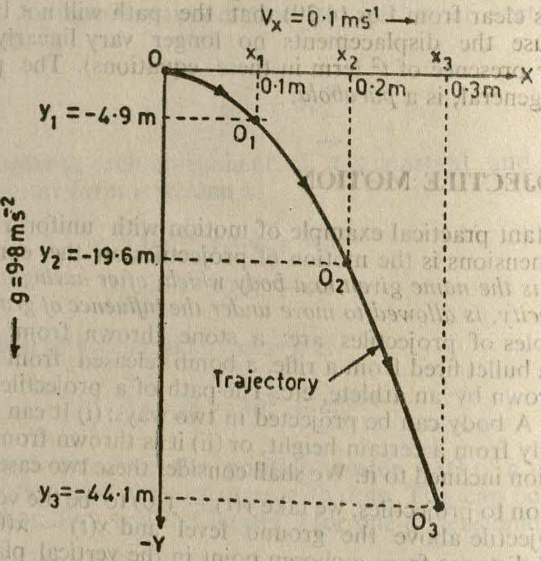


Fig. 4.20 Path of the projectile

and the downward acceleration does not depend on whether the body is moving or not. Clearly the distances Ox_1, Ox_2, Ox_3, etc. covered in the horizontal direction in 1, 2, 3 s, etc. are respectively 0.1 m, 0.2 m, 0.3 m, etc. ($\because v = 0.1\ \text{m s}^{-1}$). Similarly, the distances Oy_1, Oy_2, Oy_3, etc. covered by the body in 1 s, 2 s, 3 s, etc. are (since initial velocity in the downward direction is zero).

$$Oy_1 = -\tfrac{1}{2}gt^2 = -\tfrac{1}{2}(9.8)(1)^2 = -4.9\ \text{m}$$
$$Oy_2 = -\tfrac{1}{2}(9.8)(2)^2 = -19.6\ \text{m}$$
$$Oy_3 = -\tfrac{1}{2}(9.8)(3)^2 = -44.1\ \text{m}$$
$$\vdots$$

and so on

Since the body possesses these two motions simultaneously, it is simultaneously under the influence of two displacements at right angles to each other (one along OX and the other alone OY). It is clear from the diagram that the resultant displacements at the end of 1 s, 2 s, 3 s, etc. are given by vectors $\overrightarrow{OO_1}$, $\overrightarrow{OO_2}$, $\overrightarrow{OO_3}$, etc. with their tail at O and arrowheads at O_1, O_2, O_3, etc. These vectors are not shown in the diagram, they are the diagonals of the parallelograms $Ox_1O_1y_1$, $Ox_2O_2y_2$, $Ox_3O_3y_3$, etc. respectively. Hence the positions of the body after 1s, 2s, 3s, etc. will be O_1, O_2, O_3, etc. The trajectory of the body is $OO_1O_2O_3$.

Since the horizontal and vertical motions are independent of each other, the time taken to reach the ground by the body projected horizontally is the same as if it were just dropped freely from that height.

Nature of the trajectory Let us suppose that the body is projected at $t = 0$ horizontally with a velocity v_0 (so that $v_x(0) = v_0$) from a point O above the ground (Fig. 4.21).

Then the horizontal and vertical distances covered are

$$x(t) = v_0 t$$

and

$$y(t) = -\tfrac{1}{2} g t^2$$

Eliminating t from these equations, we get

$$y(t) = -\frac{g}{2v_0^2} x^2(t) = kx^2(t)$$

where $k = -\dfrac{g}{2v_0^2}$ is a constant. The equation $y(t) = kx^2(t)$ represents a parabola. Thus the trajectory of a body projected horizontally from any height above the ground is a parabola. This is shown in Fig 4.21.

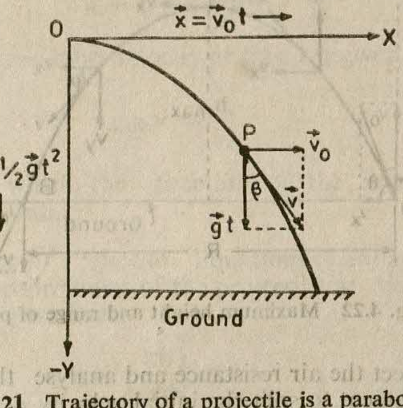


Fig. 4.21 Trajectory of a projectile is a parabola

Resultant velocity The resultant velocity of the body at any time, say, t seconds, after it is projected, can be calculated by determining the resultant of the horizontal and vertical velocity vectors at that time. Let us say that at time t the body has reached point P (Fig. 4.21). The horizontal velocity, being uniform throughout, remains v_0. Since the initial downward velocity is zero, the vertical velocity directed downwards at time t is gt, where g is the acceleration due to gravity. Therefore, the magnitude of the resultant velocity of the body at time t is given by

$$v = (v_0^2 + g^2t^2)^{1/2} \tag{4.23}$$

Let β be the angle which the resultant makes with the vertical direction. Then

$$\tan\beta = \frac{v_0}{gt}$$

Thus, the direction of the resultant velocity is inclined with the vertical direction at an angle given by

$$\beta = \tan^{-1}\left(\frac{v_0}{gt}\right) \tag{4.24}$$

Body Projected at an Angle with the Horizontal

Let us suppose that a body is projected with an initial velocity v_0 at an angle θ with the horizontal direction OX (see Fig. 4.22). The body describes a trajectory OAB and hits the ground at B. Let $h_{\max}$ be the maximum height attained.

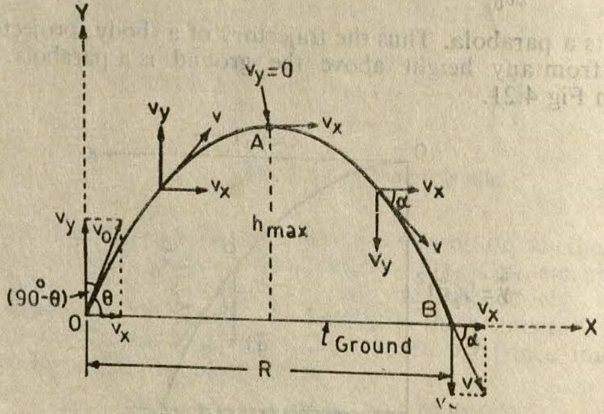


Fig. 4.22 Maximum height and range of projectile

We shall neglect the air resistance and analyse the motion of the body. It is convenient to split the initial velocity v_0 along two directions mutually at right angles to each other, one along the horizontal direction OX and the other along the vertical direction OY. The components along OX and OY are respectively given by (see Fig. 4.22).

$$v_x = v_0 \cos\theta$$

and

$$v_y = v_0 \sin\theta$$

Thus, the trajectory of the body can be regarded as the combination of the two component velocities v_x and v_y. The horizontal component v_x remains constant throughout, along the horizontal. The vertical component v_y, however, continuously changes due to the effect of gravity. This component is initially directed upward and decreases with time, becoming zero when the body is at the highest point A of its path. After this, the vertical component v_y is the directed downward and increases with time till the body strikes the ground at B.

Equations of motion We have seen that the magnitude of velocity along the horizontal direction OX is,

$$v_x = v_0 \cos \theta \tag{4.25}$$

which remains constant. The distance x moved in the horizontal direction in time t is, therefore, given by

$$x(t) = (v_0 \cos \theta)\, t \tag{4.26}$$

The velocity along the verticle direction OY at time t, is

$$v_y = v_0 \sin \theta - gt \tag{4.27}$$

The distance moved along the vertical direction in time t is given by

$$y(t) = (v_0 \sin \theta)\, t - \tfrac{1}{2} gt^2 \tag{4.28}$$

The magnitude of the resultant velocity at time t is given by

$$v = (v_x^2 + v_y^2)^{1/2} \tag{4.29}$$

The direction of the resultant velocity at time t is given by

$$\tan \alpha = \frac{v_y}{v_x} \tag{4.30}$$

where α is the angle which the direction of the resultant velocity makes with the horizontal.

Equation of Trajectory of Projectile Equations (4.26) and (4.28) give us $x(t)$ and $y(t)$, the coordinates of the projectile at time t from the start. From Eq. (4.27) we have,

$$t = \frac{x(t)}{v_0 \cos \theta}$$

Substituting in Eq. (4.28), we get

or $$y(t) = (\tan \theta)\, x(t) - \frac{g\, x^2(t)}{2v_0^2 \cos^2 \theta}$$

or $$y(t) = px(t) - qx^2(t) \tag{4.31}$$

where $p = \tan \theta$ and $q = g/(2v_0^2 \cos^2 \theta)$ are constants of the motion. Equation (4.31) is the equation of a parabola. Therefore *the trajectory of the projectile is parabolic.*

Maximum Height Attained The highest point A is reached when the magnitude of v_y becomes zero. From Eq. (4.27) this happens at a time t given by

$$0 = v_0 \sin \theta - gt$$

or

$$t = \frac{v_0}{g} \sin \theta$$

At time t, the magnitude of $y(t)$ is $= h_{\max}$, the maximum height attained. Substituting this value of t in Eq. (4.28) and setting the magnitude $y = h_{\max}$, we get

$$h_{\max} = (v_0 \sin \theta) \frac{(v_0 \sin \theta)}{g} - \frac{1}{2} g \frac{(v_0 \sin \theta)^2}{g^2}$$

$$= \frac{v_0^2}{g} \sin^2 \theta - \frac{1}{2} \frac{v_0^2}{g} \sin^2 \theta$$

or

$$h_{\max} = \frac{v_0^2 \sin^2 \theta}{2g} \tag{4.32}$$

Time of Flight It is the time taken by the projectile from the instant it is released till it strikes a point on the same horizontal plane as the point of projection. Thus, the time of flight t_f is the time taken by the body to describe its trajectory from O to B via A. The value of t_f can be obtained if we use the fact that when the body is at B, its y-coordinate is zero. Setting $y(t) = 0$ in Eq. (4.28) and putting $t = t_f$, we have

$$0 = (v_0 \sin \theta)\, t_f - \frac{1}{2} g\, t_f^2$$

or

$$t_f = \frac{2v_0 \sin \theta}{g} \tag{4.33}$$

Horizontal Range It is the maximum horizontal distance between the point of projection and the point where it strikes the horizontal plane containing the point of projection. Thus, horizontal range R is the distance OB as shown in Fig. 4.22.

It is clear that the x-coordinate at t_f, the time of flight, is the horizontal range R. Setting $x(t) = R$ and $t = t_f$ in Eq. (4.27), we have

$$R = (v_0 \cos \theta)\, t_f$$

Substituting for t_f from Eq. (4.33), we get

$$R = (v_0 \cos \theta) \frac{(2v_0 \sin \theta)}{g}$$

$$= \frac{v_0^2}{g} (2 \cos \theta \sin \theta)$$

or

$$R = \frac{v_0^2}{g} \sin 2\theta \tag{4.34}$$

It follows from Eq. (4.34) that, for a given speed of projection v_0, the range of a projectile is maximum if sin 2θ is maximum. Since the sine function has a maximum value of 1 for an angle of 90°, it is clear that R is maximum when

$$\sin 2\theta = 1$$

or
$$\sin 2\theta = \sin 90^\circ$$

or
$$2\theta = 90^\circ$$

or
$$\theta = 45^\circ$$

Hence, for a given speed of projection, the angle of projection for maximum range is 45°. The maximum range $R_{\max}$ is thus given by

$$R_{\max} = \frac{v_0^2}{g} \sin 90^\circ = \frac{v_0^2}{g} \tag{4.35}$$

Figure 4.23 shows the range of a projectile fired with the same speed (v_0) at different angles with the horizontal direction.

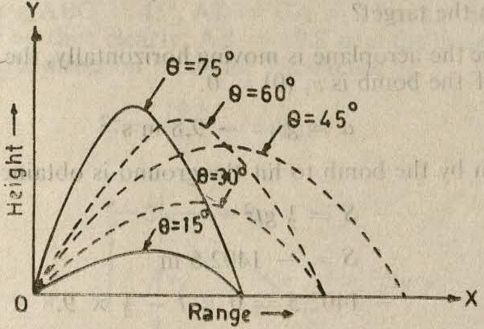


Fig. 4.23 Range of a projectile for different angles of projection; the speed of projection being the same

Further, when the angle of projection is $\left(\frac{\pi}{2} - \theta\right)$, it follows from Eq. (4.34) that the horizontal range is given by

$$R = \frac{v_0^2}{g} \sin \left\{2 \left(\frac{\pi}{2} - \theta\right)\right\} = \frac{v_0^2}{g} \sin (\pi - 2\theta)$$

or
$$R = \frac{v_0^2}{g} \sin 2\theta \qquad [\because \sin (\pi - 2\theta) = \sin 2\theta]$$

Thus, we conclude that for a given v_0, the horizontal range R is the same for angles of projection θ and $\left(\frac{\pi}{2} - \theta\right)$, i.e. $(90^\circ - \theta)$.

We see from Fig. 4.23 that the value of R is the same when

$$\theta = 15^\circ, 75^\circ \text{ and when } \theta = 30^\circ, 60^\circ$$

It is worth noting that we have obtained the equations for a projectile under the following approximations. We have assumed that

1. the earth is flat.
2. the acceleration due to gravity remains the same at all heights and does not change from place to place, and
3. the friction due to air is negligible.

It is interesting to see that the direction of the velocity $\vec{v}$ of a projectile is different at different points on its trajectory. The acceleration due to gravity is vertically downwards. Only when the body is at its maximum height is the acceleration perpendicular to the velocity.

EXAMPLE 4.7 A bomb is dropped from an aeroplane when it is directly above a target at a height of 1402.5 m. The aeroplane is moving horizontally with a speed of 500 km h^{-1}. By how much distance will the bomb miss the target?

Solution Since the aeroplane is moving horizontally, the initial downwards speed of the bomb is $v_y(0) = 0$.

$$a = g = -9.8 \text{ m s}^{-2}$$

The time taken by the bomb to hit the ground is obtained from

$$S = \tfrac{1}{2} gt^2$$

where $S = -1402.5$ m

Thus, $$-1402.5 = 0 \times t - \tfrac{1}{2} \times 9.8 \times t^2$$

or $$t = \sqrt{\frac{1402.5}{4.9}} = 15 \text{ s}$$

Now the horizontal velocity of the plane (and also of the bomb) is

$$500 \text{ km h}^{-1} = \frac{500 \times 1000}{60 \times 60} = \frac{1250}{9} \text{ ms}^{-1}$$

$\therefore$ Horizontal range = Horizontal velocity × Time

$$= \frac{1250}{9} \times 15$$

$$= 2083.3 \text{ m}$$

EXAMPLE 4.8 A ball is projected horizontally from the top of a building 19.6 m high. How long will the ball take to hit the ground? If the line joining the point of projection to the point where it hits the ground is 45° with the horizontal; what must be the initial velocity of the ball?

With what vertical velocity does the ball strike the ground? Given $g = 9.8\ \text{m s}^{-2}$.

Solution We know that the time taken to reach the ground is independent of the horizontal velocity of projection, v_0. Since the initial downward velocity is zero, we have

$$v_y(0) = 0$$

$$g = -9.8\ \text{ms}^{-2}$$

$$\text{Height} \quad h = -19.6\ \text{m}$$

The time of flight is given by

$$h = \tfrac{1}{2} gt^2$$

which gives $$t = \sqrt{\frac{2h}{g}} = \left(\frac{2 \times 19.6}{9.8}\right)^{1/2} = 2\ \text{s}$$

Referring to Fig. 4.24, let B be the point where the ball hits the ground. Since ∠ABO = 45°, AB = OA = 19.6 m. Let the horizontal velocity be v_0, then clearly; AB = 19.6 m is the distance travelled in the horizontal direction in a time 2 s. Hence

$$v_0 = \frac{19.6}{2} = 9.8\ \text{ms}^{-1}$$

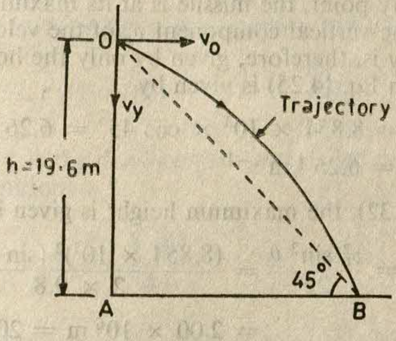


Fig. 4.24

Let v_y be the vertical component of the velocity of the ball when it hits the ground. Now

$$v_y^2 = 2gh$$

or

$$v_y = (2gh)^{1/2}$$

$$= (2 \times -9.8 \times -19.6)^{1/2}$$

$$= 19.6\ \text{ms}^{-1}$$

EXAMPLE 4.9 An intercontinental ballistic missile (ICBM) is fired for maximum range at your city from a place in the enemy country at a distance of 8000 km from your city. Suppose the missile is first detected at its halfway point:

(i) How much warning time will you have?
(ii) How fast will the missile be travelling when detected?
(iii) What will be its maximum height?
(iv) With what velocity will it strike the target?

Solution Since the missile is fired for maximum range, the angle θ (see Fig. 4.22) is 45°.

$$R_{\max} = 8000 \text{ km} = 8 \times 10^6 \text{ m}$$

The initial velocity of the missile (v_0) can now be obtained from Eq. (4.35) which gives

$$v_0 = (g \times R_{\max})^{1/2} = (9.8 \times 8 \times 10^6)^{1/2}$$
$$= 8.854 \times 10^3 \text{ ms}^{-1}$$

(i) Since the missile is first detected at its halfway point, the warning time t_w is half the total time of flight t_f. From Eq. (4.33), we have

$$t_w = \tfrac{1}{2}\, t_f = \frac{v_0 \sin\theta}{g} = \frac{8.854 \times 10^3 \times \sin 45°}{9.8}$$
$$= 639 \text{ s} \cong 10\tfrac{1}{2} \text{ minutes}$$

(ii) At its halfway point, the missile is at its maximum height (point A in Fig. 4.22). The vertical component v_y of the velocity at this point is zero; the velocity is, therefore, given by only the horizontal component v_x which from Eq. (4.25) is given by

$$v_x = v_0 \cos\theta = 8.854 \times 10^3 \times \cos 45° = 6.26 \times 10^3 \text{ ms}^{-1}$$
$$= 6.26 \text{ km s}^{-1}$$

(iii) From Eq. (4.32), the maximum height is given by

$$h_{\max} = \frac{v_0^2 \sin^2\theta}{2g} = \frac{(8.854 \times 10^3)^2 (\sin 45°)^2}{2 \times 9.8}$$
$$= 2.00 \times 10^6 \text{ m} = 2000 \text{ km}$$

(iv) The final velocity is the same as the velocity of projection v_0. Thus, the velocity with which the missile will strike the target is $8.854 \times 10^3 \text{ m s}^{-1} = 8.854 \text{ km s}^{-1}$. Notice that at its maximum height of 2000 km, the missile is somewhere in the stratosphere.

4.9 GENERAL POSITION-VELOCITY-ACCELERATION RELATIONS FOR MOTION IN TWO DIMENSIONS

So far we have discussed two kinds of motion in a plane, namely, one with uniform velocity (zero acceleration) and the other with constant acceleration. In the first the position vector $\vec{r}(t)$ changes with time but the velocity vector $\vec{v}$ remains constant, i.e. the magnitude and direction of $\vec{v}$ do not change with time. In Sec. 4.6 both $\vec{r}(t)$ and $\vec{v}(t)$ change with time but the acceleration vector $\vec{a}$ remains constant, i.e. either the magnitude or the direction or both the magnitude and direction of $\vec{v}(t)$ change at a constant rate with time.

The most general motion in a plane is the one in which the position vector $\vec{r}(t)$, velocity vector $\vec{v}(t)$ and the acceleration vector $\vec{a}(t)$ all change with time in some way.

Suppose the position vectors $\vec{r}(t)$ of an object moving in a plane are known for all times. Then, from this knowledge, can we find the velocity vector $\vec{v}(t)$ and the acceleration vector $\vec{a}(t)$ at any instant of time? The answer is yes. We use the calculus method developed in Chapter 3 in the general analysis of motion in one dimension.

Calculation of Velocity Vector

To calculate $\vec{v}(t)$, we consider two positions P and Q (of the object in a plane) very close to each other at times t and $(t + \Delta t)$ where Δt is very small and assume that the velocity is constant during Δt. Let $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$ be the corresponding position vectors so that

$$\Delta \vec{r}(t) = \vec{r}(t + \Delta t) - \vec{r}(t)$$

is small, where $\Delta \vec{r}$ is the displacement in time interval Δt. The velocity vector, by definition, is

$$\vec{v}(t) \simeq \frac{\Delta \vec{r}(t)}{\Delta t} \qquad (4.36)$$

The error in the calculation of $\vec{v}(t)$ from relation (4.36) will become smaller and smaller if the time interval Δt is made smaller and smaller, as explained in Chapter 3. Relation (4.36) will become exact in the limit Q → P or $\Delta t \rightarrow 0$ (see Fig. 4.25a)

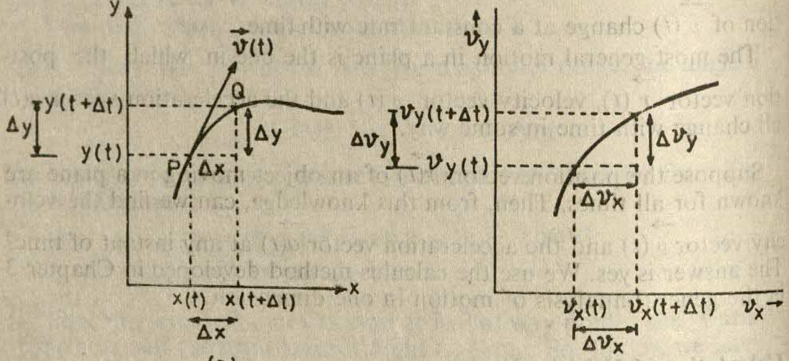
Thus, we have

$$\vec{v}(t) = \underset{\Delta t \rightarrow 0}{\text{Limt}} \frac{\Delta \vec{r}(t)}{\Delta^{t}} = \frac{d\vec{r}(t)}{dt} \qquad (4.37)$$

In terms of rectangular components, we have

$$v_x(t) = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta x\,(t)}{\Delta t} = \frac{dx\,(t)}{dt}$$

and $$v_y(t) = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta y\,(t)}{\Delta t} = \frac{dy(t)}{dt} \tag{4.38}$$



(a) (b)

Fig. 4.25 (a) Calculation of $\vec{v}(t)$ from $\vec{r}(t)$
(b) Calculation af $\vec{a}(t)$ from $\vec{v}(t)$

Notice that the velocity vector $\vec{v}\,(t)$ at any time t is along the tangent at time t (i.e. at point P) to the curve.

Calculation of Acceleration Vector

Having calculated $\vec{v}(t)$ for all values of t, we can now calculate $\vec{a}(t)$ in exactly the same way. We take two velocity vectors $\vec{v}(t)$ and $\vec{v}(t+\Delta t)$ at two close instants of time and assume that the acceleration is practically constant during the small time interval $\Delta\, t$. Let

$$\Delta\,\vec{v}(t) = \vec{v}\,(t + \Delta\, t) - \vec{v}(t)$$

Then, by definition, the acceleration is given by

$$\vec{a}(t) \simeq \frac{\Delta\,\vec{v}\,(t)}{\Delta\, t}$$

In the limit $\Delta\, t \to 0$, this becomes (see Fig. 4.25b)

$$\vec{a}\,(t) = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\,\vec{v}\,(t)}{\Delta\, t} = \frac{d\,\vec{v}\,(t)}{dt} \tag{4.39}$$

In terms of the rectangular components, we have

$$a_x(t) = \frac{dv_x(t)}{dt}$$

and $$a_y(t) = \frac{dv_y(t)}{dt} \tag{4.40}$$

It is worthwhile to recall that for the cases $\vec{a} = 0$ and $\vec{a} =$ constant dealt with in Sec. 4.6 we first found how $\vec{v}(t)$ changed with time and calculated the path of the moving object described by $\vec{r}(t)$ from the knowledge of $\vec{v}(t)$. We are now adopting the opposite point of view, i.e. knowing $\vec{r}(t)$ we calculate $\vec{v}(t)$ and $\vec{a}(t)$ as described above. We will use this point of view in the following case of circular motion where the path of the object in circular motion is known.

4.10 UNIFORM CIRCULAR MOTION

If an object moves in a circle at a constant speed, it is said to be in uniform circular motion, for example, the moon revolving round the earth, a satellite in circular orbit round a planet or a cyclist moving at constant speed on a circular track, etc.

Let us choose the centre of the circle as the origin in the plane. Let R be the radius of the circle (see Fig. 4.26). The *time period* (T) of circular motion is the time taken by the object to go once round the circle. The frequency (ν) of the motion is the number of revolutions completed in one second. Obviously $\nu = 1/T$ if is measured in seconds. The *angular velocity* (or *angular frequency*) of the motion is the angle swept out by the radius vector per second. If the radius vector sweeps out an angle $\Delta\,\theta$ (measured in radians) in a time interval $\Delta\,t$, then angular velocity (usually denoted by ω) is given by

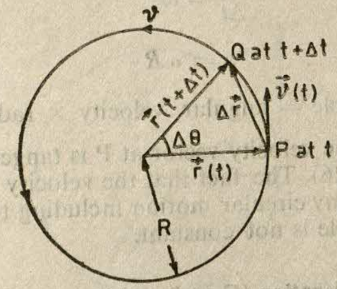


Fig. 4.26 Circular motion with constant speed.

$$\omega = \frac{\Delta\,\theta}{\Delta\,t}$$

if $\Delta t = T$, then $\Delta \theta = 2\pi$ radians. Hence

$$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{4.41}$$

Angular velocity is expressed in radian per second (rad s^{-1}).

Calculation of Velocity Vector

To find the velocity vector $\vec{v}(t)$ at time t when the object at P, we divide the displacement $\vec{PQ} = \Delta\vec{r}$ by a small time interval Δt during which the object moves from P to Q by the time interval Δt and take the limit $\Delta t \to 0$ (see Fig. 4.26).

Let us find the magnitude of $\Delta\vec{r}$. Arc PQ subtends an angle $\Delta\theta$ at the centre of the circle. Hence

$$\text{Arc PQ} = R\,\Delta\theta$$

where $\Delta\theta$ is measured in radians. As Q approaches P or as $\Delta t \to 0$, the arc PQ becomes closer and closer to the chord PQ and in the limit $\Delta t = 0$, they become equal. Hence the magnitude of velocity vector $\vec{v}(t)$ is given by

$$v = |\vec{v}(t)| = \frac{|\Delta\vec{r}|}{\Delta t} = \frac{R\,\Delta\theta}{\Delta t}$$

But

$$\frac{\Delta\theta}{\Delta t} = \omega$$

Hence

$$v = \omega R \tag{4.42}$$

i.e. speed along circle = angular velocity × radius.

The direction of the velocity vector at P is tangential to the circle at point P (see Fig. 4.26). The fact that the velocity vector is along the tangent is true for any circular motion including the one in which the speed along the circle is not constant.

Calculation of Acceleration (Centripetal Acceleration)

Consider an object moving in a circle with a constant speed v. Since the direction of motion changes continuously, so does the velocity of the object, although its magnitude (i.e. its speed) remains constant.

Now, according to Newton's first law of motion (see Chapter 5) a force must be acting on the object, for otherwise its velocity could not change. Thus, a force must act continuously on an object moving in a circle so as to bring about a change in its direction of motion.

What is the direction of this force? It is obvious that this force could not be acting along the path followed by the object (i.e. along the circumference of the circle) because, if it did, then according to Newton's second law of motion, it would produce an acceleration along the circle and the circular motion would not be uniform any more. Hence the direction of the force must be such that it has no effect on the direction of motion of the object. This can happen only if the force acts at right angles to the direction of motion, because if it acted in any other direction, it would have a component along the direction of motion which would change the speed of the object.

It is evident (see Fig. 4.27) that if this force is removed, the object (due to inertia) will tend to fly off in the direction of the tangent to the circle at that point with a uniform speed. Thus the direction of the velocity is tangential to the circle at every point. It follows that the force must act at right angles to the tangent to the circle at every point. Now, we know that the radius is perpendicular to the tangent. Hence the force must act along the radius of the circle at every point, i.e. the force is always directed towards the centre of the circular path as shown in Fig. 4.27. Such a force is called *centripetal force*. The word "centripetal" in Greek means "centre-seeking". Now according to Newton's second law of motion, a force produces an acceleration in a body of finite mass. It is obvious that the direction of the acceleration is towards the centre of the circle. This acceleration is called *centripetal acceleration* which is not due to the change in magnitude (speed) but due to the change in the direction of the velocity vector. We shall now calculate the magnitude of centripetal acceleration $\vec{a}$. The magnitude of centripetal force is given by $F = ma$, where m is the mass of the particle.

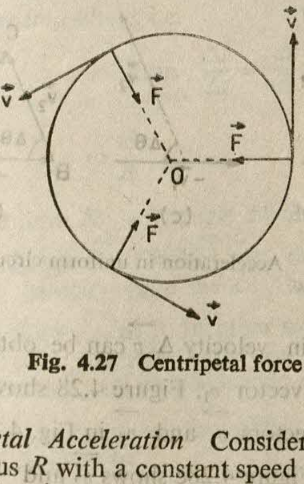


Fig. 4.27 Centripetal force

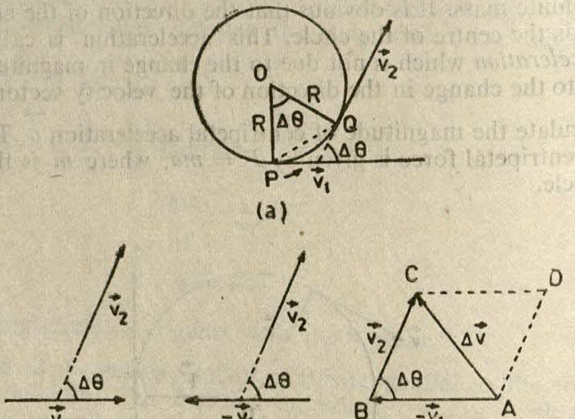
Magnitude of Centripetal Acceleration Consider a parlicle moving around a circle of radius R with a constant speed v. Suppose it moves from P to Q in small time interval Δt. then the length of arc PQ covered in this time is given by (see Fig. 4.28 a).

$$\widehat{PQ} = v\,\Delta t$$

The velocity of the object at P is represented by the vector $\overrightarrow{v_1}$ drawn along the tangent at P. Similarly, $\overrightarrow{v_2}$ represents the velocity at Q. It must be remembered that the magnitude of $\overrightarrow{v_1}$ is the same as that of $\overrightarrow{v_2}$, but their directions are different. Let $\Delta\theta$ be the angle $\angle$ POQ (in radians). Since the tangent is perpendicular to the radius, the angle between two tangents is equal to that between the two corresponding radii. Thus the angle between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is also $\Delta\theta$. Now the change in velocity $\Delta \overrightarrow{v}$ (due to change in direction) as the particle moves from P to Q in time Δt is clearly given by the difference between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. In other words,

$$\Delta \overrightarrow{v} = \overrightarrow{v_2} - \overrightarrow{v_1}$$

$$= \overrightarrow{v_2} + (-\overrightarrow{v_1})$$



(a)

(b) (c) (d)

Fig. 4.28 Acceleration in uniform circular motion

Thus, the change in velocity $\Delta \overrightarrow{v}$ can be obtained if we add vector $\overrightarrow{v_2}$ to the negative of vector $\overrightarrow{v_1}$. Figure 4.28 shows $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ in a separate diagram. The vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ in Fig. 4.28a and Fig. 4.28b are equal and parallel. Figure 4.28c shows $\overrightarrow{v_2}$ and $-\overrightarrow{v_1}$ which is the negative of $\overrightarrow{v_1}$. Now we use the method of vector addition to add $\overrightarrow{v_2}$ to $-\overrightarrow{v_1}$. Shift $\overrightarrow{v_2}$ parallel to itself until its tail is at the head of $-\overrightarrow{v_1}$. Thes is shown in Fig. 4.28d. Complete the parallelogram ABCD. The diagonal AC is the resultant vector $\Delta \vec{v}$.

Now triangles POQ and ABC are similar because they have the same vertex angle. Hence

$$\frac{\text{AC}}{\text{PQ}}=\frac{\text{BC}}{\text{OQ}}$$

or

$$\text{AC}=\frac{\text{BC}}{\text{OQ}}\,\text{PQ}$$

or

$$\frac{\Delta v}{\text{PQ}}=\frac{v}{R}$$

where Δv is the magnitude of $\Delta \vec{v}$ and v is the magnitude of $\vec{v_2}$ or $\vec{v_1}$.

If Δt is small, the chord PQ becomes very nearly equal to the arc PQ i.e.

$$\text{PQ}=\widehat{\text{PQ}}=v\Delta t$$

Therefore, we have

$$\frac{\Delta v}{v\Delta t}=\frac{v}{R}$$

or

$$\frac{\Delta v}{\Delta t}=\frac{v^2}{R}$$

When $\Delta t \to 0$, $\dfrac{\Delta v}{\Delta t}=\dfrac{dv}{dt}$ gives the magnitude of the acceleration $\vec{a}$ of of the object moving in a circle, i.e.

$$a=\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}=\frac{dv}{dt}$$

Thus

$$a=\frac{v^2}{R} \tag{4.43}$$

Notice that in the limit $\Delta t \to 0$, angle $\Delta\theta$ also approaches zero and as $\vec{\text{BA}}=\vec{\text{BC}}$, the change $\vec{\text{AC}}\ (=\Delta \vec{v})$ will be at right angles to $\vec{\text{AB}}$. Thus, the change in velocity (and hence the acceleration) will be at right angles to the velocity $\vec{v_1}$ at P. In other words, the acceleration is along the radius, directed towards the centre. This acceleration is called centripetal acceleration and its magnitude is given by Eq. (4.43).

Centripetal Force If the mass of the object is m, the magnitude of the centripetal force is given by

$$F=ma=\frac{mv^2}{R} \tag{4.44}$$

In terms of angular frequency ω, we have (since $v = R\omega$)

$$F = mR\,\omega^2 \tag{4.45}$$

EXAMPLE 4.10 A satellite is revolving around the earth in a circular orbit at an altitude of 6×10^5 m where the acceleration due to gravity is 9.4 $\mathrm{ms^{-2}}$. Assuming that the earth is a sphere of radius 6.4×10^6 m, calculate the time period of the satellite.

Solution Let m be the mass of the satellite. The centripetal force necessary to keep the satellite in orbit is provided by the gravitational force mg. Thus, we have,

$$\omega^2 mR = \frac{4\pi^2 mR}{T^2} = mg$$

or $$T = 2\pi\sqrt{\frac{R}{g}}$$

Here $$R = (6.4 \times 10^6 + 0.6 \times 10^5) = 7.0 \times 10^6 \text{ m}$$

$$g = 9.4 \text{ ms}^{-2}$$

$\therefore$ Time period of $T = 2 \times 3.14 \times \left(\dfrac{7.0 \times 10^6}{9.4}\right)^{1/2} = 5.42 \times 10^3$ s

$$= 1.5 \text{ h}$$

EXAMPLE 4.11 A string can withstand a tension of 25 N. What is the greatest speed at which a body of mass 1 kg can be whirled in a horizontal circle using a 1 m length of the string?

Solution Since the body is whirled in a horizontal circle, the gravity, acting vertically towards, has no effect on the motion.

Here $$F = 25 \text{ N}$$

$$m = 1 \text{ kg}$$

$$R = 1 \text{ m}$$

If v is the greatest speed with which the body can be whirled, the maximum centripetal force (or tension) in the string will be $\frac{mv^2}{R}$ which must balance the minimum tension F the string can withstand.

Thus $$F = \frac{mv^2}{R}$$

or $$25 = \frac{1 \times v^2}{1}$$

or $$v = 5 \text{ ms}^{-1}$$

EXAMPLE 4.12 A body of mass 0.5 kg is whirled in a vertical circle at an angular frequency of 10 rad s^{-1}. If the radius of the circle is 0.5 m, find the tension in the string when the body is (i) at the top of the circle, and (ii) at the bottom of the circle. Take $g = 9.8$ ms^{-2}.

Solution

Here $m = 0.5$ kg

$R = 0.5$ m

$\omega = 10$ rad s^{-1}

$g = 9.8$ ms^{-2}

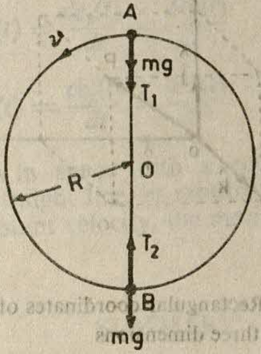


Fig. 4.29

(i) Referring to Fig. 4.29, it is clear that when the body is at A, the top of the circle, its weight mg and the tension T_1 in the string act downwards towards the centre O of the circle and the sum of the two provides the necessary centripetal force. Thus

$$T_1 + mg = mR\,\omega^2$$

$$\therefore \quad T_1 = m(R\omega^2 - g)$$

$$= 0.5\,[0.5 \times (10)^2 - 9.8]$$

$$= 20.1 \text{ N}$$

(ii) When the body is at the bottom of the circle (point B), the tension T_2 is opposite to weight mg and the difference $(T_2 - mg)$ provides the necessary centripetal force. Hence

$$T_2 - mg = mR\omega^2$$

$$\therefore \quad T_2 = m\,(R\omega^2 + g)$$

$$= 0.5\,[0.5 \times (10)^2 + 9.8]$$

$$= 29.9 \text{ N}$$

4.11 MOTION OF OBJECTS IN THREE-DIMENSIONAL SPACE

Objects do not always move in one or two dimensions. An object moving in space, such as a kite flying in the sky, has a three-dimensional motion corresponding to the three coordinate axes x, y and z of a rectangular coordinate system. The position, velocity and acceleration vectors have three rectangular components—two in the x-y plane (as in the case of two-dimensional motion) and one perpendicular to this plane, as shown in Fig. 4.30.

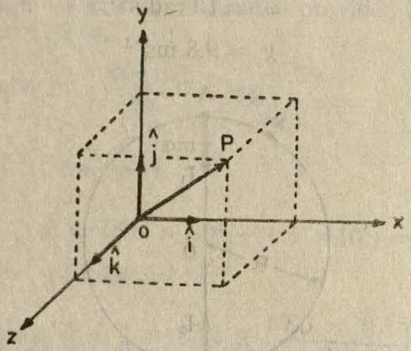


Fig. 4.30 **Rectangular coordinates of a point in three dimensions**

To describe any physical quantity of the motion in three dimensions, e.g. the position vector $\vec{r}\,(t)$ of a moving object, we choose an origin at some point O in space and three unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ at O which are mutually perpendicular. Using the ideas of the two dimensional case, we can write $\vec{r}\,(t)$ as

$$\vec{r}\,(t) = x\,(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

where $x(t)$, $y(t)$ and $z(t)$ are the rectangular coordinates of the position vector $\vec{r}\,(t)$. Notice that the z-direction (the third dimension) is perpendicular to the $x-y$ plane (the plane of the paper).

The corresponding expressions for the velocity vector $\vec{v}\,(t)$ and the acceleration vector $\vec{a}\,(t)$ are:

$$\vec{v}\,(t) = v_x(t)\;\hat{i} + v_y(t)\,\hat{j} + v_z(t)\,\hat{k}$$

and

$$\vec{a}\,(t) = a_x(t)\;\hat{i} + a_y(t)\,\hat{j} + a_z(t)\;\hat{k}$$

Their three components are:

$$v_x(t) = \frac{dx\,(t)}{dt}$$

$$v_y(t) = \frac{dy\,(t)}{dt}$$

$$v_z(t) = \frac{dz\,(t)}{dt}$$

and

$$a_x(t) = \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2}$$

$$a_y(t) = \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

$$a_z(t) = \frac{dv_z(t)}{dt} = \frac{d^2z(t)}{dt^2}$$

If the object moves in space with a constant velocity, i.e. with a constant speed in a straight line, its motion is unaccelerated. If it does not move with a constant velocity, the motion certainly has some kind of acceleration.

SUMMARY

Quantities which have a magnitude as well as a direction are called vectors. Scalars have only a magnitude but no direction. Scalars add algebraically but vectors add geometrically by using the triangle or parallelogram law of vector addition. Vectors in a plane can be resolved along any two directions. Vectors in space are resolved in three directions. In Cartesian coordinates, these directions are mutually perpendicular to each other.

Motion in two and three dimensions is described in terms of the position vector $\vec{r}\,(t)$, the velocity vector $\vec{v}\,(t)$ and the acceleration vector $\vec{a}\,(t)$ of the moving object. They are related as

$$\vec{v}(t) = \frac{d\,\vec{r}\,(t)}{dt}$$

$$\vec{a}(t) = \frac{d\,\vec{v}\,(t)}{dt} = \frac{d^2\,\vec{r}\,(t)}{dt^2}$$

If $\vec{v}\,(t)$ is given, we can calculate $\vec{r}\,(t)$ and $\vec{a}\,(t)$. If $\vec{r}\,(t)$ is given, we can calculate $\vec{v}\,(t)$ and $\vec{a}\,(t)$. The first method has been used to analyse projectile motion and the second to discuss circular motion.

EXERCISES

1. What are scalar and vector quantities? Pick out scalars and vectors from the following list: mass, volume, surface area, speed, velocity, distance, displacement, acceleration, density, angular velocity, number of moles, temperature, pressure, impulse, time, power, energy, coefficient of friction, charge, potential, gravitational field, force, momentum, work, current, magnetic moment, and relative velocity.
2. What are equal vectors? What is the negative of a vector?
3. A particle is subjected to two vector displacements $\vec{A}$ and $\vec{B}$. How will you find the magnitude and the direction of the resultant displacement if (a) $\vec{A}$ and $\vec{B}$ are perpendicular to each other, and (b) $\vec{A}$ and $\vec{B}$ are inclined at an angle?
4. Show that vector addition is (a) commutative, and (b) associative.
5. What do you mean by the resolution of a vector? How will you resolve a vector in a plane into two components (a) along two arbitrary directions, and (b) two mutually perpendicular directions?
6. Distinguish clearly between average velocity and instantaneous velocity.
7. What is a zero vector? State its properties. Why is it needed?
8. Obtain the equations that describe the motion of an object in two dimensions. Discuss the cases of (a) uniform velocity, and (b) uniform acceleration.
9. What is a projectile? Show that when a body is projected horizontally from a certain height above the ground, its path is a parabola. Deduce an expression for the magnitude and direction of its velocity at any instant of its motion.
10. A projectile is fired at an angle θ with the horizontal.
    (a) show that its trajectory is a parabola.
    (b) Obtain expressions for (i) the maximum height attained, (ii) the time of its flight, and (iii) the horizontal range.
    (c) At what value of θ is the horizontal range maximum?
    (d) Prove that, for a given velocity of projection, the horizontal range is the same for θ and $(90° - \theta)$.
11. What is angular velocity? What are its units? Establish a relation between angular and linear velocity.
12. What is centripetal acceleration? How will you account for it? Deduce an expression for it in the case of a body moving around a circle with a uniform speed.
13. How do we describe the motion of objects in three-dimensional space?

PROBLEMS

1. An aeroplane takes off at an angle of 30° to the horizontal. If the component of its velocity along the horizontal is 200 kmh^{-1}, what is its actual velocity? What is the vertical component of velocity?
2. A body is given a displacement of 12 m in the east and 5 m in the north and then 9 m vertically upwards. Find the magnitude of the resultant displacement.

3. Rain is falling vertically downwards at a speed of 30 ms^{-1}. A body is running at a speed of 10 ms^{-1} in the north-to-south direction. In which direction must he hold his umbrella?
4. From the top of a building 19.6 m high, a ball is projected horizontally. After how long will it strike the ground? If the line joining the point of projection to the point where it strikes the ground is 60° with the horizontal, what is the velocity with which the ball is projected?
5. A shell is fired horizontally with a velocity of 200 ms^{-1} from the top of a tower. In hits a target on the ground after 2 s. (i) What is the height of the tower? (ii) Where is the target situated? (iii) With what vertical velocity does the shell strike the target?
6. A body of mass 0.5 kg is whirled in a vertical circle of radius 0.5 m. If the tension in the string when the body is at the top of the circle is 20.1 N, what is the time period of revolution?
7. A string of length 0.5 m is used to whirl a 0.5 kg stone in a horizontal circle with a speed of 5 ms^{-1}. Calculate the tension in the string.
8. A ball leaves a bat at an angle of 37° to the horizontal at a speed of 50 ms^{-1}. How high does it go? How much forward is it at its highest point? If the ball is caught at a horizontal distance of 175 m from the point where it was struck, find how far above the level at which the ball was struck, was it caught?
9. A bullet is fired at an angle of 45° with the horizontal and with a velocity of 900 ms^{-1}. How high will it rise and what will be its horizontal range? What can you say about the horizontal range of the bullet if it is fired at an angle of (i) 33° and (ii) 57°?
10. An aeroplane flying horizontally with a speed of 490 ms^{-1}, releases a bomb when at a height of 1960 m. When will the bomb hit the ground and what will be its velocity then? If the bomb was aimed at a target at a horizontal distance of 10 km from the point vertically below the point of its release, by how much distance will the bomb miss the target?
11. A projectile is fired with a muzzle velocity of 400 ms^{-1} at an angle of elevation of 15° above the horizontal. At what height will it strike a vertical cliff distant 5000 m horizontally from the cliff? What will be its velocity when it strikes the cliff?
12. A water hose on the ground shoots a stream of water at an angle of 40° to the horizontal with a speed of 20 ms^{-1}. Find the height at which the water stream will strike a wall 8 m away.
13. A ball has to be thrown horizontally from a window 152.4 m above the ground in one tall building so as to enter a window 15.24 m above the ground in a second tall building that is at a horizontal distance of 90.96 m from the first building. What is the speed with which the ball must be thrown?
14. An observer on the ground finds that an object is falling vertically with a constant acceleration g. How is the motion of the object described by an observer in the train moving along a straight track with a constant velocity v? Assume that the 'origins' of the two observers are coincident at $t = 0$.

15. A string of length 1 m is fixed at one end and carries a mass of 100 g at the other end. The string makes $2/\pi$ revolutions per second about a vertical axis passing through its second end. Calculate (a) the angle of inclination of the string with the vertical, (b) the tension in the string and (c) the linear velocity of the mass.
16. A small body slides down a frictionless track after being released from rest from a point at a height h. The track ends in a circular loop of diameter D in the vertical plane. The body remains in contact with the loop as it moves along it. Find the relation between h and D.

    (*Hint*: Use conservation of energy).

5

Law of Motion

5.1 INTRODUCTION

In the last two chapters we studied the motion of a body in one and two dimensions. We described it in terms of the position, velocity and acceleration of the body. Our discussion was largely geometrical. Such a description of motion in which we are concerned with the measurements of position, velocity and acceleration of bodies and their variation with time is called *kinematics*. In the last two chapters, we have thus studied the kinematics of motion.

In this chapter we will be concerned with the *causes* that influence the motion of bodies. We will introduce two new concepts—*mass* and *force*. These words have a special meaning when used in a scientific sense. For the layman, however, mass is the quantity of matter in a body and force is some kind of a pull or push.

The aspect of motion concerning the causes of motion is called *dynamics*. In this chapter we will study the dynamics of motion.

The subject of motion was placed on a sound scientific footing by two great men, Galileo Galilei (1564-1642) and Sir Isaac Newton (1642-1727). Galileo laid the foundation of the modern scientific method. He was of the opinion that every theory of nature should be put to an experimental test. He studied the simplest examples of the natural phenomenon of motion and set up simple experiments to examine them. Only after Galileo had clarified the idea of motion was the world of physics ready for the great work of Isaac Newton. In 1687 he published a work called the *Principia* in which, for the first time, he gave a scientific meaning to the concepts of mass and force. He used these concepts to formulate the three fundamental laws governing all motions. These laws are called *Newton's laws of motion*. The greatness and genius of Newton can be assessed from the fact that his laws of motion are applicable not only to the motion of the bodies on or near the surface of the earth but also to that of heavenly bodies.

5.2 GALILEO'S EXPERIMENTS ON THE MOTION OF OBJECTS

Galileo studied the simplest examples of the motion of objects and set up simple experiments to examine their motion. On the basis of his experiments (described below he came to the following conclusion:

"*A body continues to move with the same velocity* (*i.e. with the same speed and in the same direction*) *if no unbalanced force acts on it*".

Nobody at that time, believed this conclusion to be valid because it is contrary to common experience. A ball rolled on the ground moves some distance and then comes to rest. But now we know that the above conclusion is absolutely correct. How did Galileo arrive at this revolutionary conclusion? Let us look at some of his observations in his experiments described as follows.

Galileo's Experiments with Inclined Plane

Galileo first studied the motion of objects on an inclined plane. He observed that when an object moves down an inclined plane, its speed increases (Fig. 5.1a). But when the objects is moved up the inclined plane, its speed decreases (Fig. 5.1b).

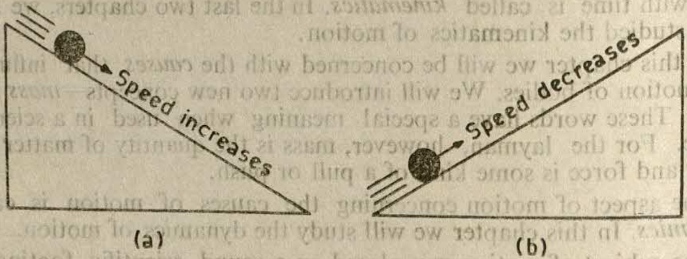


Fig. 5.1 Galileo's inclined plane experiment

He also observed that when the inclined plane was made less steep, the increase in the speed of the object was less than when the inclined plane was steeper (Fig. 5.2a).

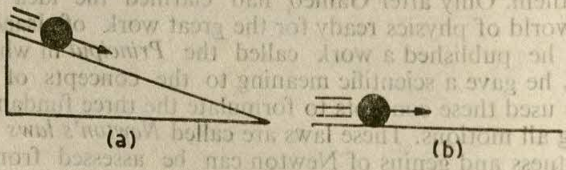


Fig. 5.2

He then asked himself the following question: What would happen if the plane were made less and less steep until it became a flat surface (Fig. 5.2b). The obvious answer was that the increase in the speed would become less and less until it (i.e. increase in speed) became zero on a flat surface. In other words, *the speed of a body, moving on a flat surface, would remain unchanged.*

Galileo's Thought (Imaginary) Experiment

Galileo next thought of an imaginary experiment not with real balls or planes but with imaginary ones. He had earlier observed the motion of a pendulum (a lamp hanging from the celling of a church). He observed that the pendulum rose to the same height on either side of the rest position (see Fig. 5.3).

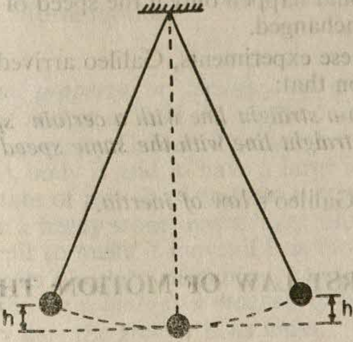


Fig. 5.3

Considering this observation, he argued that if a ball were allowed to move along the side of a hemispherical vessel, it would reach the other side of the vessel and rise to the same height on that side also (Fig. 5.4).

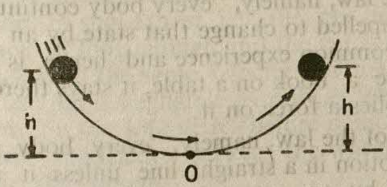
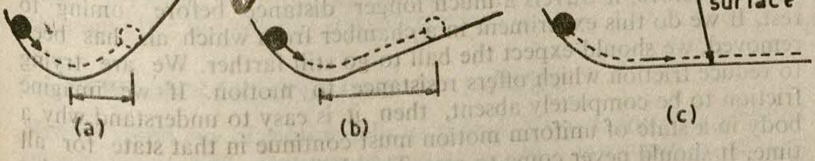


Fig. 5.4

He then asked himself what would happen if the slope of the vessel to the right of O were reduced as shown in Fig. 5.5.



(a) (b) (c)

Fig. 5.5

The obvious answer was that the ball would rise to the same height, but it would have to travel a longer distance to reach that height. This thought experiment shows that *the more we decrease the slope at the other side of the vessel, the longer would be the distance the ball has to travel to reach the same height* (see Figs 5.5a and b).

He finally wondered what distance the ball would have to travel to attain the same height if the slope on the other side of the vessel were decreased to zero, i.e. if that side of the vessel were flattened? This is shown in Fig. 5.5c. The obvious answer would be that the ball would have to travel an exceedingly large (infinite) distance to attain the same height. This could happen only if the speed of the ball on the flat surface remained unchanged.

On the basis of these experiments, Galileo arrived at the then unbelievable conclusion that:

"*A body moving in a straight line with a certain speed will continue moving in the same straight line with the same speed in the absence of an external force*".

This is known as Galileo's *law of inertia.*

5.3 NEWTON'S FIRST LAW OF MOTION: THE LAW OF INERTIA

Newton's first law of motion states that "*every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by an external unbalanced force*".

Interpretation of First Law

The first part of the law, namely, "every body continues in its state of rest unless it is compelled to change that state by an external unbalanced force", is a common experience and hence is easy to understand. If we place a book on a table, it stays there unless someone removes it, i.e. applies a force on it.

The second part of the law, namely, "every body continues in its state of uniform motion in a straight line unless it is compelled to change that state by an external unbalanced force" is difficult to understand because it is contrary to our common experience. A stone solled along a road comes to rest after going only a short distance. We then replace the stone with a smoother body, e.g. a ball bearing and the road by a smoother surface, e.g, a long sheet of glass. Now if we roll the ball bearing on the glass sheet with the same push or force as before, it travels a much longer distance before coming to rest. If we do this experiment in a chamber from which air has been removed, we should expect the ball to go still farther. We are trying to reduce friction which offers resistance to motion. If we imagine friction to be completely absent, then it is easy to understand why a body in a state of uniform motion must continue in that state for all time. It should never come to rest. The closest approximation to this ideal situation is motion of a star in free space. Observations taken over many decades show that the motion of the stars is uniform and will continue to be so. In other words, the speed of a body can be changed only if a force acts on it.

The phrase "in a straight line", which occurs in the statement of the first law, must be carefully noted. The direction as well as the speed of a moving body will remain the same (i.e. the velocity remains the same) unless a force acts on it.

Hence, from the first law, we may qualitatively define force as something which, while acting on a body, changes or tends to change its state of rest or of uniform motion.

Mass and Inertia

Inertia is the inherent property or tendency of a body to resist any change in its state of rest or of uniform motion in a straight line. The name inertia was given by Galileo. The first law is sometimes called the law of inertia. A body is said to have a large inertia if it is difficult to change its state of rest or of uniform motion. An object, such as a heavy truck or a heavy stone, has a large amount of inertia. If it is at rest, it is difficult to make it move; if it is moving, it is difficult to stop it or change its direction of motion, i.e. it is difficult to change its velocity. Thus, the term *mass is a measure of inertia of a body.* The more inertia a body has, the greater is its mass.

The following examples illustrate the inertia of rest or of motion.

1. We place a coin on a card which is placed on a tumbler and flip the card quickly with a finger. The coin neatly drops into the tumbler. This shows the inertia of the state of rest of the coin.

2. We place a toy car on smooth steel tracks. If we jerk the track quickly to the left or right, the wheels of the car turn but the car itself has a tendency to stay at rest, i.e. the car has inertia of rest. Now, if the track is long enough and the car is given a jerk, it will roll along the track with the same velocity for a considerable length of time, if the friction is negligible. A railway wagon detached from the engine can travel a few kilometres before coming to rest. It would never come to rest if the friction between the wheels and the track and that offered by air were absent.

3. We have the first-hand experience of the inertia of our body when we travel in a bus. When a moving bus suddenly stops we are thrown forward. The reason is that our body is in motion with the bus and tends to stay in motion even after the bus has stopped. On the other hand, when the bus suddenly starts, we are thrown backwards, because, in this case, our body is at rest and tends to stay at rest even after the bus has started moving.

5.4 NEWTON'S SECOND LAW OF MOTION

Newton's first law of motion tells us that, if no net force is exerted on a body, it will continue to stay at rest (if it is already at rest) or continue to move with a constant velocity (if it is already moving).

Newton's second law of motion tells us what happens when a force is actually exerted on a body. Newton introduced a new concept called *momentum*. Before we give the statement of the second law, it is necessary to understand the meaning of the term momentum.

Momentum

Suppose we have two cars, a toy and a real car. Imagine that they are moving with the same velocity. We try to stop them by pulling at a rope tied to the back of each car. The effort (or force) required to stop the real car is obviously much more than that required to stop the toy car. The reason is that the real car has a greater mass (or inertia) than the toy car. Thus, we conclude that the force required to stop a moving object depends on its mass. It is a matter of common experience that the force required to stop a moving body depends also on its velocity. Suppose the real car is moving very slowly, a small force will be needed to stop it. But if it is moving with a high velocity, a much greater force is needed to stop it.

Thus, we have seen that the force required to stop a moving body depends on two factors: (i) its mass and (ii) its velocity. Newton defined *momentum as the product of the mass and velocity of a body.*

In symbols, the momentum $\vec{p}$ of a body of mass m moving with a uniform velocity $\vec{v}$ is given by

$$\vec{p} = m\,\vec{v} \tag{5.1}$$

Since the body is moving a straight line, $\vec{p}$ is called the linear momentum to distinguish it from angular momentum (see Chapter 7).

Momentum $\vec{p}$ is a vector quantity since it is the product of a scalar m and a vector $\vec{v}$. The direction of $\vec{p}$ is the same as that of $\vec{v}$.

The unit of momentum in the SI system is kg ms^{-1}. A body of mass 1 kg moving with a constant velocity of 1 ms^{-1} has a linear momentum of 1 kg m s^{-1}.

Statement of the Second Law

Newton's remarkable genius and foresight is evident from the fact that he was the first to recognise that a force will change the momentum of a body and that the change will take place in accordance with a law which is known as Newton's second law of motion. The law states that, *the rate of change of momentum of a body is directly proportional to the applied force and the change takes place in the direction in which the force acts.*

Interpretation of the Second Law

Suppose a force $\vec{F}$ acts on a body of mass m. The velocity and hence the momentum of the body changes. We know that the momentum is given by

$$\vec{p} = m\,\vec{v}$$

where $\vec{v}$ is the velocity at a certain instant of time. Differentiating this equation with respect to time, we get

$$\frac{d\vec{p}}{dt} = m\,\frac{d\vec{v}}{dt} \qquad (\because \; m \text{ is constant})$$

$$= m\vec{a}$$

where $\vec{a} = \frac{d\vec{v}}{dt}$ is the acceleration produced. But according to Newton's second law the rate of change of momentum $\frac{d\vec{p}}{dt}$ is proportional to the impressed force $\vec{F}$, i.e.

$$\vec{F} \propto m\,\vec{a}$$

or

$$\vec{a} \propto \frac{\vec{F}}{m}$$

We, therefore, conclude that the effect of force on a body is to produce an acceleration which is directly proportional to the applied force and inversely proportional to the mass of the body, the acceleration taking place in the direction in which the force acts. We can now give a physically meaningful and a quantitative definition of *force as something that produces an acceleration in a material body.*

Alternative Statement of the Second Law

The acceleration produced by an unbalanced force acting on a body is directly proportional to the magnitude of the applied force and inversely proportional to the mass of the body; the acceleration taking place in the direction in which the force acts.

Unit of Force

We have seen that

$$\vec{F} \propto m\,\vec{a}$$

$$\vec{F} = km\,\vec{a}$$

where k is a constant of proportionality, whose numerical value will depend on the units used for $\vec{F}$, m and $\vec{a}$. For convenience, we choose a unit of force in such a way that a unit force produces a unit acceleration in a unit mass, i.e.

$$1 = k \times 1 \times 1$$

which gives $k = 1$. If the unit is fixed in this manner,

$$\vec{F} = m\,\vec{a} \tag{5.2}$$

In the SI system, the unit of force is the newton (symbol N). *A newton is defined as that force which produces an acceleration of 1 ms^{-2} in a body of mass 1 kg.* The dimensions of force are MLT^{-2}.

The first law is a special case of the second law. If no net force acts on a body, the value of $\vec{F} = 0$. Substituting in Eq. (5.2), we have, $\vec{a} = 0$, i.e. the body is not accelerated. In other words, the velocity of the body remains constant. If the velocity is zero, it remains zero and the body stays at rest. On the other hand, if the velocity is finite, say, 10 ms^{-1}, it will remain 10 ms^{-1}. Thus, if no net force acts on a body, its state of rest or of uniform motion in a straight line remains unchanged. But this is the first law of motion. Thus, the first law is a special case of the second law because it can be directly deduced from the second law by putting $\vec{F} = 0$.

5.5 CONCURRENT FORCES

Forces acting at the same point on a body are called concurrent forces. The resultant of these forces is equal to their vector sum, obtained according to the method explained in Chapter 4. The resultant $\vec{R}$ of several forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots$ is

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots$$

If the forces are coplanar, say in the x-y plane, the x and y components of the resultant are given by

$$R_x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \ldots$$

and $$R_y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + F_3 \sin \alpha_3 + \ldots$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ respectively are the angles the forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots$ subtend with the positive x-axis.

The magnitude of the resultant vector $\vec{R}$ is given by

$$R = \sqrt{R_x^2 + R_y^2}$$

and its direction is given by

$$\tan \alpha = \frac{R_y}{R_x}$$

where α is the angle subtended by the resultant with the positive x-axis. Look at Example 5.1 below.

If the magnitude of the resultant turns out to be zero, the concurrent forces are said to be in *equilibrium* and no net force acts on the body. But, if the magnitude of the resultant is finite, the body will accelerate in the direction of the resultant unbalanced force.

Example 5.1 Find the magnitude and the direction of the force acting on a body at point O (Fig. 5.6). Force $F_1 = 5$ N, $F_2 = 4$ N, $F_3 = 1$ N and $F_4 = 3$ N. The directions are shown in the figure.

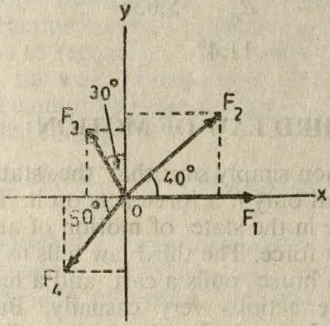


Fig. 5.6

Solution The x-component of the resultant vector $\vec{R}$ is given by

$$R_x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + F_4 \cos \alpha_4$$

where $\alpha_1 = 0°, \alpha_2 = 40°, \alpha_3 = 90° + 30° = 120°$ and

$$\alpha_4 = 180° + 50° = 230°.$$

Notice that the angle of each force is measured in the counter-clockwise sense with respect to the positive x-axis.

The y-component of the resultant vector $\vec{R}$ is given by

$$R_y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + F_3 \sin \alpha_3 + F_4 \sin \alpha_4$$

Substituting the values of forces and angles we get

$$R_x = 5 \cos 0° + 4 \cos 40° + 1 \cos 120° + 3 \cos 230°$$
$$= 5 + 3.064 - 0.5 - 1.928$$
$$= 5.636 \text{ N}$$
$$R_y = 5 \sin 0° + 4 \sin 40° + 1 \sin 120° + 3 \sin 230°$$
$$= 0 + 2.571 + 0.866 - 2.298$$
$$= 1.139 \text{ N}$$

The magnitude of the resultant is given by

$$R = \sqrt{R_x^2 + R_y^2}$$
$$= \{(5.636)^2 + (1.139)^2\}^{1/2} = 5.75 \text{ N}$$

The angle α which the resultant subtends with positive x-axis is given by

$$\tan\alpha = \frac{R_y}{R_x} = \frac{1.139}{5.636} = 0.202$$

giving $\alpha = 11.4°$

5.6 NEWTON'S THIRD LAW OF MOTION

The first law of motion simply says that the state of motion of an object can be changed only if a force acts on it. The second law tells us how much change in the state of motion of an object can be brought about by a given force. The third law tells us how forces are exerted. We know that a horse pulls a cart and a hammer drives a nail, etc. We observe these actions very casually. But Newton carefully analysed what he observed. In these examples, one object exerts a force and the other feels it—the horse exerts a force and the cart feels it; a hammer exerts a force and the nail feels it. But, is it really so simple? Is one body the pusher and the other body, the pushed? Newton could not believe that nature would have preferences. He was led to believe that the two bodies must be treated on an equal footing. This led him to the third law of motion which states that, *whenever one body exerts a force on a second body, the second body exerts an equal and opposite force on the first*, or *to every action there is an equal and opposite reaction.*

Action and Reaction Forces

The most important point to recognise is that the action and reaction forces act on *different* bodies. We are easily confused if we make the mistake of assuming that action and reaction forces act on the same body. If they were to act on the same body, they would add up to zero (because they are equal and opposite). Let us illustrate this by the following example.

No action can take place in the absence of a reaction. Everyone who has played the game of tug-of-war knows from experience that he can exert a force only if there is somebody on the other end of the rope. No force can be exerted if the other end is free. One team provides the action force and the opposite team provides the reaction force. Now replace the opposing team by a wall by fixing the other end of the rope, to the wall. The third law says that if you exert a force on the wall, the wall will exert an equal and opposite reaction force on you. Now remove the rope and exert a force directly on the wall by pushing the palm of your hand against it. Now look at your hand. It is distorted a little. Only a force can distort the shape of an object. But you are not exerting any force on your hand, you are in fact exerting a force on the wall. Then which force has distorted your hand? It is the force exerted by the wall on your hand.

How can the wall exert a force on your hand? The wall can exert a force because it is elastic. It is because of elasticity that a non-living object can exert a force. A spring, for example, has elasticity. We take a ball and press it against a spring fixed on a table. The spring is compressed by the force exerted on it. When the ball is released, it flies off due to the reaction force of the spring which is called into play as the spring begins to regain its original size. Similarly, when we press against the wall, the wall is compressed. The compression of the wall is very little and cannot be noticed. Due to this compression, a reaction force is called into play.

More examples of the third law Some more examples of the third law are as follows.

1. When an ice-skater strikes against the wall of the skating-rink, he is pushed backwards. When he strikes against the wall, he exerts a force on it. The wall in turn exerts an equal and opposite reaction force on him.
2. While swimming, a person pushes the water backwards and as a result he himself is pushed forward by the reaction force exerted by the water on him.
3. The motion of rockets depends on action and reaction. A rocket expels gases by exerting a force on them. The gases in turn exert an equal and opposite force in the rocket. A space vehicle can accelerate or change the direction of its motion just by firing rockets in the proper direction (see the Supplement at the end of this Chapter).
4. We would not be able to walk if there were no reaction force. In order to walk, we push our foot against the ground. The earth in turn exerts an equal and opposite force. This force is inclined to the surface of the earth. The verrical component of this force balances our weight and the horizontal component enables us to walk forward. (See Fig. 5.7).

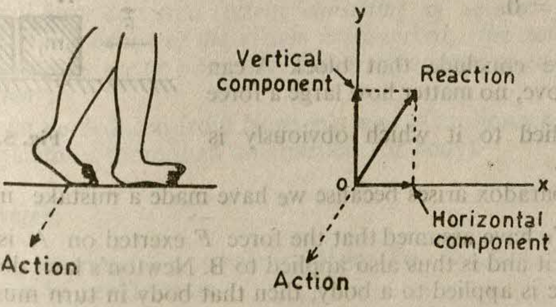


Fig. 5.7

Two apparent paradoxes 1. We push a wooden block of mass m against solid wall with a force $\vec{F}$ as shown in Fig. 5.8.

According to Newton's second law, the acceleration $\vec{a}$ is (from the equation $\vec{F} = m\,\vec{a}$),

$$\vec{a} = \frac{\vec{F}}{m}$$

Thus, the block should accelerate and start moving. But from experience we know that the block will not move. What is wrong? The mistake is that in the equation $\vec{F} = m\,\vec{a}$, $\vec{F}$ is not just the force exerted on the object but the net force experience by it. In addition to the applied force $\vec{F}$, there is a second force $\vec{F'}$ exerted by the wall on the block. The net force on the block is $(\vec{F} + \vec{F'})$. Now according to Newton's third law $\vec{F'}$ is equal and opposite to $\vec{F}$, i.e. $\vec{F'} = -\vec{F}$. The net force experienced by the block is $\vec{F}_{\text{net}} = (\vec{F} + \vec{F'}) = \vec{F} + (-\vec{F}) = 0$. Hence, the acceleration of the block is

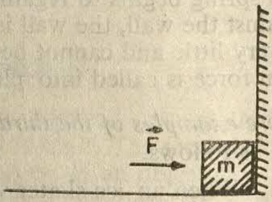


Fig. 5.8

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m} = 0$$

2. We place two blocks A and B of masses m_1 and m_2 respectively on a frictionless surface as shown in Fig. 5.9 and apply a force $\vec{F}$ to A. The block B also experiences a force due to its contact with A. By Newton's third law, the block B must exert an equal and opposite force $\vec{F}$ on block A. Thus the net force on A is $\vec{F} + (-\vec{F}) = 0$, so that

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m} = 0.$$

Thus, we conclude that block A can never move, no matter how large a force $\vec{F}$ is applied to it which obviously is not true.

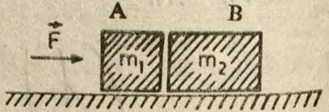


Fig. 5.9

This paradox arises because we have made a mistake in our argument. We have assumed that the force $\vec{F}$ exerted on A is transmitted through it and is thus also applied to B. Newton's laws do not say that if a force is applied to a body, then that body in turn must exert the same force on another body in contact with it. Let us, therefore, say that A exerts a force $\vec{F'}$ on B. From the third law B will exert a force $-\vec{F'}$ on A. Thus, the net force on A is $\vec{F} - \vec{F'}$ and its acceleration $\vec{a}$ is given by

$$\vec{F} - \vec{F'} = m_1 \vec{a}$$

Since B is in contact with A, it will have the same acceleration $\vec{a}$ which is given by

$$\vec{F'} = m_2 \vec{a}$$

Adding the two equations, we get

$$\vec{F} = m_1 \vec{a} + m_2 \vec{a} = (m_1 + m_2) \vec{a}$$

or $$\vec{a} = \frac{\vec{F}}{m_1 + m_2}$$

i.e. $$\text{Acceleration} = \frac{\text{Applied force}}{\text{Total mass}}$$

Which is what we expect.

5.7 THE LAW OF CONSERVATION OF LINEAR MOMENTUM

In physics we come across a variety of conservation laws. The law of conservation of momentum is the most fundamental conservation law. We might ask why Newton took the trouble of defining a new quantity called the momentum $\vec{p}$ as the product of mass m and velocity $\vec{v}$. The reason is that this particular combination is useful for understanding of the laws of nature. What makes momentum so important is the fact that it is conserved (i.e. it remains constant).

The *law of conservation of linear momentum* may be stated as '*when no net external force acts on a system consisting of several particles, the total linear momentum of the system is conserved, the total linear momentum being the vector sum of the linear momentum of each particle in the sytem*'.

This law can be deduced from Newton's laws of motion when generalised to a system of more than one particle (or body).

One Body System

For a system consisting of a single particle, Eq. (5.2) of Newton's second law can be rewritten as

$$\vec{F} = m\,\vec{a}$$

$$= m\,\frac{d\vec{v}}{dt} \quad (\because \; a = \frac{d\vec{v}}{dt} \text{ where } \vec{v} \text{ is the velocity of the body})$$

$$= \frac{d}{dt}(m\,\vec{v}) \quad (\because \; \text{mass } m \text{ is constant})$$

or $$\vec{F} = \frac{d\vec{p}}{dt}$$

where $\vec{p} = m\,\vec{v}$ is the linear momentum of the particle. Now if the external force $\vec{F}$ applied to the particle is zero, it is clear that

$$\frac{d\vec{p}}{dt} = 0$$

or $\vec{p}$ = constant, showing that in the absence of an external force, the linear momentum of a particle remains constant.

2. Two-body System

Consider a system of two particles of masses m_1 and m_2 moving with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively in the same straight line. These particles may collide with each other, and as a result their velocities may change. At any instant of time, we have

Linear momentum of mass m_1 $(\vec{p_1}) = m_1\,\vec{v_1}$

Linear momentum of mass m_2 $(\vec{p_2}) = m_2\,\vec{v_2}$

Thus the total linear momentum $(\vec{p})$ = vector sum of the individual linear momentum of each particle

or $$\vec{p} = m_1\,\vec{v_1} + m_2\,\vec{v_2}$$

Differentiating with respect to time, we get

$$\frac{d\vec{p}}{di} = \frac{d}{dt}(m_1\,\vec{v_1} + m_2\,\vec{v_2})$$

$$= m_1\,\frac{d\vec{v_1}}{dt} + m_2\,\frac{d\vec{v_2}}{dt}$$

$$= m_1 \vec{a}_1 + m_2 \vec{a}_2 \quad \left(\because \vec{a} = \frac{d\vec{v}}{dt}\right)$$

$$= \vec{F}_1 + \vec{F}_2 \qquad (\because \vec{F} = m\vec{a}) \qquad (5.3)$$

where $\vec{F}_1$ and $\vec{F}_2$ are the forces acting on m_1 and m_2 respectively. In a system consisting of two particles, there can be two kinds of forces: (i) internal forces and (ii) external forces. *Internal* forces are the forces that the particles exert on each other during their interactions (e.g. collisions). From Newton's third law, these forces always occur in pairs of action and reaction. Since these forces are equal and opposite, they bring about equal and opposite changes in the momentum of the particles. Thus, internal forces cannot bring about any net change in the momentum of a particle. The external forces, on the other hand, are the forces exerted from outside the system of two particles. From Newton's second law, these external forces will change the momentum of the particles. Thus, we conclude that forces $\vec{F}_1$ and $\vec{F}_2$ in Eq. (5.3) are actually the external forces acting on the particles.

The resultant external force $\vec{F}_{ext}$ is given by the vector sum

$$\vec{F}_{ext} = \vec{F}_1 + \vec{F}_2$$

Hence, Eq. (5.3) becomes

$$\frac{d\vec{p}}{dt} = \vec{F}_{ext}$$

If $\vec{F}_{ext} = 0$, we have

$$\frac{d\vec{p}}{dt} = 0$$

or $\vec{p}$ = constant

This implies that the vector sum $(\vec{p} = \vec{p}_1 + \vec{p}_2)$ of the linear momentum of the particles remains constant, if the net external force is zero.

Many-body System

The law can be extended to any number of particles. These particles may be the particles of a rigid body in which their positions are fixed or they may form an assembly and be free to move freely in any direction.

Let $m_1, m_2, \ldots, m_n$ be the masses of the different particles and, in addition to internal forces due to their own interactions with each other, let there also be external forces acting on them, so that they acquire velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ respectively. Then, clearly their total momentum $\vec{p}$ is equal to the vector sum of their individual momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$, i.e.

$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n$$

$$= m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_n \vec{v}_n$$

$$(\because \ \vec{p} = m\vec{v})$$

Differentiating with respect to t we have

$$\frac{d\vec{p}}{dt} = m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}}{dt} + \ldots + m_n \frac{d\vec{v}_n}{dt}$$

$$= m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots + m_a \vec{a}_n \left(\because \quad \vec{a} = \frac{d\vec{v}}{dt}\right)$$

where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the respective accelerations. Now from Newton's second law, $\vec{F} = m\vec{a}$. Thus

$$\frac{d\vec{p}}{dt} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

where $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ are the forces acting on individual particles. These forces include both internal and external forces. As mentioned earlier, the internal forces, which arise due to mutual inter-particle interactions, always occur in pairs. Since these action and reaction forces are equal and opposite, they bring about equal and opposite changes in the momentum of the particles of the system. Thus, the internal forces cannot change the momentum of the particles. We, therefore, conclude that $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ are only the external forces acting on the particles. The net external force $\vec{F}_{\text{ext}}$ is clearly equal to the vector sum of all these forces, i.e.

$$\vec{F}_{\text{ext}} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

Thus $$\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}}$$

If $\vec{F}_{ext} = 0$, it follows that $\frac{d\vec{p}}{dt} = 0$ or $\vec{p} =$ constant, i.e. the total linear momentum (which is equal to the vector sum of the momenta of individual particles) of the system is constant, if the net external force acting on the system is zero. This is the law of conservation of linear momentum.

A system which is free from the influence of any external forces is called an *isolated* system.

We have seen that the law of conservation of linear momentum can be derived from Newton's second law of motion. We shall now show that Newton's third law is a consequence of the law of conservation of linear momentum.

Deduction of Newton's Third Law of Motion from the Law of Conservation of Momentum

Consider an isolated system of two bodies of masses m_1 and m_2 mutually interacting with each other. These interactions cause a change in the velocities and hence a change in the momentum of each body. Let $\Delta \vec{p_1}$ and $\Delta \vec{p_2}$ denote the change in momentum of m_1 and m_2 respectively, brought about in a time interval Δt.

Now, according to the law of conservation of linear momentum, the net change in the linear momentum in an isolated system must be zero, i.e.

$$\Delta \vec{p_1} + \Delta \vec{p_2} = 0$$

or

$$\Delta \vec{p_2} = - \Delta \vec{p_1}$$

Dividing by Δt and taking the limit as $\Delta t \to 0$, we have

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{p_2}}{\Delta t} = - \lim_{\Delta t \to 0} \frac{\Delta \vec{p_1}}{\Delta t}$$

or

$$\frac{d\vec{p_2}}{dt} = - \frac{d\vec{p_1}}{dt}$$

i.e. rate of change of momentum of m_2 = − rate of change of momentum m_1

or

$$\text{Force on } m_2 = - \text{Force on } m_1$$

or

$$\text{Action} = - \text{Reaction}$$

i.e. the action and reaction are always equal and opposite. This is Newton's third law of motion.

Some Illustrations of Conservation of Momentum

The following examples illustrate the law of conservation of momentum.

Recoil of a Gun The gun and the bullet constitute a single system. Before the gun is fired, both the gun and the bullet are at rest. Therefore, their total momentum is zero. When the gun is fired, the bullet moves forward and the gun recoils backwards. If m_b and m_g are respectively the masses of the bullet and the gun and if $\vec{v_b}$ and $\vec{v_g}$ are their respective velocities, the total momentum of the bullet-gun system after firing is $m_b\vec{v_b} + m_g\vec{v_g}$. According to the law of conservation of momentum, the total momentum before firing must be equal to that after firing, i.e.

$$m_b\vec{v_b} + m_g\vec{v_g} = 0$$

or

$$\vec{v_g} = -\frac{m_b}{m_g}\vec{v_b}$$

It is clear from this equation that $\vec{v_g}$ is directed opposite to $\vec{v_b}$. Knowing m_b, m_g and $\vec{v_b}$, the recoil velocity $\vec{v_g}$ of the gun can be determined.

Rockets The propulsion of rockets and jet planes is an interesting application of the conservation law. When the fuel of the rocket is exploded, gases escape with a large velocity and hence a large momentum. The escaping gases, in turn, impart an equal and opposite momentum to the rocket.

Explosion of a bomb Suppose a bomb is at rest before it explodes. Its total momentum is zero. When it explodes, it breaks up into many parts, each part having a particular momentum. For a part flying in one direction with a certain momentum, there is another part moving in the opposite direction with the same momentum. If the bomb explodes into two equal parts, they will fly off in exactly opposite directions with the same velocity (since each part has the same mass).

EXAMPLE 5.2 A rocket consumes fuel at the rate of 100 kg s^{-1}. The exhaust gases are ejected at a speed of 4.5×10^4 ms^{-1}. What is the thrust experienced by the rocket?

Solution Mass of the fuel consumed and, therefore, ejected = 100 kg s^{-1}. Velocity of ejected gases = 4.5×10^4 ms^{-1}

$\therefore$ Change of momentum of exhaust gases per second

= Mass ejected per second × velocity

$$= 100\ \frac{\text{kg}}{\text{s}} \times 4.5 \times 10^4\ \frac{\text{m}}{\text{s}} = 4.5 \times 10^6\ \text{kg ms}^{-2}$$

From Newton's second law

$$\text{Force} = \text{change in momentum per second}$$
$$= 4.5 \times 10^6 \text{ N}$$

The rocket exerts this force on the escaping gases and, in turn, will experience a thrust equal to 4.5×10^6 N.

EXAMPLE 5.3 A truck of mass 1000 kg is pulling a trailer of mass 2000 kg as shown in Fig. 5.10. The retarding (frictional) force on the truck is 500 N and that on the trailer is 1000 N. The truck engine exerts a force of 6000 N. Calculate (i) the acceleration of the truck and the trailer, and (ii) the tension in the connecting rope.

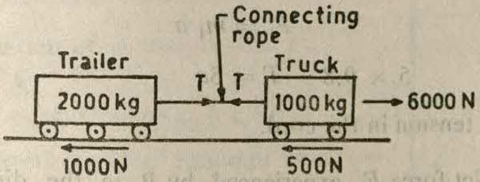


Fig. 5.10

Solution Referring to Fig. 5.10, the net force F_1 exerted on the trailer in the direction of motion is $F_1 = (T - 1000)$ N where T is the tension in the connecting rope. Using $F = ma$, the acceleration of the trailer is given by

$$T - 1000 = 2000\,a \tag{a}$$

It is clear from the figure that the net force F_2 exerted by the engine of the truck is $F_2 = (6000 - 500) - T$ in the direction of motion. Since the mass of the truck is 1000 kg, its acceleration a (which must be the same as that of the trailer) is given by

$$6000 - 500 - T = 1000\,a \tag{b}$$

(i) Adding Eqs (a) and (b) to eliminate T, we get

$$4500 = 3000\,a$$

or

$$a = \frac{4500}{3000} = 1.5 \text{ ms}^{-2}$$

(iii) Using the value of a in Eq. (a) we have

$$T = 1000 + 2000 \times 1.5 = 4000 \text{ N}$$

EXAMPLE 5.4 Two objects A and B of masses 5 and 3 kg respectively are connected by a light cord passing over a frictionless pulley. Calculate (i) the acceleration of each mass and (ii) the tension in the cord. Take $g = 9.8\ \text{ms}^{-2}$.

Solution Referring to Fig. 5.11, we see that the object A moves downwards and object B upwards each with an acceleration, say, a.

Object A: Net Force F_1 experienced by A in the direction of its motion is given by ($m_1 = 5$ kg)

$$F_1 = 5 \times 9.8 - T$$

Hence, its acceleration a is given by (net force = mass × acceleration)

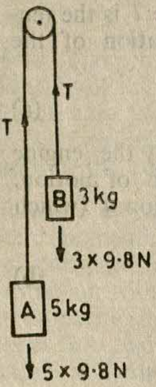
$$F_1 = m_1\, a$$

$$5 \times 9.8 - T = 5a \qquad \text{(a)}$$

where T is the tension in the cord.

Object B: Net force F_2 experienced by B in the direction of its motion is given by ($m_2 = 3$ kg)

$$F_2 = T - 3 \times 9.8$$



Its acceleration a is given by

$$T - 3 \times 9.8 = 3a \qquad \text{(b)}$$

(i) Adding Eqs (a) and (b) we have

$$5 \times 9.8 - 3 \times 9.8 = 8a$$

or
$$a = \frac{2 \times 9.8}{8} = 2.45\ \text{ms}^{-2}$$

(ii) Using this value of a in either Eq. (a) or (b) we get

$$T = 36.75\ \text{N}$$

Fig. 5.11

EXAMPLE 5.5 A shell explodes into three fragments of equal masses. Two fragments fly off at right angles to each other, with a velocity of $9\ \text{ms}^{-1}$ and $12\ \text{ms}^{-1}$. What is the direction and speed of the third fragment?

Solution The momentum of the shell before explosion is zero. The total momentum of the three fragments after explosion must also be zero. Let m kg be the mass of each fragment and $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ their velocities so that $\vec{p}_1, \vec{p}_2$ and $\vec{p}_3$ are their respective momenta (see Fig. 5.12).

$$\vec{p}_1 = m\vec{v}_1$$

$$\vec{p}_2 = m\vec{v}_2$$

$$\vec{p}_3 = m\vec{v}_3$$

where $\vec{v}_1 = 9 \text{ ms}^{-1}$ along, say, the x-direction and $\vec{v}_2 = 12 \text{ ms}^{-1}$ along the y-direction as shown.

The resultant of $\vec{p}_1$ and $\vec{p}_2$ has a magnitude give by

$$p = (p_1^2 + p_2^2)^{1/2} = m(v_1^2 + v_2^2)^{1/2}$$

and the direction of the resultant is given by

$$\tan\theta = \frac{p_2}{p_1} = \frac{mv_2}{mv_1}$$

$$= \frac{12}{9} = 1.33$$

$$\theta = 53^\circ\, 8' \quad \text{(approx)}$$

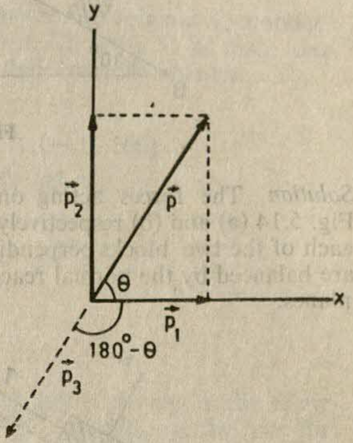


Fig. 5.12

Now since the total momentum is zero, it is clear that

$$\vec{p}_3 + \vec{p} = 0$$

or

$$\vec{p}_3 = -\vec{p}$$

∴ The magnitude of $\vec{p}_3$ is equal to that of $\vec{p}$ but the direction is opposite, (180° − θ) with x-axis. Therefore, magnitude of p_3 is

$$\begin{aligned} p_3 &= \text{magnitude of } \vec{p}_{1/2} \\ &= m\,(v_1^2 + v_2^2)^{1/2} \\ &= m\,(9^2 + 12^2)^{1/2} \\ &= 15\,m \text{ kg ms}^{-1} \end{aligned}$$

∴ The velocity of the third fragment is $= \frac{p_3}{m} = 15 \text{ ms}^{-1}$.

The direction of this velocity is $(180^\circ - \theta) = 180^\circ - 53^\circ 8'$

$= 126^\circ 52'$ inclined with the x-axis.

EXAMPLE 5.6 Two blocks connected by an inextensible string passing over a light frictionless pulley rest on two smooth inclined planes as shown in Fig. 5.13. Determine the acceleration of the blocks and the tension in the strings. Assume the string to be massless.

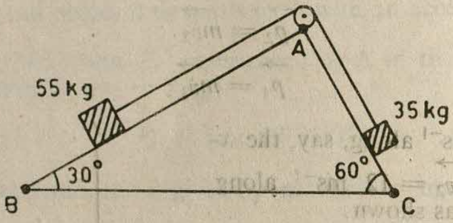
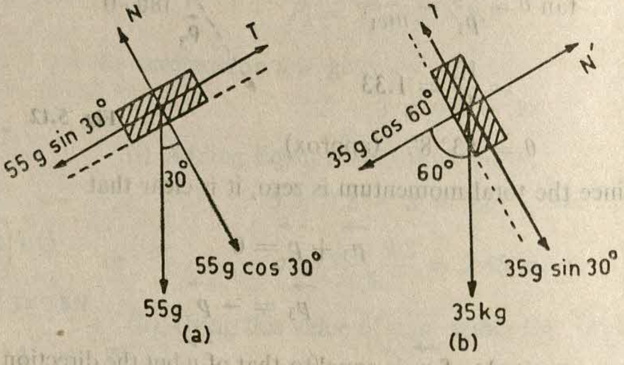


Fig. 5.13

Solution The forces acting on each of the two blocks are shown in Fig. 5.14 (a) and (b) respectively. The components of the weights of each of the two blocks perpendicular to their respective inclined planes are balanced by the normal reactions N and N' respectively at these planes.



(a)

(b)

Fig. 5.14 (a) and (b)

The tension in the string and the acceleration of both the blocks are the same, as the blocks are connected by a light and inextensible string passing over a frictionless pulley. Writing the equation of motion for the two masses, we have

$$55\ g \sin 30^\circ - T = 55 \times a$$

and
$$T - 35\ g \sin 60^{0} = 35 \times a$$

Solving, we get

$$g\,(55 \sin 30^\circ - 35 \sin 60^\circ) = (55 + 35)\,a$$

Therefore $$a = \frac{9.8(55 \times 0.5 - 35 \times 0.866)}{90}$$

$$= \frac{9.8(27.5 - 30.311)}{90}$$

$$= -\ 0.306\ \text{ms}^{-2}$$

The negative sign shows that, contrary to our original assumption, it is the 35 kg mass that moves downwards. Thus the 35 kg mass descends and the 55 kg mass ascends from their respective planes.

Tension T is given by

$$\begin{aligned} T &= 55\,g \sin 30° - 55 \times (-\ 0.\ 306) \\ &= 55\,(9.8 \times 0.5 + 0.306) \\ &= 55 \times 5.206 \\ &= 286.33\ \text{N} \end{aligned}$$

5.8 IMPULSE

Consider a collision between two bodies A and B moving in the same straight line. Let t be the duration of the collision, i.e. the time for which the bodies were in contact during which time the transfer of momentum took place. We assume that the bodies continued moving in the same straight line after the collision with velocities different from their initial velocities.

During collision, the force exerted by A on B or vice versa is usually not constant but varies with time in a complicated way. In such cases, the net effect of the force is measured by what is called the *impulse* of the force.

Let us consider the force exerted on body B during a collision of duration t during which its momentum changes from say, $\vec{p}_1$ to $\vec{p}_2$. From Newton's second law, the force acting on B at any instant of time is given by the rate of change of momentum at that instant, i.e.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

or $$d\vec{p} = \vec{F}\,dt$$

Integrating we have

$$\int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = \int_0^t \vec{F}\,dt \quad \text{or} \quad \left|\vec{p}\right|_{\vec{p}_1}^{\vec{p}_2} = \int_0^t \vec{F}\,dt$$

or $$\int_0^t F dt = \vec{p_2} - \vec{p_1}$$

We find that the integral $\int_0^t \vec{F}\, dt$ is equal to the change in momentum of the body brought about in time t and is a measure of the *impulse* of the force. It is denoted by symbol $\vec{J}$. Thus.

$$\vec{J} = \int_0^t \vec{F}\, dt = p_2 - p_1$$

Since the dependence of $\vec{F}$ on t is not known, this integral connot be evaluated.

If $\vec{F}_{av}$ measures the average force during collision, we can then treat it as constant and write

$$\int_0^t \vec{F}\, dt = \int_0^t \vec{F}_{av}\, dt = \vec{F}_{av} \int_0^t dt = \vec{F}_{av} \cdot t$$

Thus $$\vec{J} = \vec{F}_{av}\, t = \vec{p_2} - \vec{p_1}$$

Thus, *impulse of a force is the product of the average force and the time for which the force acts and is equal to the change in momentum of the body during that time.* Impulse is a vector and is measured in kg ms^{-1} or Ns.

EXAMPLE 5.7 A cricket ball of mass 150 g is moving with a velocity of 12 ms^{-1} and is hit by a bat so that the ball is turned back with a velocity of 20 ms^{-1}. The force of the blow acts for 0.01 s. Find the average force exerted on the ball by the bat.

Solution The impulse of the force exerted by the bat is given by the change in the momentum of the ball. Now

$$\text{Initial momentum of the ball} = \frac{150}{1000} \times 12 \text{ kg ms}^{-1}$$
$$= 1.8 \text{ kg ms}^{-1}$$

$$\text{Final momentum of the ball} = -\frac{150}{1000} \times 20 \text{ kg ms}^{-1}$$
$$= -3.0 \text{ kg ms}^{-1}$$

$$\text{Change in momentum of the ball} = [1.8 - (-3.0)] \text{ kg ms}^{-1}$$
$$= 4.8 \text{ kg ms}^{-1}$$

This equals the impulse of the force exerted by the bat. Since

$$\text{Impulse} = \text{force} \times \text{time}$$

we have

$$\text{Average force exerted} = \frac{\text{impulse}}{\text{time}}$$

$$= \frac{4.8 \text{ kg ms}^{-1}}{0.01 \text{ s}} = 480 \text{ kg ms}^{-2}$$

$$= 480 \text{ N}$$

5.9 FORCES IN NATURE

We come accross various kinds of forces in nature. They are

1. mechanical force
2. gravitational force
3. frictional force
4. electrical force
5. magnetic force
6. nuclear force

Mechanical force is one exerted by active or living objects, e.g. human beings, animals, engines, moving objects, springs, etc. Frictional force comes into play when the surface of a body moves over the surface of another. Mechanical and frictional forces are *contact forces*. The body exerting a force comes into contact with the body experiencing it. The other forces mentioned above *act at a distance*. The body exerting one of these forces need not come into actual contact with the body which experiences it.

In this chapter we will discuss frictional force in some detail. In Chapter 8, we will deal with gravitational force.

The force between charged bodies is electrical and that between magnets is magnetic. These forces are not independent phenomena. We know that a moving charge (i.e. current) has an associated magnetic field. Electrical and magnetic forces are two facets of a more general *electro-magnetic force*.

The nucleons in the nucleus of an atom are held together by a strong attractive force called the *nuclear force*. This is the strongest force in nature; the gravitational force is the weakest.

5.9 FRICTION

Place a block of wood on a table and give it a push. The block moves a little distance and then comes to rest. According to Newton's first law, the block should have continued to move with a uniform velocity. But this is not what we observe. Newton's second law tells us that a retarding force must be acting on it as it moves. This force opposes motion. It is always tangential to the surface in contact, acting in a direction opposite to the motion of the body and is called the force of friction. Thus, *the force of friction* (*or simply friction*) *is the force which* *comes into play when one body slides or rolls on the surface of another and acts in a direction tangential to the surfaces in contact and opposite to the direction of motion of the body.*

It must be remembered that when the surface of one body slides over that of another, each body exerts a frictional force on the other. The direction of this force on each body is opposite to its direction of motion relative to the other. In our example of a block sliding over a horizontal surface, if the block slides in the direction AB (see Fig. 5.15a) the force of friction (f) acts in the opposite direction. If the direction of motion is reversed and the block moves in the direction AC the force of friction (f) is reverse and acts opposite to AC (Fig. 5.15b).

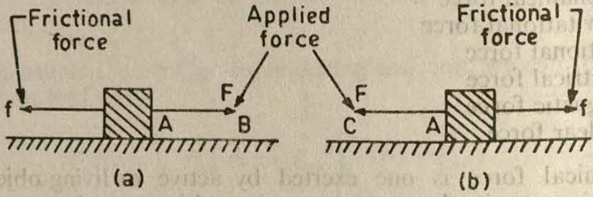


Fig. 5.15 Friction opposes motion

The Origin of Friction

Friction is a surface phenomenon. The surfaces of bodies are never perfectly smooth. Even a highly polished surface, when viewed under a microscope, reveals dents and irregularities of the surface (see Fig. 5.16). When a body is placed over the surface of another body, there is some degree of *interlocking* of these irregular projections on the two surfaces. When a force is applied on one body to make it slide over the surface of the other, the relative motion is resisted due to the interlocking of the surface irregularities and an opposing force is developed. This is the force of friction. As the applied force is increased, the resistive force also increases, but a stage is reached when the body begins to slide over the surface of another.

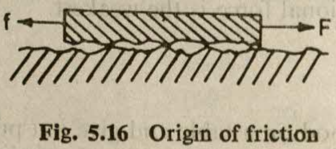


Fig. 5.16 Origin of friction

Limiting Friction

A rectangular block of wood is placed on a flat surface and one end of a string is attached to the block, the other end of which carries a pan, as shown in Fig. 5.17. Now if we place a small mass in the pan, a horizontal force F acts on the block. We will see that the pan does not move. The force of friction *f* which comes into play completely balances the force *F* applied. Only so much force of friction is called into play as is sufficient to balance the applied force. These forces are equal and opposite.

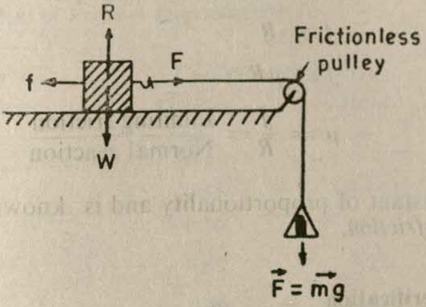


Fig. 5.17 Limiting friction

Now, if we add more mass to the pan, the block still does not move. The force of friction increases to balance the increased applied force. If the mass added in the pan is gradually increased, a stage is reached when the block just begins to slide over the horizontal surface. At this stage, the force of friction (which is equal to the applied force) is given by the total weight (= mg), where m is the mass of the pan and the mass added to it. After this stage, the force of friction cannot increase any further because it has just been overcome by the applied force. *The maximum or limiting value of the force of friction which comes into play when a body just begins to slide over the surface of another body is known as limiting friction.* Thus, the force of friction has a remarkable property of adjusting its magnitude so as to become exactly equal to the applied force tending to produce motion.

It may be mentioned that the block exerts its weight W (= Mg, where M is the mass of the block) on the table in the downward direction and the table exerts an equal and opposite reaction force R vertically upwards normal to the surface. The magnitude R of the reaction force acting normally to the surface is usually called the normal reaction. These forces are in equilibrium. There is no other horizontal force on the block except the applied force F.

The Laws of Limiting Friction

From the results of the various experiments relating to friction between different surfaces, the following three empirical laws are found to be obeyed by limiting friction.

1. The magnitude of limiting friction depends upon the nature of the surfaces in contact and on their roughness (or smoothness). It is independent of the areas of the surfaces in contact.

2. The force of friction is tangential (or parallel) to the surfaces in contact and its direction is opposite to the direction of motion of the body.

3. The value of limiting friction between two given surfaces is proportional to the normal reaction between them, or

$$f \propto R$$

or $$f = \mu R$$

i.e. $$\mu = \frac{f}{R} = \frac{\text{Limiting friction}}{\text{Normal reaction}}$$

where μ is a constant of proportionality and is known as the *coefficient of limiting friction.*

Experimental Verification

The experiment illustrated in Fig. 5.17 can be used to verify the laws of friction.

Take a fairly smooth regular block of wood and place it on a horizontal smooth surface. Weigh the block in a balance. Let its mass be M. The normal reaction R is given by $R = Mg$, where g is the acceleration due to gravity.

Now gradually add weights in the pan until the block just begins to slide on the surface. Weigh the pan along with its contents. Let the total mass be m. The applied force, $F = mg$. The value of F is the limiting static friction f_s. The coefficient of limiting static friction is given by

$$\mu_s = \frac{f_s}{R}$$

For fairly smooth wooden surfaces (i.e. wood on wood), μ_s is found to be nearly 0.2.

Now place another identical block on the first block. The normal reaction is doubled. It will be seen that twice the force is required to just make them slide over the same surface. In other words, the limiting friction is proportional to the normal reaction. The ratio $\mu = f/R$ remains unchanged.

Now, connect the two blocks together and place them one behind the other. The surface areas in contact are doubled. It is seen that the value of the limiting friction remains unchanged showing that it is independent of the areas of the surfaces in contact.

Finally, take blocks of identical surface area but of different materials such as wood and steel. The value of the limiting friction is found to be much less for steel than for wood, provided their surfaces are equally smooth. This shows that limiting friction depends on the nature of the surfaces in contact.

Table 5.1 shows approximate experimental values of the coefficient of limiting friction for some surfaces. The value of friction changes with the condition of surfaces, grain, contamination, moisture, etc.

Table 5.1 Coefficient of Friction (approximate)

Surfaces in contact	*Coefficient of friction* *Limiting* μ_s	 *Kinetic* μ_k
Steel on steel	0.25	0.18
Steel on glass	0.30	0.20
Steel on wood	0.40	0.22
Wood on glass	0.46	0.24
Wood on wood	0.50	0.26
Leather on wood	0.55	0.40
Car tyre on metalled road (for small speed)	0.60	0.40
Steel on steel (greased)	0.10	0.05

Kinetic or Sliding Friction

The applied force required to just make a body slide over the surface of another measures the limiting value of the static friction. We have denoted this maximum or limiting value by the symbol f_s. But *the force necessary to maintain a body in uniform motion over the surface of another body, after the motion has started* (i.e. after the limiting state friction has been overcome), *measures the kinetic or sliding friction between the two surfaces.* We denote this force by f_k. The ratio

$$\mu_k = \frac{f_k}{R}$$

is called the *coefficient of kinetic friction* and is usually denoted by the symbol μ_k (see Table 5.1). Notice that μ_k is always less than μ_s.

Rolling Friction

Rolling friction is a form of kinetic friction and it comes into play when one body rolls over the surface of another. The frictional force developed is called the *force of rolling friction* and the corresponding coefficient is called the *coefficient of rolling friction* (μ_r). It is a matter of common experience that it is much easier to roll an object on a surface than to slide it. For example, you require less effort to roll a drum full of coal-tar than to slide it over the surface of the road. Rolling friction is much less than sliding friction, i.e. $\mu_r < \mu_k$. It is for this reason that all vehicles are provided with wheels.

Methods of Reducing Friction

Friction can never be completely eliminated. It can only be reduced. The various methods used for reducing friction are polishing of surfaces, lubrication with oil or grease, use of ball bearings, etc.

Friction, a Necessary Evil

Friction produces heat. The excessive heat produced in a machine may cause damage; hence there is a need to reduce or minimize friction. But friction plays a very important role in our daily life. If it were absent, we would not be able to walk; we would not be able to travel in vehicles, for the wheels of the vehicles would rotate without moving, and if somehow, we succeded in moving we could not have stopped because the brakes would not function; we would not be able to light a match-stick, etc. Thus, friction is a necessary evil.

EXAMPLE 5.8 An aeroplane requires a speed of 80 km per hour for take-off; the run on the ground being 100 m. The mass of the plane is 10,000 kg and the coefficient of friction between the plane and the ground is 0.2. Assume that the plane accelerates uniformly during the take-off. What is the minimum force required by the engine of the plane for take-off?

Solution The engine needs force to (i) accelerate the plane from rest to a speed of 80 km/hour ($= 200/9\ \text{ms}^{-1}$) over a distance of 100 m and (ii) to overcome the force of friction ($= \mu R = \mu mg$). The total force needed is the sum of the forces needed in (i) and (ii).

To calculate the force needed for accelerating the plane, we note that, since $v^2 - u^2 = 2\,aS$, we have

$$\left(\frac{200}{9}\right)^2 - 0^2 = 2 \times a \times 100$$

$$a = \frac{200 \times 200}{9 \times 9 \times 200}\ \text{ms}^{-2}$$

$$= 2.469\ \text{ms}^{-2}$$

The force needed for providing this acceleration is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$= 10{,}000 \times 2.469\ \text{N}$$

$$= 24690\ \text{N}$$

The force needed to overcome the force of friction is

$$\mu mg = 0.2 \times 10{,}000 \times 9.8 \text{ N}$$
$$= 19600 \text{ N}$$

Hence the total force needed $= (24690 + 19600)$ N

$$= 44290 \text{ N} = 4.429 \times 10^4 \text{ N}$$

EXAMPLE 5.9 An object is first sliding down an inclined plane of inclination θ with a constant velocity v. It is next projected up the plane with an initial velocity u. Calculate the distance upto which it will rise before coming to rest. What will happen to the object after it has come to rest?

Solution Let m be the mass of the object. The forces acting on it when it is (a) moving down. and (b) moving up the 'incline, are as shown in Figs 5.18 (a) and (b) respectively.

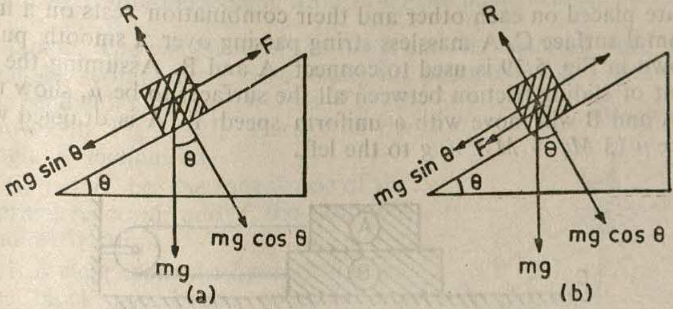


Fig. 5.18 (a) and (b)

Case (a): Since the object is sliding down with a constant velocity, the net force acting on it must be zero. Thus

$$F = mg \sin \theta$$

Also
$$F = \mu_k R = \mu_k mg \cos \theta$$

where, μ_k is the coefficient of friction (kinetic). Therefore,

$$\mu_k mg \cos \theta = mg \sin \theta$$

or
$$\mu_k = \tan \theta$$

Case (b): Since the body is projected upwards with an initial velocity u, it experiences a downward acceleration of magnitude

$$(mg \sin \theta + \mu_k mg \cos \theta)/m \text{ or } (g \sin \theta + \mu_k g \cos \theta)$$

But $\mu_k = \tan\theta$, therefore,

$$\text{Downward acceleration} = g\sin\theta + \tan\theta \cdot g\cos\theta$$
$$= 2g\sin\theta$$

Let x be the distance moved up the plane before the object comes to rest. We then have

$$0 - u^2 = 2 \times (-2g\sin\theta)\,x$$
$$x = u^2/4g\sin\theta$$

Once the object comes to rest, the frictional force to be overcome is that of static friction. The object can, therefore, slide down the plane if its inclination is $\tan^{-1}\mu_s$. Since $\mu_s > \mu_k$ and the inclination of the given plane is $\tan^{-1}\mu_k$, the object will stay at rest and will not slide down again.

EXAMPLE 5.10 Two blocks A and B of masses M_1 and M_2 respectively are placed on each other and their combination rests on a fixed horizontal surface C. A massless string passing over a smooth pulley as shown in Fig. 5.19 is used to connect A and B. Assuming the coefficient of sliding friction between all the surfaces to be μ, show that both A and B will move with a uniform speed if A is dragged with a force $\mu(3M_1 + M_2) \times g$ to the left.

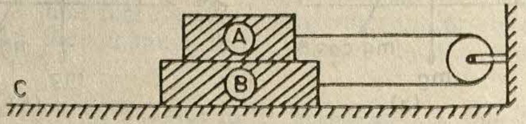
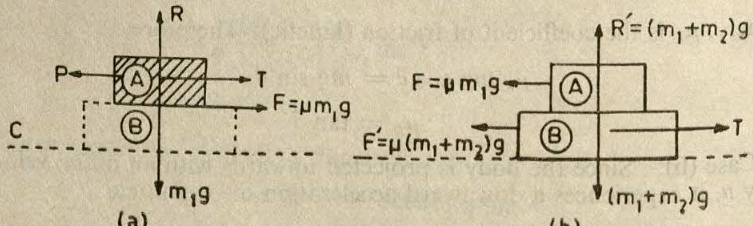


Fig. 5.19

Solution When A is dragged to the left with, say, a force P, the block B starts moving to the right. The forces acting on A and B are, therefore as shown in Fig. 5.20 (a) and (b) respectively. Here T is the tension in the connecting string.



(a)

(b)

Fig. 5.20 (a) and (b)

When both A and B move with a uniform speed, the net force acting oneach must be zero. We, therefore, have

$$P = T + \mu M_1 g$$

and
$$T = \mu (M_1 + M_2) g + \mu M_1 g$$
$$= \mu (2 M_1 + M_2) g$$

Hence,
$$P = \mu (2 M_1 + M_2) g + \mu M_1 g$$
$$= \mu (3 M_1 + M_2) g$$

EXAMPLE 5.11 A block of mass 5 kg is resting on an inclined plane. The inclination of the plane to the horizontal direction is gradually increased. It is found that, when the angle of inclination is 30°, the block just begins to slide down the plane. (i) What is the coefficient of friction between the surface of the block and that of the inclined plane? (ii) What must be the minimum force applied parallel to the plane that would just make it move up the plane? Take $g = 9.8 \text{ ms}^{-2}$.

Solution The weight W of the block can be resolved into two rectangular components, one along the plane ($W \sin \theta$) and the other perpendicular to it ($W \cos \theta$), as shown in Fig. 5.21. Here θ is the angle of inclination.

(i) Let R be the magnitude of the normal reaction and f_s the force of sliding friction.

It is clear from the figure that when the block just begins to slide, the forces are in equilibrium, i.e.

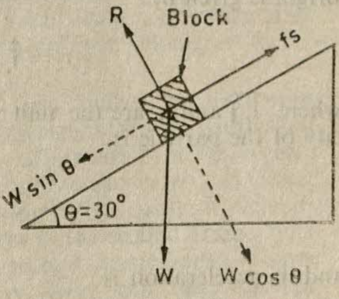


Fig. 5.21 A block on an inclined plane

$$f_s = W \sin \theta$$

$$R = W \cos \theta$$

$$\therefore \quad \mu_s = \frac{f_s}{R} = \tan \theta$$

$$= \tan 30°$$
$$= 0.577$$

(ii) Let F be the value of this minimum force (applied parallel to the plane) needed to move the block up the plane. The force F must not only overcome the force f_s of limiting friction but also the component $W \sin \theta$ of the weight W along the plane. Thus

$$F = f_s + W \sin \theta$$

But
$$f_s = W \sin \theta$$

$$\therefore \quad F = 2 W \sin \theta$$

Now $W =$ Mass of block × Acceleration due to gravity

$$= 5 \times 9.8 = 49 \text{ N}$$

$$\therefore \quad F = 2 \times 49 \times \sin 30^\circ = 2 \times 49 \times \tfrac{1}{2} = 49 \text{ N}$$

We have seen that there is a particular minimum value of the angle of inclination θ_s at which sliding begins. This angle is called the *angle of limiting friction* and its value depends upon the nature of the surfaces in contact. It may be noted that the coefficient of limiting friction μ_s is given by the relation

$$\mu_s = \tan \theta_s$$

5.11 INERTIAL AND NON-INERTIAL FRAMES

In Chapter 3 we have learnt how we can define the position of a particle in two or three dimensions in terms of a system of coordinate axes called a *frame of reference*. The simplest frame of reference is the *Cartesian system of coordinates* in which the position of the particle is specified by its three coordinates x, y and z along the three perpendicular axes. The position vector $\vec{r}$ of the particle with respect to the origin is given by

$$\vec{r} = \hat{\mathbf{i}}\, x + \hat{\mathbf{j}}\, y + \hat{\mathbf{k}}\, z$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along the three axes. The velocity of the particle is

$$\vec{v} = \frac{d\vec{r}}{dt}$$

and its acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

We know that rest and motion are relative; there is nothing like absolute rest or absolute motion. The position or state of motion of a body may appear different from different frames of reference. For example, the passengers and everything else in a moving train are at rest in a reference frame situated in the train but they are in motion in a reference frame situated on the platform. Similarly a stone dropped by a passenger from the window of a railway carriage in uniform motion appears to him to fall vertically downwards, but to a person outside the carriage, it appears to follow a parabolic path, as shown in Fig. 5.22.

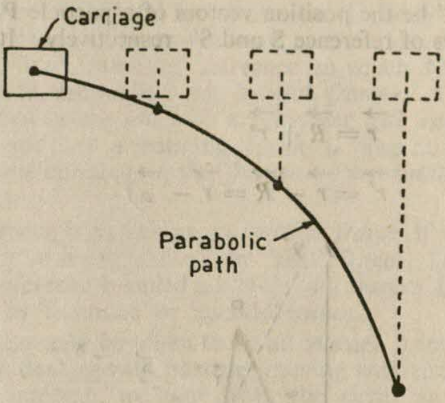


Fig. 5.22

Inertial Frames

Newton's first law of motion is not applicable in all frames of reference. It is often called the law of inertia and the reference frames in which it holds good are called *inertial frames*. Such frames are either fixed with respect to distant stars or moving at a uniform velocity with respect to them.

It is worthwhile to recall that the first law makes no distinction between a body at rest and a body with a constant velocity. Both these motions can exist in the absence of forces. The reason for this is easy to understand. Consider a body at rest in one inertial frame and viewed from a second inertial frame that is moving with a constant velocity with respect to the first frame. An observer in the second frame finds the body to be at rest; an observer in the second frame finds the same body to be moving with a constant velocity. Thus both observers find that the body has no acceleration, i.e. no change in velocity. Both conclude from the first law that no force acts on the body.

We can go a step further and say that newton's second law also holds for such frames of reference, i.e. if a body has a finite acceleration with respect to one inertial frame, it will have the same acceleration in a second inertial frame which is moving with a constant velocity with respect to the first frame.

Suppose we have an inertial frame S and another inertial frame S′ (whose coordinate axes may or may not be parallel) which is moving with a constant velocity $\vec{v}$ with respect to S. Suppose their origins O O′ coincide at time $t = 0$ and move to positions shown in Fig. 5.23 at time t so that $OO' = \vec{R} = \vec{v}\,t$, Let $\vec{r}$ and $\vec{r}'$ be the position vectors of a particle P corresponding to the two frames of reference S and S′ respectively. It is clear from the figure that

$$\vec{r} = \vec{R} + \vec{r}'$$

or

$$\vec{r}' = \vec{r} - \vec{R} = \vec{r} - \vec{v}\,t$$

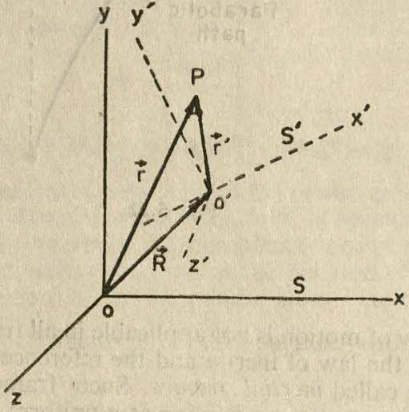


Fig. 5.23

Differentiating with respect to time we have (since $\vec{v}$ is constant)

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}$$

and

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2}$$

i.e. acceleration in frame S′ = acceleration in frame S.

Thus we conclude that *a frame of reference moving with a constant velocity with respect to an inertial frame is also an inertial frame*. The first two laws of motion hold only for such frames.

Non-Inertial Frames

Consider a frame of reference S′ which is not moving with a constant velocity with respect to an inertial frame S but is rotating with respect to it. If a body is at rest in frame S, it will appear to move in a circle in frame S′, i.e. its motion appears to be accelerated in S′. We know that a circular motion is an example of accelerated motion (centripetal acceleration). Thus no force acts on the body in frame S but an unreal or fictitious force (called centripetal force) acts on the body in frame S′. This implies that Newton's first two laws of motion do not hold in the reference frame S′. Such frames of reference in which Newton's first two laws do not hold, are called *non-inertial frames. A non-inertial frame may be defined as one which is accelerating with respect to an inertial frame.* Notice that a rotating frame is also an accelerating frame. Thus we conclude that *inertial frames are non-rotating and non-accelerating frames.*

A frame of reference is known as an *inertial frame* if all accelerations of any particle in it are caused by 'real' forces. On the other hand, a frame of reference is called a *non-inertial frames* if the accelerations are caused by 'fictitious' or 'pseudo' forces.

A distant fixed star may be taken to be an inertial frame. For most practical situations, dealing with particles moving with small velocities and for small time intervals, we may treat the earth as an inertial frame.

SUPPLEMENT

Rocket Propulsion

The propulsion of a rocket is one of the most interesting examples of Newton's third law and momentum conservation. In a rocket, gases at high temperature and pressure, produced by combustion of fuel, are ejected from a nozzle. The reaction of the escaping gases provides the necessary thrust for the launching and flight of the rocket.

The gases are ejected at an almost constant velocity with respect to the rocket which implies that the rate of change of momentum is nearly constant. But the mass of rocket-fuel system keeps on decreasing due to the escaping mass of the gases. Hence the acceleration will not remain constant; it will keep on increasing.

A quantitative estimate of the acceleration of the rocket at any instant of time during its flight can be made as follows.

We will neglect the effect of gravity. Let M_0 be the initial mass of the (rocket + fuel), and let $\frac{\Delta m}{\Delta t}$ be the rate of loss of mass due to escaping gases. After a time interval t, the mass of the rocket will be

$$\left(M_0 - \frac{\Delta m}{\Delta t} \cdot t\right)$$

Let us say that during the next time interval Δt, gases of mass Δm escape with a velocity u with respect to the rocket. We assume that the velocity u remains constant during firing. This mass Δm of escaping gases gives an additional velocity Δv to the rocket while its mass is reduced by an amount Δm. Hence the forward momentum gained by the rocket is

$$\left\{M_0 - \left(\frac{\Delta m}{\Delta t}\right) t - \Delta m\right\} \Delta v$$

and the backward momentum of the gas is $(\Delta m)\, u$. From the principle of momentum conservation, we have

$$\left\{M_0 - \left(\frac{\Delta m}{\Delta t}\right)t - \Delta m\right\}\Delta v = (\Delta m)u$$

If $\Delta m \ll M_0$, this reduces to

$$\left\{M_0 - \left(\frac{\Delta m}{\Delta t}\right)t\right\}\Delta v = (\Delta m)u$$

Dividing both sides by Δt, we have

$$\frac{\Delta v}{\Delta t} = \frac{\Delta m}{\Delta t}\,\frac{u}{\left\{M_0 - \left(\frac{\Delta m}{\Delta t}\right)t\right\}}$$

Now $\frac{\Delta v}{\Delta t}$ is the acceleration of the rocket which, from Newton's second law, should be equal to force divided by mass. The force term, therefore, is $u\left(\frac{\Delta m}{\Delta t}\right)$ which is the product of a constant velocity and the rate of change of mass. The mass term is

$$\left\{M_0 - \left(\frac{\Delta m}{\Delta t}\right)t\right\}$$

which appears in the denominator. Since the rocket keeps ejecting gases, the mass term keeps decreasing with time. Hence, although the force remains constant, the acceleration of the rocket increases due to decrease in mass. Thus the velocity and acceleration of the rocket both increase with time.

SUMMARY

Newton's first law of motion states if no net external force acts on a body, its state of rest or of uniform motion in a straight line cannot change. The tendency of a body to resist any change in its state of motion is known as inertia. The mass of a body is a measure of its inertia.

Newton's second law of motion tells us what happens when a force is actually exerted on a body. It changes the momentum of the body producing an acceleration. The rate of change of momentum is directly proportional to the applied force and the change takes place in the direction in which the force acts.

Newton's third law of motion tells us how forces are exerted. It states that whenever one body exerts a force on a second body, the second body exerts an equal and opposite force on the first, or to every action there is an equal and opposite reaction.

If a system consists of several particles, the total linear momentum of the system is equal to the vector sum of the linear momenta of all the particles of the system. The principle of conservation of linear momentum states that when no net external force acts on a system, the total linear momentum of the system is conserved.

Forces acting at the same point on a body are called concurrent forces. Their resultant is obtained by the method of vector addition.

Impulse of a force is the product of the average force and the time for which the force acts and is equal to the change in momentum of the body during that time. Impulse is a vector and is expressed in kg ms^{-1} or Ns.

Friction always opposes motion. The force of friction between two surfaces in contact depends on the roughness and material of the surfaces and on whether or not there is a relative motion between them. The rolling friction is much less than the sliding friction.

Inertial frames are non-accelerating frames. Accelerations of particles in inertial frames are caused by real forces. Accelerating frames are called non-inertial frames. In such frames, the accelerations of particles are also caused by fictitious forces responsible for accelerating the frames.

EXERCISES

1. State Newton's first law of motion. Why it is called the law of inertia? Give two illustrations.
2. State Newton's second law of motion. How does it help measure force? Define the SI unit of force.
3. State the law of conservation of linear momentum. How will you deduce it from Newton's second law of motion? Give two illustrations of the law.
4. State Newton's third law of motion. How will you deduce it from the law of conservation of linear momentum? Give two examples to illustrate the third law.
5. Define the impulse of a force. Show that the total change in the momentum of a body measures the impulse of a force acting on it.
6. What are concurrent forces? How will you find the resultant of such forces? Explain with the help of an example.
7. Name the different kinds of force we come across in nature.
8. What is frictional force? What is it due to?
9. What do you understand by the term 'limiting friction'? State the laws of limiting friction. How are the laws experimentally verified?
10. Explain what is meant by kinetic friction.
11. 'Friction is a necessary evil'. Comment.
12. What are inertial and non-inertial frames? Explain with the help of examples.

PROBLEMS

1. A rocket with a lift-off mass of 2×10^6 kg developed an initial acceleration of 2 ms^{-2} at lift-off. Calculate the initial thrust developed.

2. A man whose mass is 70 kg is standing on a weighing machine inside a lift. What is the reading in the machine if the lift starts moving with (i) an acceleration of 2.2 ms^{-2}, (ii) a retardation of 4.9 ms^{-2} and (iii) a uniform velocity?
3. A bullet of mass 0.01 kg is fired horizontally with a speed of 200 ms^{-1} into a 4 kg wooden block at rest on a horizontal surface. The bullet remains embedded in the block and the combination moves a distance of 20 m before coming to rest. Calculate the coefficient of kinetic friction between the bullet and block.
4. Two blocks of mass 1 kg and 5 kg are connected by a string and placed on a horizontal smooth surface. The combination is pulled to the right with a force which produces an acceleration of 2 ms^{-2}. Find the force and the tension in the string.
5. A jet engine takes in 50 kg of air per second at a velocity of 50 ms^{-1} and burns 2 kg of fuel per second. If the exhaust gases are ejected at a velocity of 400 ms^{-1} relative to the aircraft, find the thrust developed in the engine.
6. Figure 5.24 shows a block of mass m_1 on a smooth horizontal surface. It is connected to another block of mass m_2 by a string which passes over a frictionless pulley. Find the acceleration of the system and tension in the string in terms of m_1, m_2 and g where g is the acceleration due to gravity.

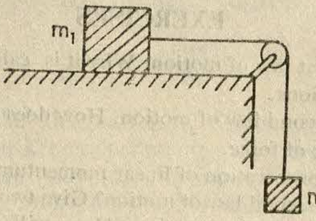


Fig. 5.24

7. Two masses m_1 and m_2 are connected by a string which passes over a frictionless pulley as shown in Fig. 5.25. If $m_2 > m_1$, find the tension in the string and the acceleration of the masses.

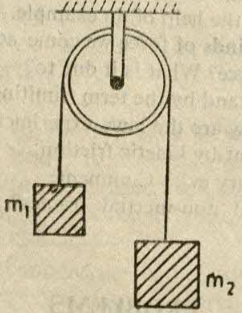


Fig. 5.25

8. A car is moving on a horizontal road with a speed of 20 ms^{-1}. If the coefficient of friction between the tyres and the road is 0.4, what is the shortest distance within which the car can be stopped? Take $g = 10$ ms^{-2}.
9. The head of a hammer moving at a speed of 10 ms^{-1} strikes a nail and drives it into a wooden block, and the duration of the impact is 0.005 s. If the hammer head weighs 1.5 kg, find the average impact force and the distance the nail has penetrated into the wood.
10. A disc of mass 20 g is kept floating horizontally by throwing 20 marbles per second against it from below. Assuming that the marbles strike the disc normally and rebound downwards with the same speed, calculate the speed, with which the marbles strike the disc.
11. Two masses m and 2 m are connected by a massless string which passes over a light frictionless pulley. The masses are initially held with equal lengths of the string on either side and are initially 13.08 m above the ground. Find the velocity of masses at the instant the lighter mass moves up a distance of 6.54 m. The string is suddenly cut at that instant; calculate the velocities with which the two masses hit the ground. Also calculate the time taken by each mass to reach the ground.
12. A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3/2}$
    (a) what force should be applied to the block to that it moves down without any acceleration?
    (b) what force should be applied to the block so that it moves up without any acceleration?
    (c) calculate the ratio of the power needed in the above two cases, if the block moves with the same speed in both the cases.
13. A block of mass 2 kg slides on an inclined plane making an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3/2}$. Calculate the force to be applied to the block so that it moves (a) down and (b) up without any acceleration.
14. A mass of 500 kg is kept in equilibrium on a rough inclined plane of 1 in $\sqrt{5}$ by a horizontal rope. Show that the maximum and minimum tensions in the rope are in the ratio 7 : 3 when the coefficient of friction between the surfaces is 0.2.

6

Work, Energy and Power

6.1 INTRODUCTION

Energy plays an important role in the modern world. The concept of energy, though abstract, is useful in the study of physics, chemistry, geology and biology. There are many forms of energy—heat, light, sound, mechanical, nuclear, electrical, magnetic, etc. In this chapter, we will discuss only mechanical energy.

In normal conversation, we use the words 'energy' and 'work' rather vaguely. For example, we say that a child who runs around a great deal is very 'energetic' or a person who can do a lot of 'work' has a lot of 'energy'. Energy is thus associated with motion. But we also say that food contains (or gives) energy which is used in our bodies so that we can do work. Thus energy is also associated with objects at rest, e.g. food, petrol, coal, etc. Work and energy are two very closely related concepts of vital importance in physics. In physics, work and energy have a precise definition.

6.2 SCALAR PRODUCT OF TWO VECTORS

The scalar (or dot) product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them, i.e.

$$\vec{A} \cdot \vec{B} = AB \cos \theta \tag{6.1}$$

The scalar product is represented by putting a dot between the two vectors. The scalar product of two vectors is a scalar quantity. In order to understand the meaning of a scalar product, we consider two vectors $\vec{A}$ and $\vec{B}$ represented by $\vec{OP}$ and $\vec{OQ}$ respectively and inclined to each other at an angle θ as shown in Fig. 6.1. According to the definition (6.1), we have

$$\vec{A} \cdot \vec{B} = AB \cos \theta = A\,(B \cos \theta)$$

It is clear from Fig. 6.1 that ($B \cos \theta$) is the magnitude of the component of $\vec{B}$ along the direction of $\vec{A}$. Therefore,

$$\vec{A} \cdot \vec{B} = A\,(\mathrm{OL})$$

where $\mathrm{OL} = B \cos \theta$

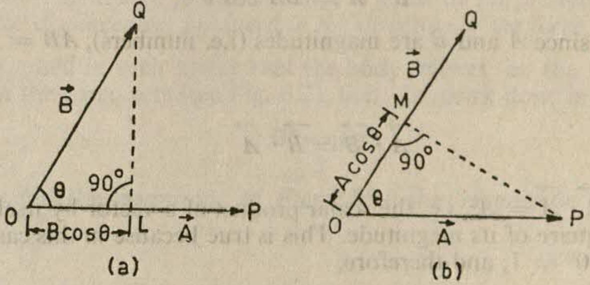


Fig. 6.1 Scalar product of two vectors $\vec{A}$ and $\vec{B}$

Notice that Eq. (6.1) may also be written as

$$\vec{A} \cdot \vec{B} = AB \cos \theta = BA \cos \theta$$

$$= B\,(A \cos \theta)$$

($\because$ A and B are magnitudes, so $AB = BA$)

But ($A \cos \theta$) is the magnitude of the component of $\vec{A}$ along the direction of $\vec{B}$ and is equal to OM (see Fig. 6.1 b)

Therefore,

$$\vec{A} \cdot \vec{B} = B\,(\mathrm{OM})$$

Hence, *the scalar product of two vectors $\vec{A}$ and $\vec{B}$ may be defined as the product of the magnitude of vector $\vec{A}$ and the magnitude of the component of vector $\vec{B}$ in the direction of vector $\vec{A}$ or vice versa.*

Work (see Sec. 6.3) is an example of a physical quantity that is a scalar product of two vector quantities.

Properties of the Scalar Product

(i) *The scalar product is commutative,* i.e. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

This follows from the definition of the scalar product, namely,

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

Therefore,

$$\vec{B} \cdot \vec{A} = BA \cos \theta$$

Now, since A and B are magnitudes (i.e. numbers), $AB = BA$.

Hence,

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

(ii) $\vec{A} \cdot \vec{A} = A^2$, i.e. the scalar product of a vector by itself is equal to the square of its magnitude. This is true because in this case, $\theta = 0°$ and $\cos 0° = 1$, and therefore,

$$\vec{A} \cdot \vec{A} = A \cdot A \cos 0° = A^2$$

or

$$A = \sqrt{\vec{A} \cdot \vec{A}}$$

(iii) If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other, then $\theta = 90°$ and we have

$$\vec{A} \cdot \vec{B} = AB \cos 90° = 0 \qquad (\because \ \cos 90° = 0)$$

Note that $\vec{A} \cdot \vec{B}$ can be zero even when neither $\vec{A}$ nor $\vec{B}$ is zero.

6.3 WORK

In physics work is defined in a specific manner, describing what is accomplished by the action of a force on an object. *Work is said to be done if the force applied on the object succeeds in moving it. If no motion takes place, no work is said to be done.* For example, a horse pulling a cart does work, an engine pulling a train does work, a person lifting a stone does work, a cyclist pedalling a bicycle does work and so on.

However, a person may exert a force and yet do no work, in the specific sense of the word, because the force may not produce any motion. For example, a man pushing against a wall does no work if the wall does not move. He may sweat and tire but he does no work if he does not succeed in moving the wall. Similarly, if you hold a heavy object in your hand and stay motionless, you do no work even though you may get tired holding it. Thus for work to be done, two conditions must be fulfilled.

(a) A force must be exerted, and
(b) The force must produce motion or displacement.

Only that force which gives rise to motion does work.

Measurement of Work

The work done by a force on a body is given by the product of the force and the displacement produced in the direction of the force. If the force $\vec{F}$ is applied in such a way that the body moves in the direction in which the force acts (see Fig. 6.2), then the work done is given by

$$W = FS$$

where F is the magnitude of $\vec{F}$ and S is the magnitude of displacement $\vec{S}$.

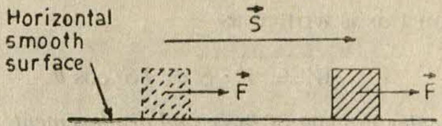


Fig. 6.2

The displacement need not always take place in the direction of the force. For example, consider a force $\vec{F}$ acting in a direction inclined at an angle θ with a horizontal smooth surface along which the body moves (Fig. 6.3a). Let $\vec{S}$ be the displacement along the surface as the body moves from position A to B. The work done by the force can be calculated in the following two equivalent ways.

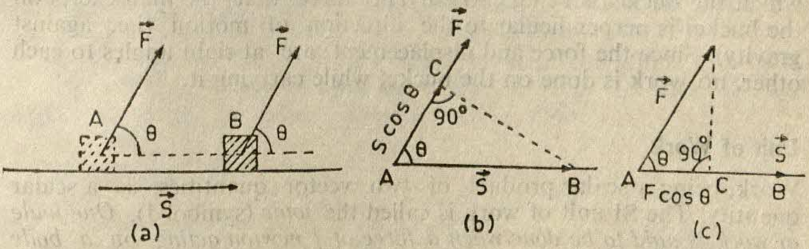


Fig. 6.3 Measurement of work

1. From Fig. 6.3b it is clear that the component of the displacement vector $\vec{S}$ in the direction of the force $= AC = S \cos \theta$. Hence, the work done is given by

$$W = \text{Magnitude of force} \times \text{component of the displacement in the direction of the force vector } \vec{F}$$

$$= F \times AC = F\,(S\cos\theta) \qquad (6.2)$$

2. From Fig. 6.3c it is clear that the component of the force $\vec{F}$ in the direction of the displacement $= AC = F\cos\theta$. Now Eq. (6.2) can also be written as

$$W = (F\cos\theta)\,S$$

$$= \text{component of the force in the direction of the displacement vector } \vec{S} \times \text{Magnitude of the force}$$

Thus, measured either way, work is given by

$$W = FS\cos\theta$$

In vector notation this is written as

$$W = \vec{F}\cdot\vec{S} = FS\cos\theta \qquad (6.3)$$

i.e. *work is the scalar product of force and displacement.*

If $\vec{F}$ and $\vec{S}$ are in the same direction, $\theta = 0$ then $W = FS$. On the other hand, if $\vec{F}$ and $\vec{S}$ are at right angles to each other, $\theta = 90°$, in which case

$$W = FS\cos 90° = 0$$

As an example, consider a man carrying a bucket of water, walking on a level road with a uniform velocity. Since the velocity is uniform, there can be no force in the horizontal direction, i.e. the direction in which the bucket is being carried. The force that the man exerts on the bucket is perpendicular to the direction of motion (i.e. against gravity). Since the force and displacement are at right angles to each other, no work is done on the bucket while carrying it.

Unit of Work

Work, being a scalar product of two vector quantities, is a scalar quantity. The SI unit of work is called the *joule* (symbol J). *One joule of work is said to be done when a force of 1 newton acting on a body moves it through a distance of 1 m in the direction of the force.*

6.4 WORK DONE BY A CONSTANT FORCE

Consider a body acted upon by a constant force $\vec{F}$. Let $\vec{S}$ be the displacement produced in the body by that constant force.

The simplest case is the one in which the displacement occurs in the direction of the force. In this case, the work done by the force on the body is simply given by the product of the magnitudes of the force and the displacement, i.e.

$$W = FS \tag{6.4}$$

However, a constant force acting on a body may not act in the direction in which the body moves (see Fig. 6.3 above). In this case, as explained above, the work is calculated from the scalar product of vectors $\vec{F}$ and $\vec{S}$, i.e.

$$W = \vec{F} \cdot \vec{S} = FS \cos \theta$$

where θ is the angle between the vectors $\vec{F}$ and $\vec{S}$.

When θ is zero, the work done is simply FS ($\because \cos 0° = 1$) in agreement with our previous equation (Eq. 6.4).

6.5 WORK DONE BY A VARIABLE FORCE

The magnitude of the force acting on a body may not always remain constant. Suppose a rocket is fired upwards from the earth. The force required to lift it and make it move away from the earth must be sufficient to overcome the gravitational attraction of the earth which decreases inversely as the square of the distance of the rocket from the earth. Thus the force necessary to keep the rocket moving away from the earth is not constant, it depends upon the position of the rocket with respect to the earth. Similarly, the force required to stretch a spring will depend on the amount of stretching. In such cases the work done by the force is calculated as follows.

Suppose a variable force F acts on a body in the x-direction. The force F is a function of x, i.e. F depends on x. Suppose the force moves the body from a position x_1 to a position x_2. What is the work done by this variable force F in moving the body from x_1 to x_2?

We plot F versus x. Let Fig. 6.4 (a) represent the graph of F versus x. We divide the total displacement into a large number of small elements, each equal to dx, as shown in Fig. 6.4b.

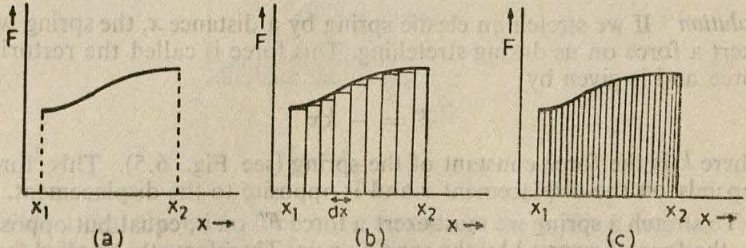


Fig. 6.4 Finding work done by a variable force

During a small displacement dx from x_1 to $x_1 + dx$, the force F can be treated to be nearly constant. Hence the work done by the force F to cause a small displacement dx is

$$dW = F\,dx \tag{6.5}$$

Likewise, during another small displacement dx from $(x_1 + dx)$ to $(x_1 + 2dx)$, the force F is nearly constant and the work done by it is $dW = Fdx$ where F is the value of the force at $x_1 + dx$. The total work done by F in displacing the body from x_1 to x_2 is clearly equal to the sum of a large number of terms like that in Eq. (6.5) in which F has a different value for each term. Thus

$$W = \sum_{x_1}^{x_2} F\,dx$$

The smaller the value of dx, the better will be the result of our calculation of W. Hence the exact result is

$$W = \lim_{dx \to 0} \sum_{x_1}^{x_2} F\,dx$$

In the language of integral calculus, this relation is expressed as

$$W = \int_{x_1}^{x_2} (F\,dx) \tag{6.6}$$

Numerically, this quantity is equal to the area between the force curve and the x-axis between the values x_1 and x_2. This area is shown shaded in Fig. 6.4c.

If the body is moved from $x_1 = 0$ to $x_2 = x$, a displacement of x, then the work done by the variable force F to cause a displacement x is given by

$$W = \int_0^x (Fdx) \tag{6.7}$$

EXAMPLE 6.1 One end of an elastic spring of force constant k is fixed to the wall and the other end is attached to a block lying on a horizontal frictionless surface. The mass is pulled to the right (along the x-direction) by a distance x. Find the expression for the work done.

Solution If we stretch an elastic spring by a distance x, the spring will exert a force on us during stretching. This force is called the restoring force and is given by

$$F = -kx$$

where k is the force constant of the spring (see Fig. 6.5). This force depends on the displacement x and is opposite to the displacement.

To stretch a spring we must exert a force F' on it, equal but opposite to the force F exerted by the spring on us. Therefore, the applied force

$$F' = kx$$

which depends upon x. The work done by the applied force in stretching the spring through a distance x is given by

$$W = \int_0^x F'\,dx = \int_0^x (kx)\,dx$$

$$= \tfrac{1}{2}kx^2 \Big|_0^x = \tfrac{1}{2}kx^2$$

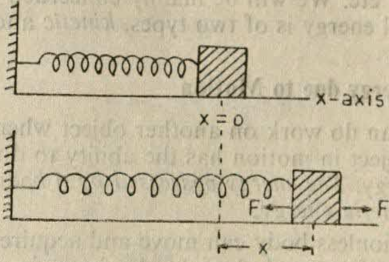


Fig. 6.5

We could also evaluate this work by finding the area under the force-displacement curve from $x = 0$ to $x = x$. This is shown shaded in Fig. 6.6. Notice that the curve of F' versus x is a straight line (since $F' = kx$). The area of the triangle of base x and altitude kx

$$= \tfrac{1}{2} \times \text{base} \times \text{altitude}$$

$$= \tfrac{1}{2} \times x \times kx$$

$$= \tfrac{1}{2}kx^2$$

in agreement with our result above.

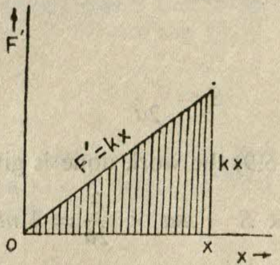


Fig. 6.6

It is evident that the work done in compressing the spring by an amount x is also given by $W = \tfrac{1}{2}kx^2$. This work is stored in the spring in the form of potential energy.

6.6 ENERGY

Energy can be defined as the capacity or ability to do work and is measured by the amount of work a body can do. So, energy is measured in the same units as work, namely, joule. Like work, energy is a scalar quantity.

Energy can exist in various forms, such as heat energy, electrical energy, sound energy, light energy, chemical energy, nuclear energy, mechanical energy, etc. We will be mainly concerned with mechanical energy. Mechanical energy is of two types, *kinetic* and *potential.*

Kinetic Energy: Energy due to Motion

A moving object can do work on another object when it strikes it. In other words, an object in motion has the ability to do work and, by definition, has energy. *The energy possessed by a body by virtue of its motion is called kinetic energy.*

An initially motionless body can move and acquire a velocity only if a force acts on it. The work done by the force in causing the body to move measures the kinetic energy (written as KE) of the moving body, i.e.

$$\text{KE} = W$$

Consider a body of mass m at rest at time $t = 0$, its initial velocity $u = 0$. Let a constant force F act on it for time t.

The body acquires a velocity v and suffers a displacement S in the direction of the force. Newton's second law gives

$$F = ma \tag{6.8}$$

where a is the constant acceleration. The distance S moved is given by

$$2aS = v^2 - u^2$$

Since $u = 0$, we have

$$S = \frac{v^2}{2a} \tag{6.9}$$

Usings Eqs (6.8) and (6.9) the work done is given by

$$W = F \times S = ma \times \frac{v^2}{2a} = \tfrac{1}{2}\, mv^2$$

or

$$\text{KE} = \tfrac{1}{2}\, mv^2 \tag{6.10}$$

Thus *the kinetic energy of a moving body is given by one-half the product of its mass and the square of its velocity.*

Although this result has been obtained for a constant force, it also holds if the force is variable. Let a variable force F act on a body and let S be the displacement produced in the direction of the force. Then the work done by the force (which measures the kinetic energy of the body) is given by

$$W = \text{KE} = \int F\,dS$$

$$= \int ma\,dS \qquad (\because F = ma)$$

$$= \int m\,\frac{dv}{dt}\,dS \qquad \left(\because a = \frac{dv}{dt}\right)$$

$$= \int m\left(\frac{dv}{dS}\cdot\frac{dS}{dt}\right)dS$$

$$= \int mv\,\frac{dv}{dS}\,ds \qquad \left(\because v = \frac{dS}{dt}\right)$$

$$= \int mv\,dv$$

or
$$\text{KE} = m\int v\,dv \qquad (\because m = \text{constant})$$

If the velocity of the body increases from zero to a value v, we have

$$\text{KE} = m\int_0^v v\,dv = \tfrac{1}{2}\,mv^2$$

which agrees with the expression for KE obtained above for a constant force.

Note: The expression $\text{KE} = \frac{1}{2}\,mv^2$ has been derived for a constant force and for a force having a variable magnitude. It may be remarked that this expression holds even for a more general case in which the force varies both in magnitude and direction. This expression, therefore, holds regardless of the way the body reaches its final velocity.

Work-Energy Principle

Suppose a body of mass m moves with an initial velocity u. A force F acts on it, as a result of which it acquires a final velocity v. The work done by the force is given by

$$W = m\int_u^v v\,dv = \tfrac{1}{2}\,m\,(v^2 - u^2)$$

$$= \tfrac{1}{2}\,mv^2 - \tfrac{1}{2}\,mu^2$$

$$= \text{final KE} - \text{initial KE}$$

$$= \text{change in KE}$$

Thus, *the work done by a force in displacing a body measures the change in its kinetic energy*. This is the work-energy principle.

Thus, when a force does work on a body, its kinetic energy increases; the increase in kinetic energy being equal to the amount of work done. The converse of this is also true. When the kinetic energy of a body is decreased by a retarding force, the decrease is equal to the work done by the body against the retarding force. Thus kinetic energy and work are equivalent quantities and are, therefore, measured in the same units, namely, joule.

Potential Energy: Energy due to Position or Configuration

An object can have energy not only by virtue of its motion, but also because of its position or configuration. *The energy possessed by a body owing to its position or configuration is called potential energy.* For example, a wound watch spring has potential energy on account of its wound state or configuration of the coils. As the spring unwinds, it does work to move the hands of the watch. Thus, a wound spring has the *potentiality* to do work.

Gravitational Potential Energy An object held at a position above the surface of the earth has potential energy by virtue of its position. When it falls from that position, it can do work. The potential energy of an object held above the earth is called *gravitational potential energy*. To calculate the energy stored in a body which has been lifted above the earth's surface against the gravitational force, we have to calculate the amount of work done in carrying it there.

Consider a body of mass m. It is lifted vertically to a height h above the earth by applying a force F vertically upward. The force F must be just enough to overcome the gravitational attraction, i.e.

$$F = mg$$

where g is the acceleration due to gravity at that place. For bodies not too far above the surface of the earth, the values of g is practically constant. Hence the work done by a constant force F in displacing a body by a height h can be calculated by the product $F \times h = mgh$. Thus gravitational potential energy of a body of mass m at a height h above the surface of the earth is mgh. The gravitational potential energy on the surface of the earth is taken to be zero.

Potential Energy of a Spring When a spring is stretched or compressed by applying a force, the work done in stretching or compressing it is stored in the spring in the form of potential energy. In Example 6.1 we have calculated the amount of work done in stretching or compressing a spring by an amount x; it is given by

$$W = \tfrac{1}{2}\,kx^2$$

where k is the force constant of the spring. Hence the potential energy of the spring is $\frac{1}{2}\,kx^2$,

6.7 POWER

The rate of doing work is called power, i.e.

$$\text{Power} = \frac{\text{Work}}{\text{Time}}$$

The faster a given amount of work is done, the greater is the power of the agent that does the work.

In the SI system, the unit of power is the *watt* (symbol W). *Power is said to be 1 watt when 1 joule of work is done in 1 second*, i.e.

$$1\ \text{W} = 1\ \text{Js}^{-1}$$

Since the watt is a small unit for the measurement of power, larger units, namely kilowatt (kW) and megawatt (MW) are often used

$$1\ \text{kW} = 1000\ \text{W} = 10^3\ \text{W}$$

$$1\ \text{MW} = 1{,}000{,}000\ \text{W} = 10^6\ \text{W}$$

The power of an agent can also be expressed is terms of the force applied and the velocity of the object on which the force is applied. Now, power P is given by

$$P = \frac{W}{t} = \frac{\vec{F}\cdot\vec{S}}{t}$$

$$= \vec{F}\cdot\vec{v}\ \left(\because\ \frac{\vec{S}}{t} = \text{rate of change of displacement} = \vec{v}\right)$$

Power is a scalar quantity as it is the ratio of two scalars W and t, or a scalar product of two vectors $\vec{F}$ and $\vec{v}$.

EXAMPLE 6.2 A bullet of mass 0.01 kg, moving horizontally with a velocity of 200 ms^{-1}, strikes a 4 kg wooden block placed at rest on a horizontal surface. The bullet is embedded in the block. If the coefficient of kinetic friction between the block and the surface is 0.25, find the distance moved by the combination before coming to rest.

Solution

Mass of the block = 4 kg

$\because$ Weight of the block (W) = $4 \times 9.8 = 39.2$ N

$$\mu_k = 0.25$$

Now $$\mu_k = \frac{f_k}{R}$$

But Normal reaction $R = W = 39.2$ N

$\therefore$ The force of friction $(f_k) = \mu_k R = 0.25 \times 39.2$

$= 9.8$ N

Now the KE of the bullet $= \frac{1}{2} mv^2$

$= \frac{1}{2} \times 0.01 \times (200)^2$

$= 200$ J

Since the bullet is brought to rest, the change in its KE

$= 200$ J

But Change in KE = Work done by the force of friction against the block

$=$ Force × Distance $= f_k \times S$

or $$S = \frac{200}{9.8} = 20.4 \text{ m}$$

EXAMPLE 6.3 A body of mass 5 kg is moving on a horizontal smooth surface at a velocity of 10 ms^{-1}. A constant force of 20 N is applied on the body for 10 s in the direction of the velocity. Find (a) the work done by the force, (b) the increase in the kinetic energy of the body and (c) the power of the agent that exerts the force.

Solution

Given
$m = 5$ kg
$F = 20$ N
$t = 10$ s
$u = 10 \text{ ms}^{-1}$

(a) The acceleration of the body is

$$a = \frac{F}{m} = \frac{20 \text{ N}}{5 \text{ kg}} = 4 \text{ ms}^{-2}$$

The distance travelled by the body is given by

$$S = ut + \tfrac{1}{2} at^2$$
$$= 10 \times 10 + \tfrac{1}{2} \times 4 \times (10)^2$$
$$= 300 \text{ m}$$

$\because$ Work done $W = F \times S$

$= 20 \text{ N} \times 300 \text{ m}$

$= 6000$ J

(b) The final velocity of the body is

$$v = u + at$$
$$= 10 + 4 \times 10 = 50 \text{ ms}^{-1}$$

$$\therefore \quad \text{Final KE} = \tfrac{1}{2}\, mv^2 = \tfrac{1}{2} \times 5 \times (50)^2$$
$$= 6250 \text{ J}$$
$$\text{Initial KE} = \tfrac{1}{2}\, mu^2 = \tfrac{1}{2} \times 5 \times (10)^2$$
$$= 250 \text{ J}$$
$$\therefore \quad \text{Increase in KE} = 6250 - 250 = 6000 \text{ J}$$

Note that Increase in KE = Work done

(c) Power
$$P = \frac{W}{t}$$
$$= \frac{6000 \text{ J}}{10 \text{ s}} = 600 \text{ W}$$

EXAMPLE 6.4 An engine pulls a car of mass 1500 kg on a level road at a constant velocity of 5 ms^{-1}. If the frictional force is 500 N, what power does the engine generate? What extra power must the engine develop to maintain the same speed up a plane incline having a gradient of 1 in 10?

Solution

Given
$$\text{Mass of car } (M) = 1500 \text{ kg}$$
$$\text{Velocity } (v) = 5 \text{ ms}^{-1}$$
$$\text{Frictional force } (f) = 500 \text{ N}$$

Since the car moves at a constant velocity, its acceleration is zero. Hence the engine has to do work only to overcome the frictional force (f). Since the distance moved in 1 second is v metres, the work done per second or the power of the engine is

$$P = f \times v$$
$$= 1500 \times 5$$
$$= 7500 \text{ W} = 7.5 \text{ kW}$$

When the car is being pulled along an inclined plane of gradient 1 in 10 $\left(\text{(i.e. } \sin\theta = \frac{1}{10}\right)$, the engine has to do extra work against the component $Mg \sin\theta$ of the weight Mg of the car (Fig. 6.7). The extra work per second or extra power the engine has to develop to maintain the same speed v is

$$= Mg \sin\theta \times v$$

$$= 1500 \times 9.8 \times \frac{1}{10} \times 5$$

$$= 7350 \text{ W}$$

$$= 7.35 \text{ kW}$$

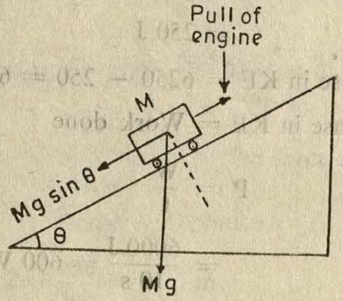


Fig. 6.7

EXAMPLE 6.5 A particle of mass m has half the kinetic energy of another particle of mass $m/2$. If the speed of the heavier particle is increased by 2.0 ms^{-1}, its new kinetic energy equals the original kinetic energy of the lighter particle. Calculate the original speeds of the two particles.

Solution Let v and v' be the original speeds of the heavier and the lighter particles respectively. We then have

$$\tfrac{1}{2} m v^2 = \tfrac{1}{2} \times \left\{ \tfrac{1}{2} \left(\frac{m}{2}\right) v'^2 \right\}$$

$$\therefore \qquad v^2 = \tfrac{1}{4} v'^2$$

or $$v' = 2v$$

When the heavier particle is speeded up by 2.0 ms^{-1}, its kinetic energy becomes $\frac{1}{2} m (v + 2)^2$. Since this equals the original kinetic energy of the lighter particle, we have

$$\tfrac{1}{2} m (v + 2)^2 = \tfrac{1}{2} (m/2) (4 v^2)$$

$$v^2 + 4 + 4v = 2v^2$$

or $$v^2 - 4v - 4 = 0$$

$$v = \frac{4 \pm \sqrt{16 + 16}}{2} = \frac{4 \pm 2\sqrt{8}}{2}$$

$$= 3.24 \text{ ms}^{-1}$$

$$\therefore \qquad v' = 2v = 6.48 \text{ ms}^{-1}$$

Thus the original speeds of the heavier and the lighter particles are 3.24 ms^{-1} and 6.48 ms^{-1} respectively.

EXAMPLE 6.6 Two identical cylindrical vessels with their bases at the same level, each contain a liquid of density ρ. The height of the liquid in one vessel is h_1 and that in the other is h_2. The area of either base is A. What is the work done by gravity in equalizing the levels when the vessels are interconnected?

Solution The work done by gravity equals the change in the potential energy of the system after the vessels are interconnected. We may regard the liquid in each vessel as equivalent to a point mass kept at their respective centres of gravity. Remembering that the mass of the liquid is given by $(A\ h\ \rho)$ and that the PE of a mass at a height h in earth's gravity is mgh, we have

$$\text{Total PE at start} = (A\ h_1\ \rho)\ g\,\frac{h_1}{2} + (A\ h_2\ \rho)\,g\,\frac{h_2}{2}$$

$$= \frac{A\ \rho\ g}{2}\,(h_1^2 + h_2^2)$$

After the vessels are connected, the height of liquid in each vessel is $(h_1 + h_2)/2$. Hence

$$\text{PE after connection} = \left\{A\ \rho\left(\frac{h_1 + h_2}{2}\right) g \left(\frac{h_1 + h_2}{2}\right)\right\}$$

$$= \frac{A\ \rho\ g}{4}\,(h_1 + h_2)^2$$

$$\text{Change in PE} = \frac{A\ \rho\ g}{4}\,\{(h_1 + h_2)^2 - 2\,(h_1^2 + h_2^2\}$$

$$= -\ \frac{A\ \rho\ g}{4}\,(h_1 - h_2)^2$$

$$= -\ A\,\rho\,g\left(\frac{h_1 - h_2}{2}\right)^2$$

This must be equal to the work done 'by' gravity on the liquid. Thus the work done 'by' gravity is

$$A\,\rho\,g\,\left(\frac{h_1 - h_2}{2}\right)^2$$

EXAMPLE 6.7 A gun weighing 5 kg fires a bullet of mass 10^{-2} kg with a speed of $5 \times 10^2\ \text{ms}^{-1}$. Obtain the kinetic energy of the bullet and the gun. Also obtain the ratio of the distance the gun moves backward (while the bullet is in the barrel) to the distance moved by the bullet.

Solution

$$K_B = \text{Kinetic energy of bullet} = \tfrac{1}{2}\,m_b \times v_b^2$$

Given

$$m_b = 10^{-2}\text{ kg}, \qquad v_b = 5 \times 10^2\text{ ms}^{-1}$$

$$\therefore \quad K_B = \tfrac{1}{2} \times 10^{-2} \times (5 \times 10)^2$$

$$= 12.5 \times 10^2\text{ J}$$

When the bullet is fired, the gun recoils with a speed v_g. From the law of conservation of linear momentum, we have

$$0 = m_b\, v_b + m_g\, v_g$$

$$v_g = -\, m_b\, v_b / m_g$$

$$= -\,\frac{10^{-2} \times 5 \times 10^2}{5}\text{ ms}^{-1}$$

$$= -\ 1\text{ ms}^{-1}$$

$$K_G = \text{kinetic energy of the gun}$$

$$= \tfrac{1}{2}\, m_g\, v_g^2$$

$$= \tfrac{1}{2} \times 5 \times (1)^2\text{ J}$$

$$= 2.5\text{ J}$$

Considering the bullet and the gun to constitute a single system, the centre of mass of the system is at rest and the linear momentum is conserved. Therefore,

$$0 = m_g\, v_g + m_b\, v_b$$

$$= m_g \frac{\Delta x_g}{\Delta t} + m_b \frac{\Delta x_b}{\Delta t}$$

where Δx_g and Δx_b stand for the distance moved by the gun and the bullet respectively. Hence

$$\left|\frac{\Delta x_g}{\Delta x_b}\right| = \frac{m_b}{m_g}$$

$$= \frac{10^{-2}}{5} = 2 \times 10^{-3}$$

6.8 TRANSFORMATION OF ENERGY

As mentioned earlier, energy exists in many forms. The various sources of energy involve transformation of one form of energy into another. A steam engine changes heat energy into mechanical energy. In an electric heater, electrical energy is converted into heat energy. A loudspeaker converts electrical energy into sound energy. In an electric bulb, electrical energy is converted into light energy. When we strike a match, the chemical energy of the chemicals is converted into heat and light.

Mass Energy Equivalence

When the nucleus of a heavy atom such as uranium is split up into nuclei of smaller size, the total mass of the components is slightly less than the mass of the parent nucleus. The mass difference is converted into energy in accordance with Einstein's *energy-mass equivalence relation*,

$$E = mc^2$$

where c is the speed of light. This process is called *nuclear fission*. Since c is large ($3 \times 10^8 \text{ ms}^{-1}$) and c^2 even larger, a small mass difference can produce enormous energy. In a reactor, electricity is generated by using the energy released in nuclear fission.

There is a similar loss of mass when two light nuclei, namely the nuclei of hydrogen atoms, fuse (or combine) to form a heavier nucleus. This process is called *nuclear fusion*. The energy emitted by the sun is due to nuclear fusion.

With the advancement in the knowledge of the atomic theory of matter during the last century, scientists have come to the conclusion that all forms of energy mentioned above are simply kinetic energy and potential energy at the atomic level. For example, according to the atomic theory, heat energy is merely the kinetic energy of the molecules (see Chapter 11). Since any form of energy is basically either potential or kinetic, a transformation from one form to another essentially involves a change of kinetic energy into potential energy or vice versa.

6.9 CONSERVATION OF ENERGY

Whenever energy is transformed from one form to another, it is found that no energy is lost in the process. This is illustrated by the following simple example.

Let a body of mass m be lifted from the ground to a position A so that it is at a height h above the ground (Fig. 6.8). The potential energy of the body at A is mgh and kinetic energy is zero, since it is at rest at A. Therefore, the total energy at A is mgh. When the body is allowed to fall, it begins to move. Let us calculate its energy when it is at position C so that it has fallen through a distance x and is at a height $(h-x)$ above the ground at a certain instant of time. At this position, the potential energy is $mg(h-x)$. Let v be the velocity when the body reaches C, then from the relation $v^2 = u^2 + 2gS$, we have (since $u = 0$) $v^2 = 2gx$. Thus, the kinetic energy at C is $\frac{1}{2}mv^2 = \frac{1}{2}m(2gx) = mgx$. Therefore the total energy at C is PE + KE $= mg(h-x) + mgx = mgh$, the same as that at A.

When the body reaches the ground, $x = h$. The potential energy at B is 0 and kinetic energy is $mgx = mgh$. Therefore, the total energy at B is mgh. Thus, although both the kinetic and potential energies change throughout the motion, the sum of the two remains constant.

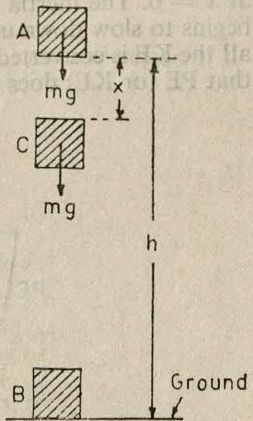


Fig. 6.8 Conservation of energy

This is depicted in Fig. 6.9 which shows the plots of KE and PE of a freely falling body. The thin continuous line shows the variation of KE with height. The variation of PE is shown by the broken line. The bold line shows the total energy (KE + PE) of a freely falling body; it remains constant at *mgh*.

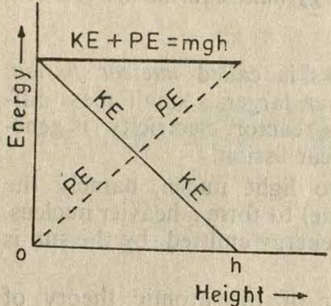


Fig. 6.9 Plots of KE and PE during the free fall of a body

The most important application of the conversion of PE into KE of falling bodies is in the generation of hydroelectric power. The large amount of water stored at a height in a dam has enormous gravitational PE. The water is made to fall on the blades of a turbine at the bottom of the dam. The KE of the falling water rotates the blades of the turbine at a very high speed. Then the mechanical energy of the rotating blades is converted into electrical energy by the generator.

In this section we have discussed the gravitational potential energy near the surface of the earth. In Chapter 8 we will discuss a more general form of gravitational potential energy.

We can also draw the PE and PE plots for an elastic spring. We have seen that the PE of a spring stretched or compressed by an amount x is given by $\frac{1}{2}kx^2$ where k is the force constant of the spring. Refer to Fig. 6.5 again. When the spring is stretched by an amount x, all its energy is PE $= \frac{1}{2}kx^2$. When it is released, it begins to resume its normal length under a restoring force. As a result the block begins to move until it reaches the position $x = 0$ where it has the maximum velocity. All the PE is now converted into KE. The block will not stop at $x = 0$. The inertia of its motion will take it to the left of $x = 0$. It begins to slow down until it momentarily stops at a position $-x$, where all the KE is converted back into PE. This is shown in Fig. 6.10. Notice that PE (or KE) does not vary linearly with x; it is proportional to x^2.

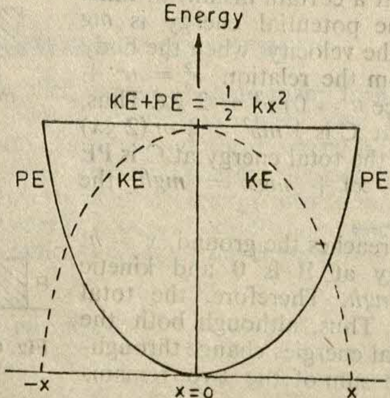


Fig. 6.10 Plots of KE and PE for an elastic spring

These simple examples illustrate one of the most important principles of physics, the *law of conservation of energy which states that energy is neither created nor destroyed, it can simply be transformed from one form to another, the total amount of energy remaining constant.*

This law has been found to be valid in every experimental situation in which it is tested. No violation of this law has yet been observed. What makes the concept of energy so important is the fact that it is conserved. Our universe is in a state of constant flux. In very complex processes (e.g. interactions), there is something that remains constant— the energy. The law of conservation of energy is the one unifying principle that brings order to the jumble of facts concerning the natural processes in our universe.

More Examples of Conservation of Energy

1. Figure 6.11 shows a spherical steel ball (such as a ball bearing) placed at the periphery of a large concave mirror or watch glass which is firmly embedded in plastic or modelling clay. If the mass of the ball is m, its potential energy when held at A is mgh. When the ball is released, it begins to move thus acquiring kinetic energy. When it reaches the bottom of the mirror, i.e. B, its energy is entirely kinetic. As it moves towards the point C, it gradually loses its kinetic energy and when it reaches C its energy is entirely potential, equal to mgh, i.e. the energy it had when it started from A.

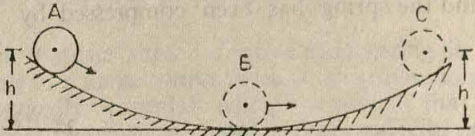


Fig. 6.11

2. When a simple pendulum is pulled to one side and released, it swings from one side to the other, reaching equal heights (= h) on

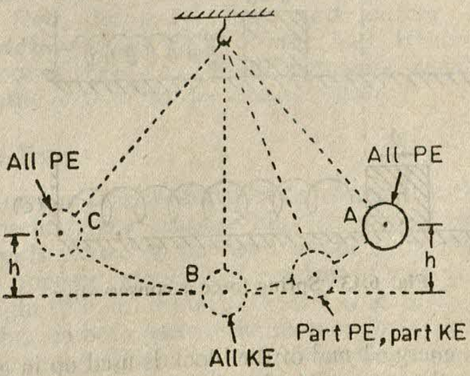


Fig. 6.12

either side (Fig. 6.12) as long as the friction is negligible. At the lowest point B, it has the maximum speed and hence the maximum kinetic energy. At the maximum height (A and C) of the swing on either side it has no kinetic energy (since it is momentarily at rest there) and the maximum potential energy. At intermediate positions, i.e. between A and B or B and C, it has both kinetic and potential energy, the sum of which remains constant throughout the swing.

In the above examples, we have neglected friction which is always present in real situations. When a body, e.g. a ball bearing or a pendulum, moves in a medium, it loses a part of its kinetic energy which appears as heat energy in the medium. Thus all the kinetic energy is not converted into potential energy. The law of conservation includes heat energy as well.

6.10 CONSERVATIVE AND NON-CONSERVATIVE FORCES

Conservative Forces

Consider an elastic spring, one end of which is fixed to the wall as shown in Fig. 6.13a. Suppose a block of mass m is directed towards the other end of the spring with a velocity v. Assume that the block moves on a horizontal frictionless surface.

When the block touches the spring, its kinetic energy does work on the spring. The kinetic energy of the block decreases until it finally comes to rest and the spring has been compressed by an amount x (Fig. 6.13b).

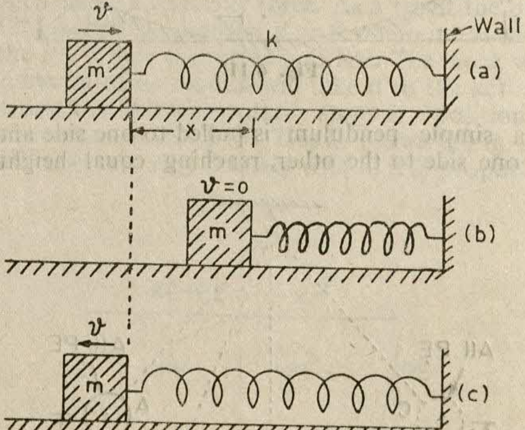


Fig. 6.13 Spring force is conservative

The kinetic energy $\frac{1}{2}mv^2$ of the block is used up in giving potential energy $\frac{1}{2}kx^2$ to the spring. The block now reverses its motion as the compressed spring expands. It gains kinetic energy and regains its initial position (Fig. 6.13c); only the direction of its motion is reversed.

The block loses kinetic energy during the first part of its motion and gains back all its kinetic energy during the second part of its motion. Since the block has regained all its kinetic energy (i.e. there is no change in its kinetic energy) in the course of the round trip, the spring has done no net work on the block. This can also be understood as follows.

During the first part of the motion of the block (Fig. 6.13b), i.e. when the spring was being compressed, the work done by the spring was negative, because the reaction force exerted by the spring on the block (directed to the left) was directed opposite to the displacement of the block (directed to the right). However, during the second part of the motion of the block (Fig. 6.13c), i.e. when the spring is expanding, the work done by the spring on the block is positive, because now the force of the spring (directed to the left) and the displacement of the block (which is also directed to the left) are in the same direction.

Since the block regains all its kinetic energy during its round trip, the negative work done by the spring in the first part of the trip must be equal to the positive work it does in the second part. Hence the net work the spring does in the entire process is zero. The ability of the block to do the work (i.e. its KE) remains the same; it is conserved.

Such a force is called a conservative force. *A force is conservative if the work done by the force on a particle that moves through any round trip is zero.*

We have seen that the force of an elastic spring is conservative. Another example of a conservative force is the gravitational force. It we throw a ball vertically upward, it will (if we assume that the air resistance is negligible) return to our hand with the same kinetic energy (i.e. the same velocity) that it had when it left our hand. During the upward journey of the ball, the work done by the force of gravity on the ball is negative, because the ball is moving upward, whereas the force of gravity is directed downwards. But during the downward journey of the ball, the force of gravity and the motion of the ball are both directed downwards. Thus, during the downward journey, the force of gravity does an equal positive work on the ball. Hence the net work done by the force of gravity on the ball during the round trip is zero. This means that the *force of gravity is conservative.*

Non-conservative Forces

To illustrate a non-conservative force let us assume that the surfaces of the block and the horizontal plane in Fig. 6.13 are not smooth so that a force of friction is exerted on the block by the horizontal plane. The frictional force opposes motion independent of whether the block is moving to the right (Fig. 6.13b) or to the left (Fig. 6.13c). The block overcomes friction in both parts of its motion during which it loses a part of its kinetic energy which appears as heat. Consequently the block returns to its original position with less kinetic energy than it had initially. The work done by the frictional force on the block is negative during both parts of its motion. Thus the frictional force does a net non-zero work on the block during its round trip. Such a force is called a non-conservative force. *A force is non-conservative if the work done by the force on a particle that moves through any round trip is not zero.* Thus *frictional forces are non-conservative.* Another example of a non-conservative force is the induction force in a cyclotron. Instead of dissipating energy, this force generates kinetic energy so that an electron moving in a circular path in a cyclotron will return to its original position with more kinetic energy than it originally had. In a round trip the electron gains kinetic energy which is precisely the function of a cyclotron (or particle accelerator).

The difference between conservative and non-conservative forces can also be understood in the following equivalent way.

Consider a body of mass *m* being raised to a height *h* vertically upward from position A to position E as shown in Fig. 6.14a. The work done is *mgh*.

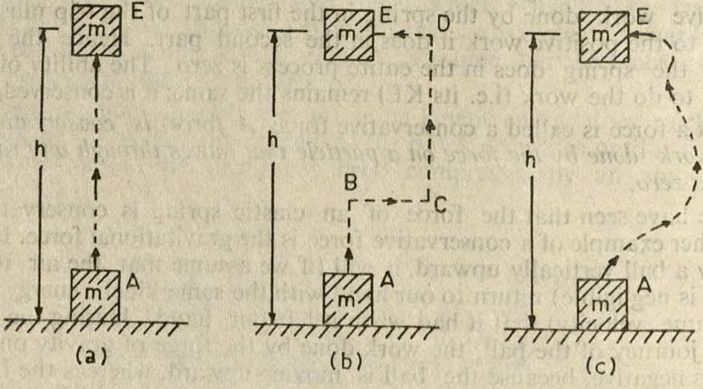


Fig. 6.14 Gravitational force is conservative

Now suppose the body is raised to the same height by taking it along the path ABCDE as shown in Fig. 6.14b. During the horizontal motions B to C and D to E, the force of gravity is at right angles to the displacement. Hence no work is done by gravity in moving the body from B to C and from D to E. The force of gravity does work only during the vertical paths A to B and C to D. Thus the net work done is still *mgh*.

The work done when the body is taken along any arbitrary path such as the one shown in Fig. 6.14c will also be *mgh*. The reason is that any path can be broken up into elementary horizontal and vertical portions. The work done along horizontal parts is zero and the work done along vertical parts adds upto *mgh*. Thus we find that the work done in raising a body against gravity is independent of the path taken and depends only on the initial and final positions of the body.

If the body is brought back to its initial position, the net work done by gravity is obviously zero.

If the work done by a force depends only on the initial and final positions of the body on which it is acting, then such a force is called a *conservative force*. Thus gravitational force is conservative. So are electrostatic and elastic spring forces.

Frictional forces, on the other hand, are non-conservative forces. If a block is moved from a position A to a position B on a rough table, the work done against frictional force will depend on the length of the path between A and B and not on the positions A and B. If the block is brought back to its initial position A, the net work done against the frictional force will not be zero.

Hence we have another equivalent definition of conservative and non-conservative forces. *A force is conservative if the work done by it on a particle that moves between two points depends only on these points and not on the path followed. A force is non-conservative if the work done by that force on a particle that moves between two points depends on the path taken between those points.*

EXAMPLE 6.8 A simple pendulum of length 1 m has a bob of mass 100 g. It is displaced from its mean position A to a position B so that the string makes an angle of 45° with the vertical. Calculate the velocity of the bob when (a) it passes through the mean position A and (b) it makes an angle of 30° with the vertical. Neglect air resistance and take $g = 10\ \mathrm{ms^{-2}}$.

Solution (a) Let A be the mean position of the bob of pendulum (Fig. 6.15). It is displaced to a position B so that $\theta = 45°$. At this position, let h_1 be the vertical height through which the bob has been raised against the force of gravity.

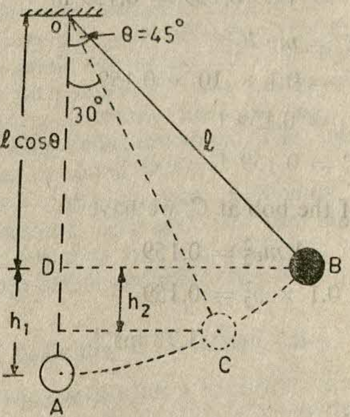


Fig. 6.15

It is clear from the figure that

$$OD = l \cos \theta$$

Hence $$h_1 = l - l \cos \theta = l(1 - \cos \theta)$$

Now $$l = 1 \text{ m and } \theta = 45°$$

$\therefore$ $$h_1 = 1\,(1 - \cos 45°) = 0.293 \text{ m}$$

$\therefore$ $$\begin{aligned} \text{PE at B} &= mg\,h_1 \\ &= 0.1 \times 10 \times 0.293 \\ &= 0.293 \text{ J} \end{aligned}$$

Since air resistance is neglected, all the potential energy will be converted into kinetic energy when the bob passes through the mean position A. Hence

$$\text{KE at A} = 0.293 \text{ J}$$

If v_1 is the velocity of the bob when it reaches A, we have

$$\tfrac{1}{2}\,mv_1^2 = 0.293$$

or $$\tfrac{1}{2} \times 0.1 \times v_1^2 = 0.293$$

giving $$v_1 = \sqrt{\frac{0.293 \times 2}{0.1}} = 2.42 \text{ ms}^{-1}$$

(b) when the pendulum makes an angle of 30° with the vertical (position C), the vertical height through which it falls while moving from B to C is

$$\begin{aligned} h_2 &= l\cos 30° - l\cos 45° \\ &= l\,(0.866 - 0.707) \\ &= 1 \times 0.159 = 0.159 \text{ m} \end{aligned}$$

$\therefore$ $$\begin{aligned} \text{Loss in PE} &= mg\,h_2 \\ &= 0.1 \times 10 \times 0.159 \\ &= 0.159 \text{ J} \end{aligned}$$

$\therefore$ $$\text{KE at C} = 0.159 \text{ J}$$

If v_2 is the velocity of the bob at C, we have

$$\tfrac{1}{2}\,mv_2^2 = 0.159$$

or $$\tfrac{1}{2} \times 0.1 \times v_2^2 = 0.159$$

which gives $$v_2 = 1.78 \text{ ms}^{-1}$$

6.11 COLLISIONS

We are familiar with many examples of collisions. We have seen two carrom coins or billiard balls colliding with each other or an autombile colliding with another. The molecules of a gas collide with one another. The collision of a neutron with an atom is well-known.

In physics the term *collision* does not necessarily mean that a particle (or a body) actually strikes another. In fact, two particles may not even touch each other and yet they are said to collide if one particle influences the motion of the other. For example, the negatively charged alpha-particles speeding towards the negatively charged nucleus of gold are simply deflected by the electro-static force of repulsion. In physics *a collision between two particles (or bodies) is said to occur if they physically strike (or collide) against each other or if the path of the motion of one is influenced by the other.*

When two bodies collide (e.g. when a bat strikes a ball), each exerts a force on the other. The two forces are exerted simultaneously and for an equal but short interval of time. According to Newton's third law of motion, each body exerts an equal and opposite force on the other at each instant of the duration of the collision. During a collision, the two fundamental conservation laws, namely, the law of conservation of momentum and that of energy are obeyed and these laws can be used to determine the velocities of the bodies after the collision.

Types of Collisions

For mathematical simplicity, two extreme cases of collision are considered. They are elastic and elastic collisions are described as follows.

Elastic Collisions: *If there is no loss of kinetic energy during a collision it is called an elastic collision.* The collision between subatomic particles is generally elastic. The collision between two steel or glass balls is nearly elastic.

Inelastic Collisions; *If there is a loss of kinetic energy during a collision, it is called an inelastic collision.* Since there is always some loss of kinetic energy in any collision, collisions are generally inelastic. If the loss is negligibly small, the collision is very nearly elastic. Perfectly elastic collisions are not possible. If two bodies stick together, after colliding, the collision is perfectly inelastic, e.g. a bullet striking a block of wood and being embedded in it. The loss of kinetic energy usually results in heat or sound energy.

It may be remembered that *the total momentum remains conserved in both elastic and inelastic collisions.* Further, since the interacting forces become effectively zero after the collision, *the potential energy remains the same both before and after the collision.*

Calculation of Final Velocities of Colliding Bodies

The velocities of the particles after a collision can be calculated by solving their equations of motion if we know their velocities before the collision and the forces acting during the collision. Unfortunately, we do not always know these interacting forces. Even if we do not know the forces involved, it is still possible to determine the final velocities in the following three cases:

1. *The collision is perfectly inelastic*, i.e. the colliding body sticks to the target body after collision.
2. *The collision is elastic but one-dimensional*, i.e. the bodies move in the same straight line before and after the collision.
3. *The collision is elastic and two-dimensional and the direction of motion of the bodies after collision is known*.

We will consider these cases separately.

Inelastic Collision Let m_1 and m_2 be the masses of the two colliding particles and $\vec{u_1}$ and $\vec{u_2}$ their respective velocities before collision. Then we have,

$$\text{Total momentum of particles before collision} = m_1\vec{u_1} + m_2\vec{u_2}$$

After collision, the two particles stick together and move with a common velocity, say $\vec{v}$. So that

$$\begin{aligned}\begin{array}{l}\text{Total momentum of particles}\\ \text{after collision}\end{array} &= \begin{array}{l}\text{Mass of the composite}\\ \text{particle} \times \text{common}\\ \text{velocity}\end{array}\\ &= (m_1 + m_2)\,\vec{v}\end{aligned}$$

Since the momentum is always conserved in a collision, we have,

Total momentum before collision = total momentum after collision

or
$$m_1\vec{u_1} + m_2\vec{u_2} = (m_1 + m_2)\,\vec{v}$$

or
$$\vec{v} = \frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1 + m_2} \qquad (6.11)$$

Thus knowing the masses of the two particles and their velocities before collision, we can calculate their common velocity after collision.

If the second particle is at rest initially (i.e. $u_2 = 0$) then we have

$$\vec{v} = \frac{m_1\vec{u_1}}{m_1 + m_2} \qquad (6.12)$$

In this case, we can easily see that there indeed is a loss in the kinetic energy. Now the kinetic energy of the particles before collision is given by (the second particle being at rest, $u_2 = 0$) $\frac{1}{2}m_1u_1^2$.

But the kinetic energy of the particles after collision

$$= \tfrac{1}{2}(m_1 + m_2)\,v^2$$

Hence,
$$\frac{\text{Kinetic energy after collision}}{\text{Kinetic energy before collision}} = \frac{\frac{1}{2}(m_1 + m_2)\, v^2}{\frac{1}{2}\, m_1 u_1^2}$$

$$= \frac{(m_1 + m_2)}{m_1} \frac{v^2}{u_1^2}$$

$$= \frac{(m_1 + m_2)}{m_1} \left(\frac{m_1}{(m_1 + m_2)}\right)^2 \quad \text{(using Eq. 6.12)}$$

$$= \frac{m_1}{m_1 + m_2}$$

$$< 1$$

This shows that kinetic energy after collision is less than that before collision. As mentioned earlier this loss in kinetic energy appears mostly in the form of heat and sound energy.

Elastic Collision in One Dimension We will now consider a collision between two bodies moving along the same straight line before and after the collision. Let us consider two bodies A and B of masses m_1 and m_2 moving along the same straight line with velocitie s$\vec{u}_1$ and $\vec{u}_2$ respectively (Fig. 6.16). Let us assume that $\vec{u}_1$ is greater than $\vec{u}_2$. A and B suffer a head-on collision when they meet and continue moving along the same straight line with new velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. Let us further assume that the collision is elastic, i.e. in addition to momentum, the kinetic energy is also conserved.

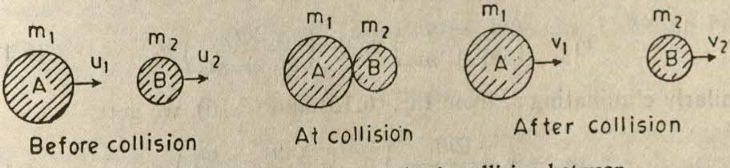


Fig. 6.16 One-dimensional elastic collision between two bodies ($u_1 > u_2$)

From the law of conservation of linear momentum, we have

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \tag{6.13}$$

Since the kinetic energy of the bodies is also conserved during this collision, we have,

$$\tfrac{1}{2}\, m_1 u_1^2 + \tfrac{1}{2}\, m_2 u_2^2 = \tfrac{1}{2}\, m_1 v_1^2 + \tfrac{1}{2}\, m_2 v_2^2$$

or

$$m_1\,(u_1^2 - v_1^2) = m_2\,(v_2^2 - u_2^2) \tag{6.14}$$

where u_1, u_2, v_1 and v_2 are the magnitudes of $\vec{u_1}$, $\vec{u_2}$, $\vec{v_2}$, and $\vec{v_2}$ respectively. Since the direction of all these velocities is the same, we can write Eq. (6.13) in terms of magnitudes as

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \tag{6.15}$$

Dividing Eq. (6.14) by Eq. (6.15), we have

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2}$$

or
$$u_1 + v_1 = v_2 + u_2$$

or
$$u_1 - u_2 = v_2 - v_1 \tag{6.16}$$

Equation (6.16) shows that *in an elastic one-dimensional collision, the relative velocity with which the two bodies approach each other before collision is equal the relative velocity with which they recede from each other after the collision.*

To find v_1 and v_2, we use Eqs (6.15) and (6.16). From Eq. (6.16) we have

$$v_2 = u_1 - u_2 + v_1$$

Substituting for v_2 in Eq. (6.15), we get

$$m_1 (u_1 - v_1) = m_2 (u_1 - u_2 + v_1 - u_2)$$

or
$$m_1u_1 - m_1v_1 = m_2u_1 - 2m_2u_2 + m_2v_1$$

or
$$v_1 (m_1 + m_2) = m_1u_1 - m_2u_1 + 2m_2u_2$$

or
$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2 \tag{6.17}$$

Similarly eliminating v_1 from Eqs (6.15) and (6.16), we get

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 \tag{6.18}$$

Now let us consider the following special cases:

(i) *If both bodies have the same mass*, then

$$m_1 = m_2 = m$$

In this case, Eqs (6.17) and (6.18) respectively give

$$v_1 = u_2$$

and
$$v_2 = u_1$$

This means that *in a one-dimensional elastic collision between two bodies of equal mass, the bodies merely exchange their velocities after the collision.*

(ii) *If one of the bodies, say m_2 is initially at rest*, then

$$u_2 = 0$$

In this case Eqs (6.17) and (6.18) reduce to

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 \tag{6.19}$$

and
$$v_2 = \left(\frac{2m_1u_1}{m_1 + m_2}\right) \tag{6.20}$$

If, in addition, $m_1 = m_2 = m$, these equations give

$$v_1 = 0 \qquad v_2 = u_1$$

Thus, *if a body suffers a one-dimensional elastic collision with another body of the same mass at rest, the first body is stopped dead, but the second begins to move with the velocity of the first.*

However, if the body at rest, namely, B is much more massive than the colliding body A, i.e. $m_2 \gg m_1$, such that m_1 is negligibly small, then Eqs (6.19) and (6.20) give

$$v_1 = -u_1$$

$$v_2 \to 0$$

Thus, *if a very light body suffers an elastic collision with a very heavy body at rest, the velocity of the lighter body is reversed on collision, while the heavier body remains practically at rest.* A common example of this type of collision is the dropping of a hard steel ball on a hard concrete floor. The ball rebounds and regains its original height from where it was dropped while the much more massive ground remains at rest.

Finally, if the body at rest is much lighter than the colliding body, i.e. if $m_2 \ll m_1$, we have

$$v_1 \simeq u_1 \qquad v_2 \simeq 2u_1$$

i.e. *the velocity of the massive body remains practically unchanged on collision with the lighter body at rest and the lighter body acquires nearly twice the initial velocity of the massive body.*

In a reactor, fast neutrons are produced in the fission of uranium atoms. These neutrons have to be slowed down so that a sustained reaction is possible. They are, therefore, made to collide with hydrogen. The hydrogen nucleus (i.e. proton) has nearly the same mass as a neutron and hence in a head-on collision with a hydrogen nucleus at rest, the neutrons are nearly stopped dead or considerably slowed down. If, instead of hydrogen, a more massive nucleus like that of lead were used as the target, the neutrons would simply bounce back from the target with the same velocity with which they struck it, and if we used a lighter target (an electron, for example), the neutrons would continue to move with their original velocity.

Elastic Collision in Two Dimensions Let a particle of mass m_1 moving with a velocity $\vec{u_1}$ collide with a particle of mass m_2, initially at rest (i.e. $\vec{u_2} = 0$) as shown in Fig. 6.17. Let us assume that the body of mass m_1 is moving along the + x-direction. After collision with m_2, let it be deflected at an angle θ_1 with the initial direction with velocity $\vec{v_1}$, where $\vec{u_1}$ and $\vec{v_1}$ lie in the x-y plane. Let us say that after collision mass m_2 is deflected at an angle θ_2 with the original direction with a velocity $\vec{v_2}$. It is clear that vector $\vec{v_2}$ will also lie in the x-y plane and will have no component in the z-direction because $\vec{u_1}$ and $\vec{v_1}$ have no z-components.

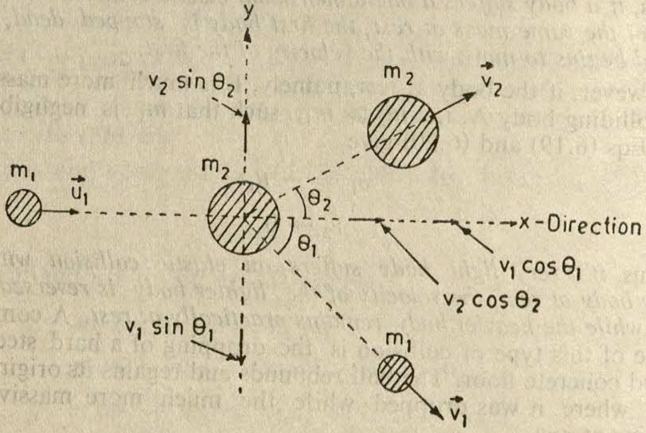


Fig. 6.17 Two-dimensional elastic collision

Taking the scalar components of $\vec{u_1}$, $\vec{v_1}$ and $\vec{v_2}$ along the x- and y-directions and applying the law of conservation of linear momentum, we have, for the x-components of motion (see Fig. 6.17)

$$m_1u_1 = m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2 \tag{6.21}$$

and for the y-components of motion

$$0 = m_1v_1 \sin\theta_1 - m_2v_2 \sin\theta_2 \tag{6.22}$$

Since the collision is elastic, kinetic energy is also conserved. Therefore,

$$\tfrac{1}{2} m_1u_1^2 = \tfrac{1}{2} m_1v_1^2 + \tfrac{1}{2} m_2v_2^2$$

or

$$m_1u_1^2 = m_1v_1^2 + m_2v_2^2 \tag{6.23}$$

In Eqs (6.21), (6.22) and (6.23) if we know m_1, m_2 and u_1, then there will be four unknown quantities, namely v_1, v_2, θ_1 and θ_2 which cannot be determined from three equations. However, if we know the directions of deflection (θ_1 and θ_2) of the concerned particles, these equations can be solved for velocities v_1 and v_2 (see Example 6.10).

EXAMPLE 6.9 A neutron of mass 1.67×10^{-27} kg moving at a velocity of 1.0×10^8 m s^{-1} collides with a deuteron at rest and sticks to it. If the mass of the deuteron is 3.34×10^{-27} kg, what will be the speed of the composite particle?

Solution Momentum of particles before collision

$$= 1.67 \times 10^{-27} \times 1.0 \times 10^{8}$$
$$= 1.67 \times 10^{-19} \text{ kg m s}^{-1}$$

Combined mass on sticking together

$$= 1.67 \times 10^{-27} + 3.34 \times 10^{-27}$$
$$= 5.01 \times 10^{-27} \text{ kg}$$

If v is the velocity of the composite particle, the momentum after collision $= (5.01 \times 10^{27})\, v$ kg m s^{-1}

Since momentum is conserved in collision, we have

$$(5.01 \times 10^{-27})\, v = 1.67 \times 10^{-19}$$

giving $$v = 3.33 \times 10^{7} \text{ m s}^{-1}$$

EXAMPLE 6.10 A steel ball of mass 0.1 kg moving with a speed of 10 m s^{-1} collides with an identical steel ball at rest. After collision the direction of each ball makes an angle of 30° with the original direction. Find the speed of each ball after collision. Is this collision elastic?

Solution

$$m_1 = m_2 = 0.1 \text{ kg}$$
$$u_1 = 10 \text{ m s}^{-1}$$
$$\theta_1 = \theta_2 = 30^\circ$$

Substituting in Eq. (6.22) we have

$$0 = 0.1 \sin 30^\circ \times (v_1 - v_2)$$

which gives $$v_1 = v_2$$

Now from Eq. (6.21) we have (since $m_1 = m_2$)

$$u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

Since $v_1 = v_2$ and $\theta_1 = \theta_2 = 30^\circ$, we have

$$u_1 = 2v_1 \cos\theta_1$$

$$\text{or} \qquad v_1 = \frac{u_1}{2\cos\theta_1} = \frac{10}{2 \times \cos 30^\circ} = 5 \times \frac{\sqrt{3}}{2}$$

$$= 4.33 \text{ m s}^{-1}$$

$$= v_2$$

$$\text{Initial kinetic energy of balls} = \tfrac{1}{2} m_1 u_1^2 + 0$$

$$= \tfrac{1}{2} \times 0.1 \times 10^2$$

$$= 5 \text{ J}$$

$$\text{Final kinetic energy of balls} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$= m_1 v_1^2 \; (\because m_1 = m_2;\, v_1 = v_2)$$

$$= 0.1 \times \left(5 \times \frac{\sqrt{3}}{2}\right)^2$$

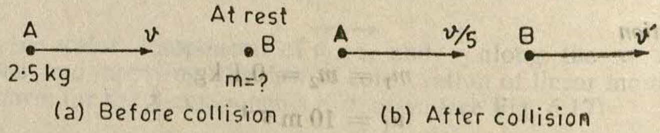
$$= \frac{15}{8} \text{ J}$$

$$\simeq 2 \text{ J}$$

The collision is not elastic since the kinetic energy is not conserved.

EXAMPLE 6.11 A ball A of mass 2.5 kg undergoes an elastic collision with another ball B that is at rest. After collision, ball A continues moving in its original direction with a speed 1/5 of its original speed. Determine the mass of ball B.

Solution The situation before and after collision is shown in Figs 6.18 (a) and (b) respectively.



(a) Before collision (b) After collision

Fig. 6.18

Since the collision is elastic, both energy and momentum must be conserved. The law of conservation of momentum gives

$$2.5\, v = 2.5 \frac{v}{5} + m\, v'$$

$$\text{or} \qquad 2\, v = m\, v' \qquad \text{(i)}$$

The law of conservation of energy yields

$$\frac{1}{2} \times 2.5\, v^2 = \frac{1}{2} \times 2.5 \left(\frac{v}{5}\right)^2 + \frac{1}{2} m v'^2$$

$$\text{or} \qquad 2.4\, v^2 = m v'^2 \qquad \text{(ii)}$$

Combining Eqs (i) and (ii), we get

$$2v = m\left(\frac{2.4}{m}\right)^{1/2} v$$

or $$2 = m^{1/2}\,(2.4)^{1/2}$$

or $$m = 4/2.4 = 1.667 \text{ kg}$$

EXAMPLE 6.12 A particle of mass M moving with a speed V undergoes an elastic (head-on) collision with a stationary particle of mass m. After the collision, both particles move along the initial direction of motion. Calculate their speeds after collision. Show that

(a) If $M = m$, the ball of mass M comes to rest whereas the ball of mass m starts moving with a speed V.

(b) If $m \gg M$, the ball of mass M bounces back with speed V whereas the ball of mass m remains stationary.

(c) If the ball of mass M is a neutron ($M = 1.8 \times 10^{-27}$ kg) and the ball of mass m is a deuteron ($m = 3.6 \times 10^{-27}$ kg), and $V = 10^6$ ms^{-1}, calculate the speeds of the neutron and the deuteron after the collision.

Solution Let A and B be particles of mass M and m respectively. Figure 6.19 (a) shows the situation before the collision. After the collision let the speeds of A and B be V' and v' respectively, as shown in Fig. 6.19 (b).

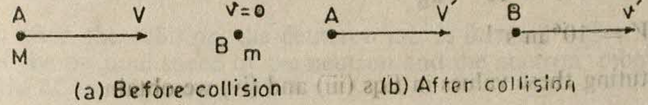


Fig. 6.19

Since the collision is perfectly elastic, both energy and momentum must be conserved.

The conservation of linear momentum gives

$$MV + m \times 0 = MV' + mv'$$

or $$MV = MV' + mv' \tag{i}$$

From the law of conservation of energy, we have

$$\tfrac{1}{2}MV^2 = \tfrac{1}{2}MV'^2 + \tfrac{1}{2}mv'^2$$

or $$MV^2 = MV'^2 + mv'^2 \tag{ii}$$

Solving Eqs (i) and (ii), we get

$$V' = \left(\frac{M-m}{M+m}\right)V \tag{iii}$$

and $$v' = \frac{2M}{M+m}V \tag{iv}$$

(a) When $M = m$, we have from Eq. (iii)

$$V' = 0$$

and from Eq. (iv)

$$v' = V$$

In other words, after the collision, the particle of mass M becomes stationary and that of mass m starts moving with a speed V.

(b) If $m \gg M$

$$M - m \simeq -m$$

$$M + m \simeq m$$

Therefore, Eqs. (iii) and (iv) reduce to

$$V' \simeq -V$$

and $$v' \simeq \frac{M}{\mathrm{m}} V \simeq 0 \qquad \left(\because \frac{M}{m} \ll 1\right)$$

In other words, after collision, the particle of mass M bounces back with the same speed and the particle of mass m remains stationary.

(c) $M = 1.8 \times 10^{-27}$ kg

$m = 3.6 \times 10^{-27}$ kg

$V = 10^{6}\ \mathrm{m\ s^{-1}}$

Substituting these values in Eqs (iii) and (iv) we obtain

$$V' = \frac{(1.8 \times 10^{-27} - 3.6 \times 10^{-27}) \times 10^{6}}{(1.8 \times 10^{-27} + 3.7 \times 10^{-27})}$$

$$= -\ 3.33 \times 10^{5}\ \mathrm{m\ s^{-1}}$$

and $$v' = \frac{2 \times 1.8 \times 10^{-27} \times 10^{6}}{(1.8 + 3.6) \times 10^{-27}}$$

$$= 6.66 \times 10^{5}\ \mathrm{m\ s^{-1}}$$

The negative sign of V' indicates that the direction of motion of the neutron is reversed after collision.

EXAMPLE 6.13 A neutron of mass 1.8×10^{-27} kg moving with a velocity of $10^{6}\ \mathrm{m\ s^{-1}}$ collides with a stationary deuteron of mass 3.6×10^{-27} kg. Assuming that the collision is head-on and elastic, determine the speed of each particle after the collision.

Solution Let v_n and v_d respectively be the velocities of the neutron and deuteron after the collision. Since the collision is head-on and elastic, we have, from the laws of conservation of linear momentum and conservation of kinetic energy,

$$m_n \times 10^6 + m_d \times 0 = m_n v_n + m_d v_d$$

and $$\tfrac{1}{2} \times m_n \times (10^6)^2 + 0 = \tfrac{1}{2} m_n v_n^2 + \tfrac{1}{2} m_d v_d^2$$

where m_n = mass of the neutron and m_d = mass of the deuteron. Substituting the valves of the masses, we have

$$1.8 \times 10^{-27} \times 10^6 = 1.8 \times 10^{-27} v_n + 3.6 \times 10^{-27} v_d$$

or $$1.8\, v_n + 3.6\, v_d = 1.8 \times 10^6$$

or $$v_n + 2\, v_d = 10^6$$

$\therefore$ $$v_n = (10^6 - 2\, v_d)$$

and $$1.8 \times 10^{-27} \times 10^{12} = 1.8 \times 10^{-27} v_n^2 + 3.6 \times 10^{-27} v_d^2$$

or $$v_n^2 + 2\, v_d^2 = 10^{12}$$

$\therefore$ $$(10^6 \times 2\, v_d)^2 + 2\, v_d^2 = 10^{12}$$

or $$6\, v_d^2 - 4\, v_d \times 10^6 = 0$$

or $$2\, v_d\,(3\, v_d - 2 \times 10^6) = 0$$

$\therefore$ $$v_d = 2/3 \times 10^6\ \text{ms}^{-1} = 0.67 \times 10^6\ \text{ms}^{-1}$$

and $$v_n = (1 - 4/3) \times 10^6\ \text{m s}^{-1} = -\ 0.33 \times 10^6\ \text{m s}^{-1}$$

Thus, after the collision, the deuteron moves forward with a speed 2/3rds of the original speed of the neutron and the neutron rebounds with 1/3rd of its original speed.

EXAMPLE 6.14 A ball A of mass 10 kg and another ball B of unknown mass are placed on a horizontal frictionless table which rests against a rigid wall as shown in Fig. 6.20. The ball A moves towards the stationary ball B with a velocity v. What should be the mass of B such that both A and B move with the same speed after A has undergone a collision with ball B and the wall? All collisions are assumed to be elastic.

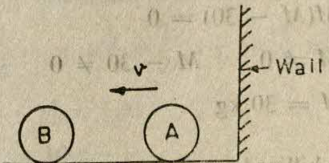


Fig. 6.20

Solution Figure 6.21 (a) shows the positions of balls A and B before the collision takes place. After the collision has occured, let V' and v'

be the respective speeds of balls A and B (see Fig. 6.21). The ball A starts moving towards the wall and undrgoes an elastic collision with the rigid wall. After collision with the wall, the velocity of A will be reversed. We are given that

$$V' = v'$$

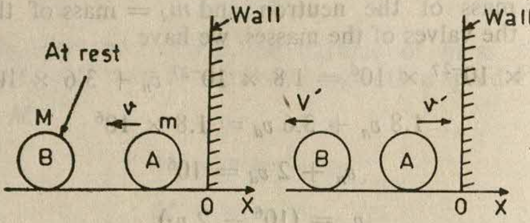


Fig. 6.21

Applying the law of conservation of linear momentum, we have

$$M \times 0 - m \times v = - M V' + m v'$$

$$m v = (M - m) v' \; (\because V' = v') \qquad \text{(i)}$$

The law of conservation of energy gives

$$\frac{1}{2} m v^2 = \frac{1}{2} M V'^2 + \frac{1}{2} m v'^2$$

or $$m v^2 = (M + m) v'^2 \qquad \text{(ii)}$$

Combining Eqs (i) and (ii) we obtain

$$m v^2 = (M + m) \frac{m^2 v^2}{(M - m)^2}$$

or $$(M - m)^2 = m (M + m)$$

We are given that $m = 10$ kg. Therefore

$$(M - 10)^2 = 10 (M + 10)$$

or $$M^2 - 30 M = 0$$

or $$M(M - 30) = 0$$

Since $M \neq 0, \quad M - 30 \neq 0$

$\therefore$ $M = 30$ kg

EXAMPLE 6.15 In a billiards game, a stationary red ball is aimed at a white ball that had been given a speed of 10 m s^{-1}. After the collision, the red ball moves at an angle of 30° with the original direction of motion of the white ball. Assuming the two balls to have the same mass, calculate the speed of the two balls and the direction of motion of the white ball. The collision is assumed to be elastic.

Solution Let m be the mass of each ball and v_1 and v_2 the speeds of the red and the white balls respectively after the collision has taken place (Fig. 6.22).

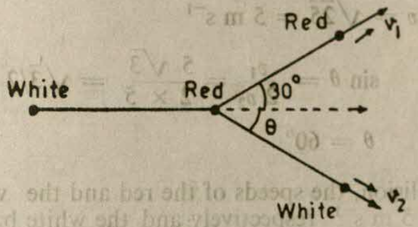


Fig. 6.22

If θ is the angle at which the white ball moves after impact, we have (equating the parallel and perpendicular components of momentum)

$$m \times 10 = mv_1 \cos 30^\circ + mv_2 \cos \theta$$

$$0 = mv_1 \sin 30^\circ - mv_2 \sin \theta$$

or $$v_1 \frac{\sqrt{3}}{2} + v_2 \cos \theta = 10$$

and $$\frac{v_1}{2} - v_2 \sin \theta = 0$$

$\therefore$ $$\sin \theta = \frac{v_1}{2\,v_2} \quad \text{and} \quad \cos \theta = \frac{20 - \sqrt{3}v_1}{2v_2}$$

Using $$\sin^2 \theta + \cos^2 \theta = 1,$$

We get $$\frac{v_1^2}{4\,v_2^2} + \frac{400 + 3\,v_1^2 - 40\sqrt{3}v_1}{4\,v_2^2} = 1$$

This gives $$v_1^2 - 10\sqrt{3}v_1 + 100 = v_2^2 \tag{i}$$

Again from the law of conservation of kinetic energy, we have

$$\frac{1}{2}\,m \times 10^2 = \frac{1}{2}\,m\,v_1^2 + \frac{1}{2}\,mv_2^2$$

$\therefore$ $$v_1^2 + v_2^2 = 100$$

or $$v_2^2 = 100 - v_1^2 \tag{ii}$$

From Eqs (i) and (ii) we get

$$v_1^2 - 10\sqrt{3}v_1 + 100 = 100 - v_1^2$$

or $$2\,v_1\,(v_1 - 5\sqrt{3}) = 0$$

Either $v_1 = 0$ or $v_1 = 5\sqrt{3}$ m s^{-1}.

Since $v_1 \neq 0$, we have $v_1 = 5\sqrt{3} = 8.66$ m s^{-1}.

Also $$v_2^2 = 100 - v_1^2 = 100 - (5\sqrt{3})^2$$
$$= 100 - 75 = 25$$

or $$v = \sqrt{25} = 5 \text{ m s}^{-1}$$

and $$\sin\theta = \frac{v_1}{2\,v_2} = \frac{5\sqrt{3}}{2 \times 5} = \sqrt{3}/2$$

$\therefore$ $$\theta = 60°$$

Thus, after collision, the speeds of the red and the white balls are 8.66 m s^{-1} and 5 m s^{-1} respectively and the white ball moves at an angle of 60° with its original direction of motion.

EXAMPLE 6.16 A perfectly elastic oblique collision occurs between a moving particle and a particle at rest. If the masses of particles are equal, show that they move at right angles to each other after the collision.

Solution Since the collision is elastic, the linear momentum and kinetic energy are both conserved. Referring to Fig. 6.17 and applying the law of conservation of momentum, we have, for the x and y components of motion [see Eqs (6.21) and (6.22)],

$$m_1 u_1 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2$$

and $$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2$$

Since $m_1 = m_2$, these equations reduce to

$$u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2 \qquad \text{(i)}$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \qquad \text{(ii)}$$

The conservation of kinetic energy gives

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

For $m_1 = m_2$, we have

$$u_1^2 = v_1^2 + v_2 \qquad \text{(iii)}$$

Equation (i) may be rewritten as

$$v_2 \cos\theta_2 = u_1 - v_1 \cos\theta_1 \qquad \text{(iv)}$$

Squaring Eqs (ii) and (iv) and adding, we get

$$v_2^2 = u_1^2 + v_1^2 - 2\,u_1 v_1 \cos\theta_1 \qquad \text{(v)}$$

Eliminating v_2 from Eqs (iii) and (v), we have

$$\cos\theta_1 = \frac{v_1}{u_1} \tag{vi}$$

Eliminating v_2 from Eqs (i) and (ii), we get

$$\tan\theta_2 = \frac{\sin\theta_1}{\left(\dfrac{u_1}{v_1} - \cos\theta_1\right)} \tag{vii}$$

Using Eq. (vi) in Eq. (vii) gives

$$\begin{aligned}\tan\theta_2 &= \frac{\sin\theta_1}{\left(\dfrac{1}{\cos\theta_1} - \cos\theta_1\right)}\\ &= \frac{\sin\theta_1\cos\theta_1}{(1-\cos^2\theta_1)}\\ &= \cot\theta_1\\ &= \tan\left(\frac{\pi}{2} - \theta_1\right)\end{aligned}$$

$$\therefore \qquad \theta_2 = \frac{\pi}{2} - \theta_1$$

or

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

i.e. the particle move at right angles to each other after the collision.

SUMMARY

The scalar product of two vectors is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them.

Work is an example of the scalar product of two vectors, the force vector $\vec{F}$ and the displacement vector $\vec{S}$

$$W = \vec{F}\cdot\vec{S} = FS\cos\theta$$

Energy is the ability of an agent to do work and is measured by the amount of work an agent can do. Energy and work are measured in the same units, the joule.

Energy can exist in various forms: mechanical, electrical, magnetic, chemical, sound, light nuclear and thermal. They are interconvertible.

Mechanical energy is of two kinds; kinetic and potential. The kinetic energy of a body is due to its motion and potential energy is due to its state or configuration.

The law of conservation of energy states that energy is neither created nor destroyed; it can only be transformed from one form to another, but the total amount of energy always remains constant in any natural process.

In elastic collisions, the total kinetic energy is conserved. In inelastic collisions, a part of kinetic energy is lost as other forms of energy such as heat and sound. Momentum is conserved in all types of collisions.

EXERCISES

1. What do you understand by the scalar product of two vectors? Show that it is commutative.
2. How is work defined and measured? Define the SI unit of work. Give two examples of a force that does no work.
3. Show that the work done by a constant force in displacing a body through a certain distance is equal to the change in the kinetic energy of the body.
4. State the law of conservation of energy. Show that the total mechanical energy of a freely falling body remains conserved throughout its fall.
5. Can you imagine situations under which
   (i) the total momentum of a system is conserved, but its total kinetic energy is not conserved,
   (ii) the total energy of a system is conserved, but its total momentum is not conserved?
6. Explain how the work done by a variable force is measured and find the work done in compressing an elastic spring of force constant k by an amount x.
7. Define kinetic energy. Deduce the expression for the kinetic energy of a body which starts from rest and acquires a velocity under the action of (a) a constant force and (b) a variable force.
8. Define potential energy. Deduce the expression for the potential energy of (a) a body raised to a height above the ground, and (b) a stretched elastic spring.
9. What do you understand by mass-energy equivalence?
10. What do yon mean by a collision? Distinguish between elestic and inelastic collisions.
11. Two bodies of masses, m_1 and m_2, are moving with velocities u_1 and u_2 in the same straight line. On collision, they are stuck to each other. Calculate the common velocity after collision.
12. Two bodies of masses, m_1 and m_2, moving with velocities u_1 and u_2 in the same straight line, suffer an elastic collision in one dimension. Derive the expression for their velocities after collision. Discuss the special cases (i) when the colliding bodies have the same mass, (ii) if one of the bodies is initially at rest, (iii) if the body at rest is very heavy, and (iv) if the body at rest is very light.
13. An elastic oblique collision occurs between a moving particle and a stationary particle. The two particles are found to move at right angles to each other after the collision. Show that their masses must be equal.

PROBLEMS

1. A ball is dropped vertically from rest at a height of 12 m. After striking the ground, it bounces back to a height of 9 m. What fraction of kinetic energy does it lose on striking the ground?
2. A man whose mass is 75 kg walks up 10 steps, each 20 cm high, in 5 s. Find the power he develops.
3. A body of mass 10 kg is dropped from a height of 20 m. Find (i) its potential energy before it is dropped, (ii) its kinetic energy when it is 8 m above the ground, and (iii) its kinetic energy when it hits the ground.
4. Two bodies of mass 1 kg and 4 kg have equal linear momentum. What is the ratio of their kinetic energies?
5. A truck tows a car of mass 500 kg at a speed of 10 m s^{-2} on a level road. If the tension in the connecting rope is 1000 N, what is the power developed by the truck? What will be the tension in the rope when the truck climbs a small hill having a slope of 1/5?
6. A bullet of mass 0.02 kg is fired horizontally with a speed of 200 m s^{-1} into a 2 kg block of wood suspended by a string 1 m long. The bullet is embedded in the block. Calculate the maximum inclination of the string to the vertical.
7. A body of mass 2 kg and moving with a velocity of 3 m s^{-1} collides head-on with a body of mass 1 kg moving in the opposite direction with a velocity of 4 m s^{-1}. After collision, the two bodies stick together and move with a common velocity. Calculate the common velocity.
8. A neutron of mass 1.67×10^{-27} kg moving at 10^8 m s^{-1} collides with a deuteron at rest. After collision, the two particles stick together and the composite particle moves with a velocity of 3.33×10^7 m s^{-1}. Calculate the mass of the deuteron.
9. A proton of mass 1.67×10^{-27} kg undergoes a head-on collision with an alpha particle initially at rest. After the collision, the alpha particle moves with a speed of 8×10^5 m s^{-1}. Calculate the velocity of the proton before and after the collision. Given, mass of alpha particle $= 6.68 \times 10^{-27}$ kg.
10. An unoccupied car of mass 10^5 kg has a speed of 15 m s^{-1} on a level road. A truck of mass 5×10^3 kg moving in the opposite direction undergoes a head-on collision with the car. Calculate the speed the truck must have so that both vehicles come to rest after the collision.
11. A ball P of mass 10 kg moving with a speed of 8 m s^{-1} collides with another ball Q of mass 20 kg initially at rest. After the collision, P and Q move in directions making angles of 30° and 45° respectively with the initial direction of motion of P. Calculate the speeds of P and Q after the collision.
12. An electron undergoes a one-dimensional collision with a hydrogen atom at rest. All motions before and after the collision are in a straight line. Calculate the fraction of electron's kinetic energy transferred to the hydrogen atom due to collision.

13. A stationary mass explodes into two parts of masses 0.4 kg and 4 kg respectively. If the larger mass has a kinetic energy of 100 J, what is the kinetic energy of the smaller mass?
14. A vehicle of mass 2000 kg travelling at 10 ms^{-1} on a horizontal surface is brought to rest in a distance of 12.5 m by the application of brakes which provide a constant retarding force. What power must the engine develop in order to take the vehicle up an incline of 1 in 10 at a constant speed of 10 ms^{-1}? The frictional force is 200 N.

7

Centre of Mass, Rigid Bodies and Rotational Motion

7.1 INTRODUCTION

We have so far treated objects as *'point masses'*. In Chapters 3 and 4 we studied the kinematics of point objects and learnt to describe their motion in one, two and three dimensions. In Chapter 5 we studied the dynamics of point objects and learnt three laws which quantitively describe the causes of motion. In Chapter 6 we studied the concepts of work, energy and power and learnt how objects can have kinetic and potential energies by virtue of their motion or configuration.

We have stated earlier that Newton's laws of motion hold for point objects, such as atoms. But any object has to be finite and consists of a large number of atoms. So no real object can be treated to be small in the absolute sense. But we have applied these laws to the motion of objects on earth and even to that of planets and other heavenly bodies. It was the remarkable foresight and genius of Newton that made him realise that his laws of motion could be reasonably applied even to the motion of finite or extended objects. This is the subject matter of this chapter. The key concept needed in the extension of Newton's laws of motion to extended objects is that of the *centre of mass* of a system of point bodies (called particles) which constitute an extended body. We first develop this concept for a system of two particles and then generalise it in the case of a large number of particles constituting the finite body.

7.2 CENTRE OF MASS OF A TWO-PARTICLE SYSTEM

A particle is defined as an object whose mass is finite but whose size and internal structure can be neglected. Consider a system consisting of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r}_1(t)$ and $\vec{r}_2(t)$ ɘq their position vectors at time *t* with respect to origin O as shown in Fig. 7.1.

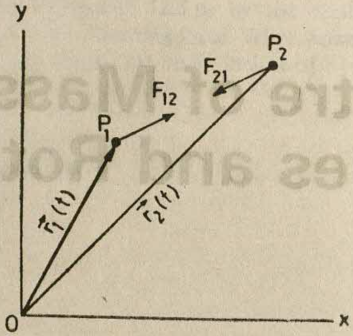


Fig. 7.1 Equations of motion of a two particle system

Then their velocity and acceleration vectors are

$$\vec{v}_1 = \frac{d\,\vec{r}_1}{dt}$$

$$\vec{a}_1 = \frac{d\,\vec{v}_1}{dt} = \frac{d^2\,\vec{r}_1}{dt^2}$$

$$\vec{v}_2 = \frac{d\,\vec{r}_2}{dt}$$

$$\vec{a}_2 = \frac{d\,\vec{v}_2}{dt} = \frac{d^2\,\vec{r}_2}{dt^2} \qquad (7.1)$$

Equations of Motion

Let $\vec{F}_1$ be the force causing acceleration $\vec{a}_1$ in particle P_1. $\vec{F}_1$ is the resultant of two kinds of force acting on particle P_1—the force $\vec{F}_{12}$ exerted by particle P_2 on particle P_1 (this force may be gravitational or electrical in nature) and a force $\vec{F}_1$ (ext) which could be due to some other particles external to the system. $\vec{F}_{12}$ is called *internal force* because it is exerted by one part of the system on another part of the same system. The net force $\vec{F}_1$ acting on particle P_1 is given by the vectorial sum of $\vec{F}_{12}$ and $\vec{F}_1$ (ext), i.e.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_1 \text{ (ext)}$$

Similarly, the net force $\vec{F_2}$ acting on particle P_2 is given by

$$\vec{F_2} = \vec{F_{21}} + \vec{F_2}\ (\text{ext})$$

where $\vec{F_{21}}$ is the force exerted by P_1 on P_2.

The equation of motion of each particle can now be written by using Newton's second law of motion (mass × acceleration = force). Thus

$$m_1\,\vec{a_1} = \vec{F_1} = \vec{F_{12}} + \vec{F_1}\ (\text{ext})$$

and

$$m_2\,\vec{a_2} = \vec{F_2} = \vec{F_{21}} + \vec{F_2}\ (\text{ext}) \tag{7.2}$$

Now from Newton's third law, the internal force $\vec{F_{12}}$ exerted by particles P_2 on particle P_1 is equal and opposite to $\vec{F_{21}}$, the force exerted by P_1 on P_2, i.e.

$$\vec{F_{12}} = -\vec{F_{21}}$$

Therefore, if we add the two equations (7.2) the forces $\vec{F_{12}}$ and $\vec{F_{21}}$ cancel out:

$$m_1\,\vec{a_1} + m_2\,\vec{a_2} = \vec{F} \tag{7.3}$$

where $\vec{F} = \vec{F_1}\ (\text{ext}) + \vec{F_2}\ (\text{ext})$ is the total external force acting on the system.

Definition of Centre of Mass

The total mass of the system consisting of two particles of masses m_1 and m_2 is

$$M = m_1 + m_2$$

Let us say that the total external force $\vec{F}$ acting on the system (whose mass is M) produces in it an acceleration $\vec{a_{\text{CM}}}$. Therefore we must have (if Newton's second law is to hold for the system)

$$M\,\vec{a_{\text{CM}}} = \vec{F} \tag{7.4}$$

From Eqs (7.3) and (7.4) we get

$$M\,\vec{a_{\text{CM}}} = m_1\,\vec{a_1} + m_2\,\vec{a_2}$$

or

$$\vec{a_{\text{CM}}} = \frac{1}{M}(m_1\,\vec{a_1} + m_2\,\vec{a_2}) \tag{7.5}$$

The acceleration $\vec{a}_{CM}$ is called the acceleration vector of the centre of mass of the system. Let $\vec{R}_{CM}$ be the position vector of the centre of mass. Then

$$\vec{a}_{CM} = \frac{d^2 \vec{R}_{CM}}{dt^2} \qquad (7.6)$$

Using $\vec{a}_1$ and $\vec{a}_2$ from Eq. (7.1) and $\vec{a}_{CM}$ from Eq. (7.6) in Eq. (7.5) we get

$$\frac{d^2 \vec{R}_{CM}}{dt^2} = \frac{1}{M}\left(m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2}\right)$$

Therefore, $\vec{R}_{CM}$ is given by

$$\vec{R}_{CM} = \frac{1}{M}\,(m_1 \vec{r}_1 + m_2 \vec{r}_2)$$

or

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad (7.7)$$

This equation defines the position of the centre of mass of a system comprising two particles of masses m_1 and m_2 and with position vectors $\vec{r}_1$ and $\vec{r}_2$. With this definition, Newton's second law is applicable to the system. Thus we conclude that Newton's second law, as applied to the individual particles of the system, will hold for the entire system provided the external force acts at the centre of mass as defined by Eq .(7.7). Newton's third law enables us to get rid of the internal forces between the particles themselves. It must be clearly understood that while applying Newton's second law to the motion of the centre of mass, only forces external to the system need be considered.

Notice that if the entire mass of the system is concentrated at one point, say the tip P_1 of vector $\vec{r}_1$ (i.e. if $m_2 = 0$) then $\vec{R}_{CM} = \vec{r}_1$. If the masses are equal (i.e. $m_1 = m_2$) then

$$\vec{R}_{CM} = \tfrac{1}{2}\,(\vec{r}_1 + \vec{r}_2)$$

which means that the centre of mass lies exactly in the middle of the line joining the two masses. The arrowhead (or tip) of vector $\vec{R}_{CM}$ lies midway on the line joining the tips of vectors $\vec{r}_1$ and $\vec{r}_2$. If, however, $m_1 \neq m_2$, R_{CM} will be the '*weighted*' average, each particle making a contribution proportional to its mass. This is shown in Fig. 7.2 where C is the centre of mass and $P_1C/CP_2 = m_2/m_1$.

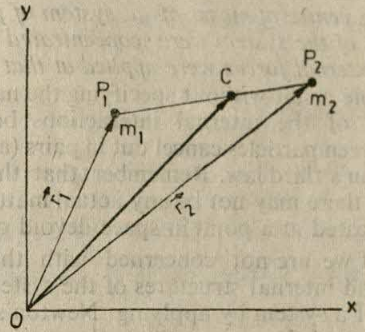


Fig. 7.2 Centre of mass for two particles: $P_1C/CP_2 = m_2/m_1$ with $m_2 > m_1$.

Extension to N Particles

We can easily extend our definition (7.7) to the case of a system consisting of N particles with masses $m_1, m_2, m_3, \ldots, m_N$ with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_N$. This can be done by adding the products $m_1 \vec{r}_1, m_2 \vec{r}_2, m_3 \vec{r}_3 \ldots, m_N \vec{r}_N$ and dividing the sum so obtained by the total mass $M = m_1 + m_2 + m_3 + \ldots + m_N$ of the system. The position vector $\vec{R}_{CM}$ of the centre of mass with respect to origin O is given by

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots + m_N \vec{r}_N}{m_1 + m_2 + m_3 + \ldots + m_N}$$

or
$$\vec{R}_{CM} = \frac{\sum_{n=1}^{N} m_n \vec{r}_n}{\sum_{n=1}^{N} m_n} = \frac{\sum_{n=1}^{N} m_n \vec{r}_n}{M} \tag{7.8}$$

Again, $\vec{R}_{CM}$ is the weighted average of all the position vectors of the particles of the system, the contribution of each particle being proportional to its mass. The motion of the centre of mass is governed by the equation

$$M \vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_N$$

or
$$M \vec{a}_{CM} = \vec{F}_{tot} \tag{7.9}$$

with
$$\vec{a}_{CM} = \frac{d^2 \vec{R}_{CM}}{dt^2}$$

where $\vec{F}_{tot}$ is the vector sum of all the external forces acting on the system and $\vec{a}_{CM}$ is the acceleration of the centre of mass. Equation (7.9) states that the *centre of mass of a system of particles moves as though all the mass of the system were concentrated at the centre of mass and all the external forces were applied at that point.* Notice that we obtain this simple result without specifying the nature of the system of the particles or of the internal interactions between them. The internal forces between particles cancel out in pairs (as explained above) by virtue of Newton's third law. Remember that the definition (7.8) holds even though there may not be any actual matter at the centre of mass; it may be located at a point in space devoid of any matter.

To summarise, if we are not concerned with the internal forces, internal motions and internal structures of the system, we can analyse the gross motion of a system by applying Newton's laws of motion to the centre of mass of the system where the entire mass of the system may be assumed to be concentrated.

Momentum Conservation and Centre-of-Mass Motion

We now consider the case when no net external force acts on the system. The particles of the system, however, move only under the influence of their mutual internal forces. In this case, setting $F_{tot} = 0$ in Eq. (7.9) we have

$$M\,\vec{a}_{CM} = 0$$

Thus

$$\vec{a}_{CM} = 0$$

or

$$\frac{d}{dt}\vec{V}_{CM} = 0$$

or

$$\vec{V}_{CM} = \text{constant}$$

where $\vec{V}_{CM}$ is the velocity of the centre of mass of the system. Thus in the absence of external forces (or if the external forces balance) the velocity of the centre of mass remains constant, i.e.

$$\vec{R}_{CM}(t) = \vec{R}_{CM}(0) + \vec{V}_{CM}\,t$$

and it moves uniformly in a straight line. This is Newton's first law of motion.

The case of an isolated system (i.e. a system on which no external forces act) is significant because, as will be evident from the following, the total linear momentum of such a system is conserved. From Eq. (7.9) we find that if $\vec{F}_{tot} = 0$, it implies that

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_N = 0$$

or

$$m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_N\vec{a}_N = 0$$

or $$\frac{d}{dt}(m_1\vec{V}_1 + m_2\vec{V}_2 + m_3\vec{V}_3 + \ldots + m_N\vec{V}_N) = 0$$

where $\vec{V}_1, \vec{V}_2, \vec{V}_3, \ldots, \vec{V}_N$ are the velocities of the particles of the system. Thus

$$m_1\vec{V}_1 + m_2\vec{V}_2 + m_3\vec{V}_3 + \ldots + m_N\vec{V}_N = \text{constant}$$

or $$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_N = \text{constant} \quad (7.10)$$

where $\vec{P}$ is the total linear momentum of the system.

Now, differentiating Eq. (7.8) with respect to time we have

$$\frac{d\vec{R}_{\text{CM}}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \ldots + m_N\frac{d\vec{r}_N}{dt}\right)$$

or $$\vec{V}_{\text{CM}} = \frac{1}{M}(m_1\vec{V}_1 + m_2\vec{V}_2 + m_3\vec{V}_3 + \ldots + m_N\vec{V}_N) \quad (7.11)$$

where $\vec{V}_{\text{CM}}$ is the velocity of the centre of mass and $\vec{V}_1$, $\vec{V}_2$, etc. are the velocities of the individual particles of the system. From Eqs (7.10) and (7.11) we have

$$\vec{P} = M\vec{V}_{\text{CM}} \quad (7.12)$$

which is an equivalent definition of the momentum of a system of particles. In words, Eq. (7.12) states that *the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.*

Equation (7.10) states that *if no net external force acts on the system, the total linear momentum of the system remains constant.* This simple and general result is called *the principle of conservation of linear momentum* which we have already used in Chapter 5 to analyse the problem of collisions.

An Important Note The conservation of linear momentum is the second of the great conservation principles that we have encountered, the first being the conservation of energy. Later in this chapter we will come across the principle of conservation of angular momentum and, in higher classes, the principle of conservation of electric charge. As stated in Chapter 1, conservation principles are of immense theoretical and practical importance in physics because they are universal. These principles state that while the system is changing, there is one aspect of the system that remains unaltered. These principles hold for all reference frames. Each observer in his own reference frame will have his own value of $\vec{P}$ which remains unchanged. Momentum is a vector quantity. Hence the conservation of momentum supplies us with three conditions (one corresponding to each coordinate direction) on the motion of the system, whereas energy (being a scalar) supplies us with only one conservation condition. The law of conservation of linear momentum holds good even at the atomic and nuclear levels where Newtonian mechanics fails. This makes the law more fundamental than Newton's laws.

Some Examples of Centre-of-Mass Motion

1. Let us first consider an example in which an external force acts on the system. Let us imagine that a shell (or a fire cracker) is projected from the earth and explodes in flight. The system is the shell, our reference frame is the earth and the external force is the force of gravity. The trajectory of the shell is shown in Fig. 7.3.

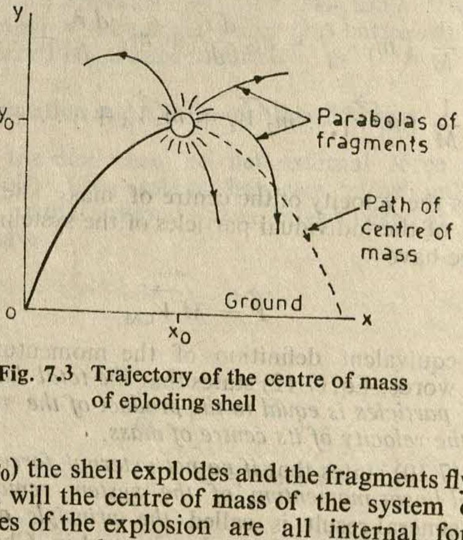


Fig. 7.3 Trajectory of the centre of mass of eploding shell

At point (x_0, y_0) the shell explodes and the fragments fly in all directions. What path will the centre of mass of the system of fragments follow? The forces of the explosion are all internal forces because they are the forces exerted by one part of the system on its other parts. These forces may change the moment a of individual fragments but cannot change the total momentum of the system. Only an external force can change the total momentum. Since the system as a whole moves as though its entire mass were concentrated at its centre of mass with the external force acting at this point (the external force is the force of gravity), it follows that the centre of mass of the system of fragments will continue moving in the parabolic trajectory (shown by a broken curve in Fig. 7.3) which is the path that the unexploded shell would have followed. The change in the total momentum, due to gravity, is the same whether the shell explodes or not.

2. We now consider the motion of the centre of mass of the earth-moon system. The moon moves round the earth in a circular orbit and the earth moves round the sun in an elliptical orbit. It is more correct to say that the earth and the moon both move in circular orbits about

their common centre of mass while the centre of mass moves in an elliptical orbit round the sun (Fig. 7.4).

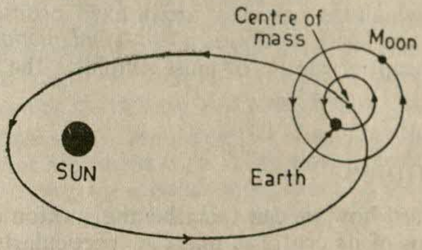


Fig. 7.4 Motion of centre of mass of earth-moon system

For the system consisting of the earth and the moon, their mutual gravitational attractions are the internal forces in the system and the sun's attraction of both earth and moon are the external forces acting on the centre of mass of the system.

3. The third example is the spontaneous decay of a radioactive nucleus into say, two fragments. The parent nucleus uranium-238 emits an alpha particle (light fragment) and decays into the thorium nucleus (heavy fragment). If the parent nucleus is initially at rest, then (since the decay is caused by internal forces) the centre of mass of the two fragments will continue to be at rest. The reason is that the total momentum is zero initially (i.e. before decay) and must remain zero after decay because the total momentum has to be conserved (there are no external forces acting on the system). This means that the light fragment (alpha particle in our example) and the heavy fragment (thorium nucleus) must fly off in opposite directions with speeds inversely proportional to their masses (Fig. 7.5).

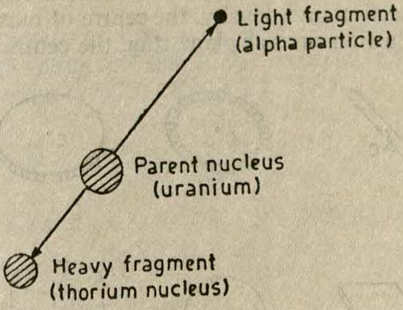


Fig. 7.5 Momentum conservation in the decay of a uranium nucleus

We can summarize our discussion of the centre of mass of a system as follows. The complete motion of a system of particles is made up of two parts; (i) the motion of the centre of mass which describes the motion of system as a whole and (ii) the motion of the particles of the system with respect to the centre of mass. If the system is a *rigid body* (in which the particles are in fixed positions with respect to one another) the latter motion is a *rotational motion.* We shall now see how the concept of centre of mass simplifies the analysis of rotational motion.

7.3• RIGID BODIES

We have discussed how we can describe the motion of a system of N particles in terms of its centre of mass. A particularly important system is what is called a *rigid body.* A body is said to be rigid if it does not undergo any change in shape by the action of a force, however large. More appropriately, a rigid body is one whose constituent particles retain their relative position even when its motion is under the influence of a force. This implies that the body cannot be deformed. If the body undergoes some displacement, every particle in it undergoes the same displacement. In actual fact, no real body can be perfectly rigid. In practice, solid bodies like steel, glass, etc. can be regarded as rigid for moderate forces.

Centre of Mass of a Rigid Body

The centre of mass of a rigid body is a point at a fixed position with respect to the body as a whole. The position of the centre of mass of a body depends on its shape and the way the mass is distributed in the body. Depending upon these two factors, the centre of mass may lie within the body or even outside it. If the body has a regular geometrical shape and if its mass distribution (i.e. density) is uniform, it is easy to locate the position of its centre of mass. For example, the centre of mass of a uniform sphere is at its geometrical centre. For a thin rod of a uniform cross-section and density, the centre of mass is at its geometrical centre. For a thin circular plane ring, the centre of mass is again

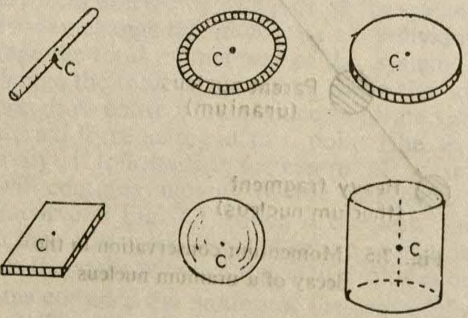


Fig. 7.6 Centres (C) of mass of symmetrical rigid bodies

at its geometrical centre where there is actually no matter. This is an example of body whose centre of mass lies outside the body. Thus, it is easy to locate the centre of mass of symmetrical bodies with uniform mass density. If we have a body which has an irregular shape or a non-uniform mass distribution, it is not easy to locate its centre of mass; we have to resort to integral calculus. Figure 7.6 shows the position of the centre of mass of some regular bodies.

The centre of mass of a thin triangular plate is at the point of intersection of the three meridians (Fig. 7.7a). For a right circular solid cone, the centre of mass is a point on its axis at a distance $h/4$ from its base, where h is the height of the cone (Fig. 7.7b).

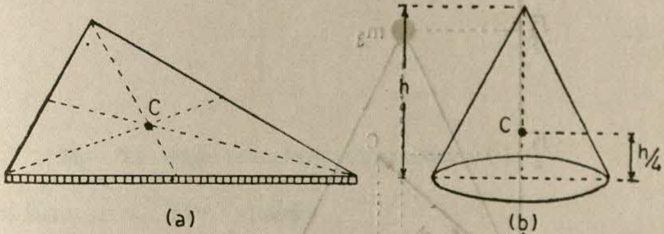


Fig. 7.7

It may be worthwhile to recall that, for every rigid body, the motion of its centre of mass is determined by Eq. (7.9) in which the internal forces do not play any role. These internal inter-particle forces are always present, for otherwise the body would fall apart. But we need not consider these internal forces while analysing the gross motion of the body, e.g. rotation and translation.

EXAMPLE 7.1 Three point masses $m_1 = 1.0$ kg, $m_2 = 2.0$ kg and $m_3 = 3.0$ kg are placed at the corners of an equilateral triangle of side 1 m. Locate the centre of mass of the system.

Solution Let C be the centre of mass of the system (Fig. 7.8). Choose the x-axis along one side of the triangle. The position vector of the centre of mass is given by [see Eq. (7.8)]

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the rectangular components of vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively and (X_{CM}, Y_{CM}) be the rectangular components of $\vec{R}_{CM}$. Then we have

$$X_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$Y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

The coordinates of mass m_1 are $x_1 = 0, y_1 = 0$, of mass m_2 are $x_2 = 1$ m, $y_2 = 0$ and of mass m_3 are $x_3 = \frac{1}{2}$ m and $y_3 = \sqrt{3}/2$ m (see Fig. 7.8)

$$\therefore \qquad X_{CM} = \frac{1 \times 0 + 2 \times 1 + 3 \times \frac{1}{2}}{1 + 2 + 3} = \frac{7}{12} \text{ m}$$

and

$$Y_{CM} = \frac{1 \times 0 + 2 \times 0 + 3 \times \sqrt{3}/2}{1 + 2 + 3} = \frac{\sqrt{3}}{4} \text{ m}$$

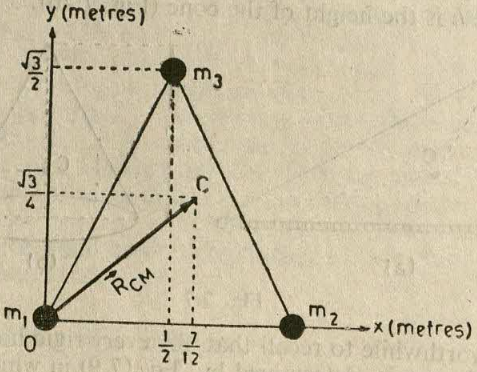


Fig. 7.8

The point C whose x and y coordinates are (7/12 m, $\sqrt{3}/4$ m) is the centre of mass of the system. Notice that C is not at the geometric centre of the triangle. The reason is that the masses at the corners are not equal.

7.4 DESCRIPTION OF ROTATIONAL MOTION IN A PLANE

So far we have discussed the translational motion of a single particle or of rigid bodies. We will now discuss the rotation of rigid bodies. Figure 7.9 shows a rigid body rotating about a fixed axis, the y-axis of our inertial reference frame. Let P be any particle (in the body) whose position vector is $\vec{r}$. *The motion of the rigid body will be purely rotatory* if every particle of the body (such as P) moves in a circle, whose centre lies on the straight line called the axis of rotation (the y-axis in Fig. 7.9).

If we draw a perpendicular from any point in the body on the axis of rotation, this line will sweep out the same angle in any given time interval as any other perpendicular line from any other particle of the body. Thus we can describe the rotatory motion of a rigid body by considering the motion of any one of the particles (such as P) that constitute the body.

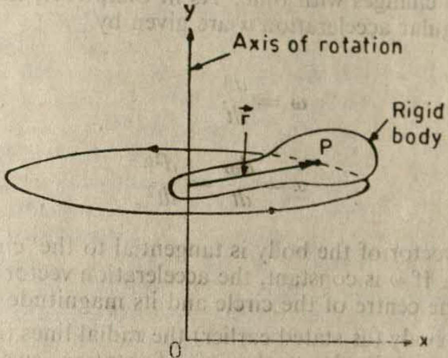


Fig. 7.9 Plane rotational motion about y-axis

Rotational Kinematics: The Variables

Figure 7.10 shows a cross-sectional veiw of a rigid body rotating counter-clokwise about the origin O in a circle of radius r. The axis of rotation is now perpendicular to the plane of the paper and we look downward on it from above, along the y-axis.

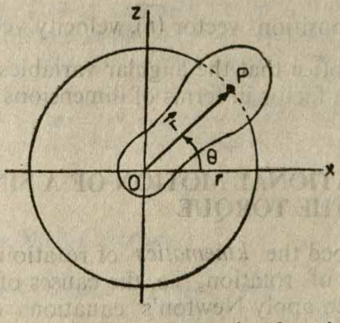


Fig. 7.10 Variables in rotational motion

We can tell exactly where the entire rotating body is if we know the location of any point of the body in our inertial reference frame. Thus, to describe a plane rotatory motion, we need to consider only the two-dimensional motion of a particles in a circle. In Chapter 4 we have already learnt how to describe the motion of a particle in a circle in a plane. The positive sense of rotation is arbitrarily chosen as the counter-clockwise sense and angle θ subtended by the position vector

$\vec{r}$ with the x-axis is measured in counter-clockwise sense as shown in the diagram. As the body rotates about origin O the angle θ (measured in radians) changes with time. As in Chapter 4, the angular velocity ω and angular acceleration α are given by

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{7.13}$$

The velocity vector of the body is tangential to the circle and has a magnitude $r\omega$. If ω is constant, the acceleration vector is always directed towards the centre of the circle and its magnitude is $r\omega^2$.

For a rigid body (as stated earlier) the radial lines from all particles of the body perpendicular to the axis of rotation sweep out the same angle in the same time. This means that the angular velocity ω about the axis of rotation is the same for each particle in the body. Thus ω is characteristic of the body as a whole. Angular velocity ω has dimensions (T^{-1}) and is expressed in rad s^{-1}. Angular acceleration α has dimensions (T^{-2}) and is expressed in rad s^{-2}.

Thus, the three kinematical variables that describe the rotational motion of a particle (or a rigid body) are angular position (θ), angular velocity (ω) and angular acceleration (α). The corresponding variables that describe the translational motion of a particle (or a rigid body) respectively are the position vector $(\vec{r})$, velocity vector $(\vec{v})$ and acceleration vector $(\vec{a})$. Notice that the angular variables differ from linear variables by a length factor in terms of dimensions.

7.5 PLANE ROTATIONAL MOTION OF A SINGLE PARTICLE: THE TORQUE

We have just described the *kinematics* of rotation. We will now discuss the *dynamics* of rotation, i.e. the causes of rotation. We have already learnt how to apply Newton's equations of motion to a system of particles. Since all rotating systems are made up of particles, rotational dynamics does not contain any fundamentally new features. In studying rotational motion, we begin with a single particle and later generalize to systems of many particles including the important case of a rigid body rotating about a fixed axis.

We know that, in translational motion, a force produces a linear acceleration in a body. In rotational motion, what is the quantity causing angular acceleration ? The quantity cannot simply be a force as is clear from the following example.

The simplest example of rotatory motion is the opening or closing of a door. When you apply a force at the handle A (Fig. 7.11), you will notice that you can close the door with very little effort. If you apply a force at a point B, somewhere midway between the handle and the hinge C, you will have to put in a much greater effort to close the door. Next try to close the door by applying a force at the hinge C. You will not be able to close the door even if you apply an extremely large force. Thus we conclude that in rotatory motion, the effect of force depends on the magnitude of the applied force and the point of application of the force. When the force is applied at point A, the distance between the point of application of force and axis of rotation (which is the line PCQ) is AC; in the second it is BC, and in the third (when the force is applied at C), zero. Since AC > BC > 0, we conclude that the rotatory effect is more if the distance between the point of application of force and the axis of rotation is larger. Further, in each case, the rotatory effect will be more if the magnitude of force is larger. The third factor is the direction in which the force is applied. The direction in which the force acts is called the *line of action* of the force. If you apply a force at the handle A of the door in a direction perpendicular to the face of the door, it is much easier to close the door than when the force is applied at an angle. In particular, if the line of action of the force applied at A is parallel to the face of the door, no rotation is possible. Thus we conclude that the rotatory effect of a force on a body, which is capable of rotation about an axis, depends upon (i) the magnitude and the direction of the force, i.e. the force vector and (ii) the axis of rotation. The rotatory effect of a force is characterised by what is called *torque* or (*moment of force*) which is the analogue of force in translatory motion.

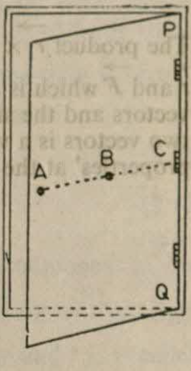


Fig. 7.11

Torque Acting on a Single Particle

Consider a particle P whose position vector with respect to origin O of an inertial reference frame is $\vec{r}$. If a force $\vec{F}$ acts on the particle, as shown in Fig. 7.12, the magnitude of torque $\vec{\tau}$ acting on the particle with respect to the origin O is defined as

$$|\vec{\tau}| = |\vec{r}|\,|\vec{F}| \sin\theta$$

or simply
$$\tau = rF\sin\theta \tag{7.14}$$

[where θ is the angle between vectors $\vec{r}$ and $\vec{F}$.

In vector notation, Eq. (7.14) is written as

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{7.15}$$

The product $\vec{r} \times \vec{F}$ is called the *vector* (*or cross*) *product* of vectors $\vec{r}$ and $\vec{F}$ which is defined as the product of the magnitudes of the two vectors and the sine of the angle between them. The cross product of two vectors is a vector (refer to supplement on 'vector product and its properties' at the end of this chapter).

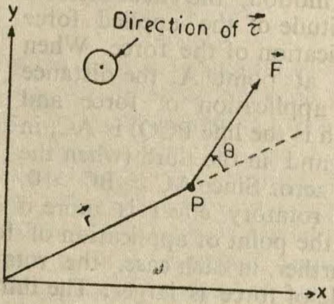


Fig. 7.12 Definition of torque

Torque $\vec{\tau}$ is a vector quantity. Its magnitude is given by $\tau = rF \sin \theta$; its direction is normal to the plane containing vectors $\vec{r}$ and $\vec{F}$ and can be determined by the *right-handed screw rule*. According to this rule, if we curl the fingers of our right hand in the direction in which vector $\vec{r}$ must be rotated to move into the position of vector $\vec{F}$ through the smaller angle between them, the extended (erect) thumb gives the direction or sense of torque $\vec{\tau}$. This is shown in Fig. 7.12 as a dot within a small circle ⊙. This symbol represents the tip of an arrow pointing towards the reader, perpendicular to the plane of the page which contains vectors $\vec{r}$ and $\vec{F}$.

Unit of Torque

Torque has the same dimensions as those of work (both being force times distance) viz. ML^2T^{-2}. The two are, however, very different quantities. Work is a scalar, torque is a vector. To distinguish between the two we express work in joules and torque in newton-metre (N m).

Dependence of Torque on the Moment Arm

From Eq. (7.15), we observe that the torque produced by a force depends not only on the magnitude and the direction of the force (i.e.

the force vector $\vec{F}$) but also on the point of application of the force relative to the origin O (i.e. on vector $\vec{r}$). In particular, when particle P is at the origin O, so that the line of application of force passes through the origin, $\vec{r}$ is zero and the torque about the origin is zero.

We may rewrite Eq. (7.14) for the magnitude of $\vec{\tau}$ as

$$\tau = (r \sin \theta) F = r_{\perp} F \tag{7.16}$$

or as

$$\tau = r (F \sin \theta) = r F_{\perp} \tag{7.17}$$

where, as shown in Fig. 7.13a, $r_{\perp}$ ($= r \sin \theta$) is the component of $\vec{r}$ at right angles to $\vec{F}$ and $F_{\perp}$ ($= F \sin \theta$) is the component of $\vec{F}$ at right angles to $\vec{r}$. Torque is also called the *moment of force* and $r_{\perp}$ is called the *moment arm*.

The moment arm is the perpendicular distance between the line of application of force and the axis of rotation. Equation (7.17) shows that only that component of $\vec{F}$ which is perpendicular to $\vec{r}$ (i.e. $F_{\perp}$)

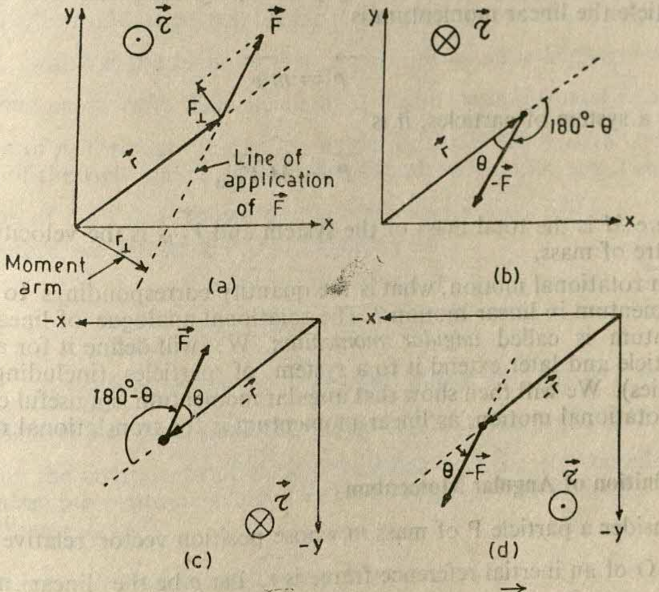


Fig. 7.13 The symbol ⊙ represents the direction of $\vec{\tau}$ and it means that $\vec{\tau}$ points towards the reader in a direction perpendicular to the plane of the page and the dot represents the tip of the arrow. The symbol ⊗ represents $\vec{\tau}$ pointing into the page and perpendicular to it, the cross representing the tail of the arrow

contributes to the torque. Thus if $\theta = 0$ or $180°$, $F\perp = F\sin\theta = 0$, then the line of application of force passes through the origin and $r\perp$ is also zero. In this case, both the equations [Eqs (7.16) and (7.17)] show that torque τ is zero.

From Fig. 7.13b, it is clear that if the direction of $\vec{F}$ is reversed, the magnitude of $\vec{\tau}$ remains unchanged but its direction is reversed. Similarly, if the direction of $\vec{r}$ is reversed, the magnitude of $\vec{\tau}$ remains unchanged but its direction is reversed (Fig. 7.13c). If both $\vec{F}$ and $\vec{r}$ are reversed the magnitude and direction of $\vec{\tau}$ remain unchanged (Fig. 7.13d). Students should verify the direction of $\vec{\tau}$ using the righthand rule in each case shown in Fig. 7.13.

7.6 ANGULAR MOMENTUM OF A PARTICLE

In Chapter 5 we introduced the concept of linear momentum in relation to the linear (or translational) motion of a single particle or of a system of particles (rigid bodies). This concept is useful because linear momentum is conserved in any process, e.g. collisions. For a single particle the linear momentum is

$$\vec{p} = m\,\vec{v}$$

For a system of particles, it is

$$\vec{P} = M\,\vec{V}_{\text{CM}}$$

where M is the total mass of the system and $\vec{V}_{\text{CM}}$ is the velocity of its centre of mass.

In rotational motion, what is the quantity correspondings to linear momentum in linear motion ? The rotational analogue of linear momentum is called *angular momentum.* We will define it for a single particle and later extend it to a system of particles (including rigid bodies). We will then show that angular momentum is a useful concept in rotational motion, as linear momentum is for translational motion.

Definition of Angular Momentum

Consider a particle P of mass m whose position vector relative to origin O of an inertial reference frame is $\vec{r}$. Let $\vec{p}$ be the linear momentum of particle. The angular momentum $\vec{L}$ of the particle with respect to origin O is defined as the vector product of $\vec{r}$ and $\vec{p}$, i.e.

$$\vec{L} = \vec{r} \times \vec{p} \tag{7.18}$$

Angular momentum is a vector. Its magnitude is given by

$$L = r\,p \sin\theta \tag{7.19}$$

where θ is the angle between vectors $\vec{r}$ and $\vec{p}$ (Fig. 7.14), and its dimensions are $\{ML^2T^{-1})$. The units of L are kg m^2s^{-1}.

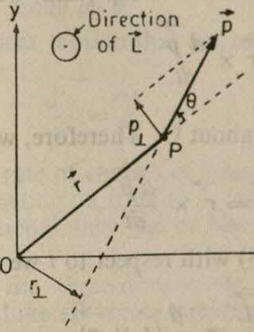


Fig. 7.14 Definition of angular momentum of a particle

The direction of $\vec{L}$ is perpendicular to the plane containing the vectors $\vec{r}$ and $\vec{p}$ (which is the plane of this page) and its sense is given by the *right-hand thumb rule.* This means that if you were to rotate $\vec{r}$ to the position of $\vec{p}$, through the smaller angle between them with curled fingers of the right hand, the extended thumb would then point in the direction of $\vec{L}$. This is shown by the symbol ⊙ in Fig. 7.14.

As before, we can write Eq. (7.19) either as

$$L = (r \sin\theta)\,p = r_\perp\, p \tag{7.20}$$

or as

$$L = r\,(p \sin\theta) = r\, p_\perp \tag{7.21}$$

where $r_\perp$ is the component of $\vec{r}$ at right angles to $\vec{p}$ and $p_\perp$ is the component of $\vec{p}$ at right angles to $\vec{r}$ (Fig. 7.14). Equation (7.21) shows that only the component of $\vec{p}$ perpendicular to $\vec{r}$ (i.e. p_1) contributes to angular momentum. When $\theta = 0$ or 180°, $p_\perp = 0$ and $r_\perp = 0$, in which case $L = 0$.

Relation between Torque and Angular Momentum

From Newton's second law, we have

$$\vec{F} = m\,\vec{a}$$

$$= m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \qquad (\because\ m = \text{constant})$$

or
$$\vec{F} = \frac{d\vec{p}}{dt} \tag{7.22}$$

where $\vec{p} = m\vec{v}$ is the linear momentum of the particle. Taking the vector product of $\vec{r}$ with both sides of Eq. (7.22) we get

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

But $\vec{r} \times \vec{F} = \vec{\tau}$, the torque about O. Therefore, we have

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} \tag{7.23}$$

Now differentiating Eq. (7.18) with respect to t we obtain

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \tag{7.24}$$

The derivative of a vector product is taken in the same way as the derivative of an ordinary product except that the order of the vectors must not be changed. Thus

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

Equation (7.24) becomes

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \tag{7.25}$$

Now $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$. Therefore

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = m(\vec{v} \times \vec{v})$$

Now $(\vec{v} \times \vec{v})$ is the vector product of two parallel vectors and is, by definition, equal to zero. Hence

$$\frac{d\vec{r}}{dt} \times \vec{p} = 0$$

Using this in Eq. (7.25), we get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Using Eq. (7.23) in this equation, we obtain

$$\frac{d\vec{L}}{dt} = \vec{\tau} \tag{7.26}$$

which states that *the rate of change of angular momentum of a particle is equal to the torque acting on it.*

The equation in linear motion that corresponds to Eq. (7.26) is

$$\frac{d\vec{p}}{dt} = \vec{F}$$

which states that the rate of change of linear momentum of a particle is equal to the force acting on it. This is Newton's second law. Thus Eq. (7.26) is the rotational analogue of Newton's second law of linear motion.

Equation (7.26) is the three-dimensional equation of motion for angular momentum. Like all vector equations, it is equivalent to three scalar equations:

$$\tau_x = \frac{dL_x}{dt} \qquad \tau_y = \frac{dL_y}{dt} \qquad \tau_z = \frac{dL_z}{dt} \tag{7.27}$$

Physical Meaning of Angular Momentum

A closer look at Eqs (7.16) and (7.20) shows that, just like torque $\vec{\tau}$, angular momentum $\vec{L}$ also depends on the position $\vec{r}$ of the particle and the point O about which it is calculated. Equation (7.16) tells us that

Torque = force × moment arm

and Eq. (7.20) says that

Angular momentum = linear momentum × moment arm

Therefore, torque is often called the *moment of force* and angular momentum the *moment of momentum*. Their physical meanings are as follows: *torque is a measure of the turning effect of a force and angular momentum is a measure of the turning motion of the object.*

Geometrical Meaning of Angular Momentum

Equation (7.21) reads

$$L = rp\perp = r\,(mv\perp) = mr\,v\perp \tag{7.28}$$

where $p\perp$ and $v\perp$ are the components of $\vec{p}$ and $\vec{v}$ respectively at right angles to $\vec{r}$.

Suppose in a small time interval Δt the particle moves a small distance from P to Q so that its displacement $\Delta \vec{r}$ is

$$\vec{PQ} = \Delta \vec{r} = \vec{v}\,\Delta t$$

In this displacement, let $\Delta\,\theta$ be the small change in angle θ (see Fig. 7.15).

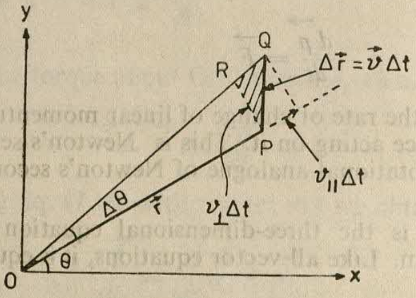


Fig. 7.15 Geometrical meaning of angular momentum

Vector $\vec{PQ} = \vec{v}\,\Delta t$ is resolved into two rectangular components $(v\perp\,\Delta t)$ and $(v_{\parallel}.\,\Delta t)$. $V\perp$ is called the *angular* component of velocity vector v, and $\vec{v}_{11}$ the *radial* component of $\vec{v}$. It is clear from the diagram that only $v\perp$ is involved in the change $\Delta\,\theta$, so that we have

$$r\,\Delta\,\theta = v\perp\,\Delta\,t$$

Now, neglecting the small contribution of the shaded portion, the area $\Delta\,A$ swept out by the radius vector $\vec{r}$ from O as the particle moves from P and Q in time $\Delta\,t$ is given by.

$$\begin{aligned}\Delta\,A &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times \text{PR} \times \text{OP}\\ &= \tfrac{1}{2} \times r\,\Delta\,\theta \times r\end{aligned}$$

But $\quad r\,\Delta\,\theta = v\perp\,\Delta\,t$. Therefore,

$$A = \tfrac{1}{2}\,r\,v\perp\,\Delta\,t$$

which, with the use of Eq. (7.28), becomes

$$\Delta A = \frac{L}{2m}\,\Delta\,t$$

or $\quad L = 2\,m\,\Delta\,A/\,\Delta\,t = 2\,m \times$ rate of sweeping out of area by radius vector

i.e. $\quad L = 2\,m \times$ areal velocity

Thus, angular momentum = 2 × mass × areal velocity. This is the geometrical meaning of angular momentum.

Conservation of Angular Momentum of a Particle

We have seen that the rate of change of angular momentum of a particle is equal to the torque produced by the total force (Eq. 7.26). If, in a certain situation, the torque itself vanishes, then it follows that the angular momentum of the particle will remain constant. This is the law of conservation of the angular momentum of a particle. One trivial situation is when the force vanishes. Then the torque vanishes too. The particle then moves freely in a straight line in accordance with Newton's first law in which case both linear and angular momenta are conserved.

A general situation is when the torque vanishes without the force itself vanishing. Equation (7.17) tells us that the torque τ will vanish if the component $F_\perp$ (the angular component) of $\vec{F}$ vanishes but the radial component $F_\parallel$ does not. The radial component $F_\parallel$ is the component of $\vec{F}$ along the radius (or position) vector $\vec{r}$. Hence, *if the force acting on the particle is purely radial* (*i.e. if it is directed along or against its position vector*) *then the torque acting on the particle vanishes and its angular momentum is conserved and so is its areal velocity.*

Kepler's Second Law of Planetary Motion The most famous example of the above result is Kepler's second law of the motion of planets round the sun. Kepler proposed his second law in 1602 and months later he proposed his first law which states that each planet moves on an ellipse with the sun at one of its foci. His second law states that the straight line from the sun to the planet sweeps out equal areas in equal intervals of time. Two such areas are shown shaded in Fig. 7.16. Thus the planet has greater speed when it is closer to the sun as indicated by the figure.

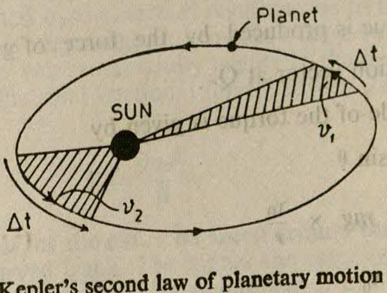


Fig. 7.16 Kepler's second law of planetary motion ($v_1 < v_2$)

This law implies that the angular momentum of the planet is conserved during its motion and the sun's force of attraction which keeps the planet in orbit must be radial, always pointing towards the sun.

A radial force produces no torque. In three dimensions, the motion of a planet is confined to a plane as a consequence of the conservation of angular momentum. Kepler's laws will be dealt with in detail in the next chapter. It may be pointed out that at large astronomical distances the planet and the sun may be treated as particles (point masses).

EXAMPLE 7.2 A particle of mass m is released from rest from point P at $x = x_0$ on the x-axis from origin O and falls vertically along the y-axis as shown in Fig. 7.17.

(a) Find the torque τ acting on the particle at a time t when it is at point Q with respect to O.

(b) Find the angular momentum L of the particle about O at this time t.

(c) Show that $\tau = \dfrac{dL}{dt}$ in this example.

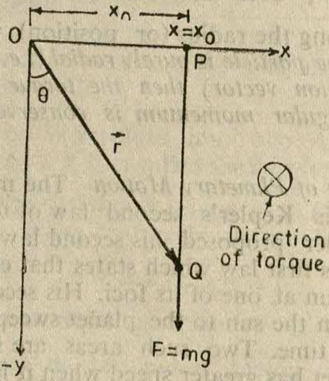


Fig. 7.17

Solution The torque is produced by the force of gravity $F = mg$, Let $\vec{r}$ be the position vector at Q.

(a) The magnitude of the torque is given by

$$= rF \sin \theta$$

$$= r \times mg \times \frac{x_0}{r} \qquad (\because \ \sin\theta = x_0/r)$$

$$= mg\, x_0 \qquad \text{(i)}$$

The direction of the torque is shown in the figure; it is directed into the page and perpendicular to it.

(b) The magnitude of the angular momentum is

$$L = rp \sin \theta$$

where $p = mv$; v being the velocity at Q which is given by

$$v = u + at = 0 + gt = gt$$

$$\therefore \quad L = r \times mgt \times \frac{x_0}{r} = mg\,x_0\,t \qquad \text{(ii)}$$

The direction of angular momentum is the same as that of torque.

(c) Differentiating (ii) with respect to t, we have

$$\frac{dL}{dt} = \frac{d}{dt}(mg\,x_0\,t) = mg\,x_0$$

$$= \tau \quad \text{[see Eq. (i)]}$$

Hence $\tau = \dfrac{dL}{dt}$ is obeyed in this example.

7.7 SOME EXAMPLES OF CIRCULAR MOTION OF RIGID BODIES

In Chapter 4 we have discussed the circular motion of a point mass. We now consider a few examples of circular motion of extended objects. The mass of such objects may be treated to be concentrated at their centre of mass. The first two examples illustrate centripetal acceleration and the third example will show how we can use the angular momentum to analyse the problem.

Banking of Round Tracks

When a car (or some other vehicle) negotiates a curved level road, the centripetal force required to keep the car in motion around the curve is provided by the friction between the road and the tyres. The weight of the car is supported by the normal reaction due to the earth. If the surface of the road is very rough, it provides a large amount of friction and hence the car can successfully negotiate the bend with a fairly high speed. If F is the total frictional force between the tyres and the road, then

$$F = \frac{mv^2}{R}$$

when m is the mass of the car, v its speed around the curve and R is the radius of the curved track. The higher the value of F, the faster is the speed at which the bend can be negotiated. If μ is the coefficient of friction between the tyres and the road, then $F \leqslant \mu N$; N = normal reaction = mg. The maximum speed which friction can sustain is

$$v \leqslant v_{\max} = (\mu Rg)^{1/2}$$

Banking of Curves

The large amount of friction between the tyres and the road would damage the tyres. To minimize the wearing out of tyres the road bed is banked, i.e. the outer part of the road is raised a little so that the road slopes towards the centre of the curved track. Suppose a car of mass m is moving around a banked track in a circular path of radius R as shown in Fig. (7.18). Let N_1 and N_2 be the reaction at each tyre due to the road. Then the total reaction is $N = N_1 + N_2$ acting in the middle of the car. If θ is the angle of the banking, the vertical component $N \cos\theta$ supports the weight mg of the car while the horizontal component $N \sin\theta$ provides the necessary centripetal force. Thus

$$N \sin\theta = \frac{mv^2}{R} - F\cos\theta$$

and $$N\cos\theta = mg + F\sin\theta$$

Also $$F = \mu N$$

when F is the force of friction acting radially inwards on the car. These equations give

$$\tan\theta = \frac{v^2 - \mu Rg}{Rg + \mu v^2} \quad \text{and} \quad v^2 = Rg\,\frac{\mu + \tan\theta}{1 - \mu\tan\theta}$$

The first equation determines the proper banking angle for given v, R and μ, and the second equation the maximum speed at which the car can successfully negotiate the curve for given R, μ and θ.

For given θ and R, there is an optimum (best) speed for negotiating a banked curve at which there will be the least wear and tear, i.e. when friction is not needed at all ($\mu = 0$). If $\mu = 0$, this speed is

$$v_0 = (Rg \tan\theta)$$

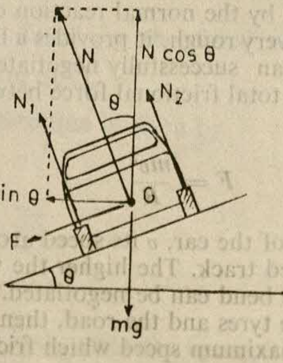
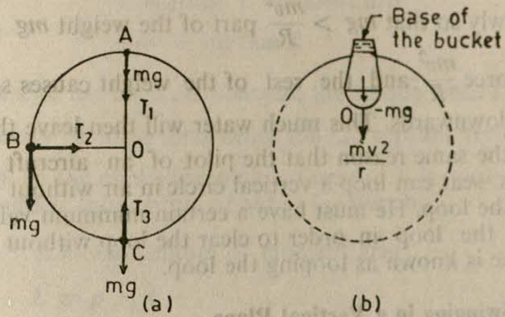


Fig. 7.18 Car on a banked curved road

Bucket of Water Whirled in a Vertical Circle

Figure 7.19 (a) shows an object of mass m whirled with a constant speed v in a vertical circle of centre O with a string of length R. When the object is at top A of the circle, let us say that the tension (force) in the string is T_1. Since the weight mg acts vertically downwards towards the centre O, we have,



(a) (b)

Fig. 7.19 Motion in a vertical circle

$$\text{Force towards centre, } F = T_1 + mg = \frac{mv^2}{R}$$

or
$$T_1 = \frac{mv^2}{R} - mg \tag{i}$$

At the point B, where OB is horizontal, the weight mg has no component along OB. Thus, if the tension in the string is T_2 at B, we have

$$\text{Force towards centre, } F = T_2 = \frac{mv^2}{R} \tag{ii}$$

At C, the lowest point of motion, the weight mg acts in the opposite direction to the tension T_3 in the string. Thus at C we have,

$$\text{Force towards centre, } F = T_3 - mg = \frac{mv^2}{R}$$

or
$$T_3 = \frac{mv^2}{R} + mg \tag{iii}$$

From (i), (ii) and (iii), we see that the maximum tension occurs at lowest point C of the motion. Here the tension T_3 must be greater than mg by $\frac{mv^2}{R}$ to keep the object in a circular path. The minimum tension T_{12} is given by (i) when the object is at the highest point A of the motion. Here part of the required centripetal force is provided by the weight and the rest by T_1.

Figure 7.19 (b) shows a bucket with water whirled in a vertical circle without water spilling out. When the bucket is vertically above the point O of support (as shown), the weight mg of water is less than the required centripetal force $\frac{mv^2}{R}$ towards the centre and so the water stays in the bucket without falling out. The rest of the centripetal force is balanced by the reaction of the base of the bucket. If the bucket is whirled slowly so that $mg > \frac{mv^2}{R}$ part of the weight mg provides the necessary force $\frac{mv^2}{R}$ and the rest of the weight causes some water to accelerate downwards. This much water will then leave the bucket.

It is for the same reason that the pilot of an aircraft who is not tied to his seat can loop a vertical circle in air without falling out at the top of the loop. He must have a certain minimum velocity at the bottom of the loop in order to clear the loop without any mishap. This exercise is known as looping the loop.

Pendulum Swinging in a Vertical Plane

This example illustrates the use of angular momentum and its relation to torque (Eq. 7.26). A string of length l, one end of which is fixed at O, carries a bob of mass m at the other. We will consider the oscillations of the pendulum bob in a vertical plane (Fig. 7.20)

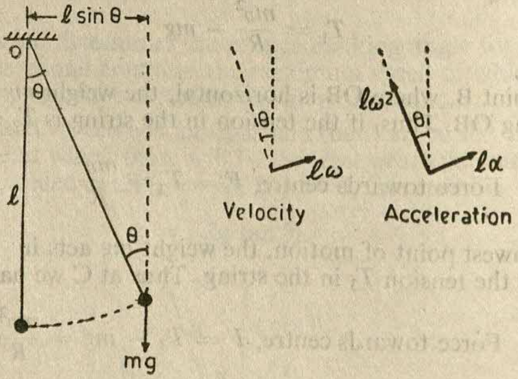


Fig. 7.20 Pendulum swinging in a vertical plane

The position of the bob is specified by the angle θ which the string makes with the vertical. The pendulum moves in a circular path of radius l. The angular velocity and angular acceleration are given by (see Eq. 7.13, see also Ch. 4).

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The velocity vector is tangential to the circle and the acceleration vector has two rectangular components, the centripetal or radial component $l\omega^2$ directed towards centre O and a tangential component $l\alpha$, as shown in the figure.

The force $F(= mg)$ of gravity acts vertically downwards. The torque produced by this force about O is given by

$$\tau = \text{force} \times \text{moment arm}$$

or

$$\tau = -\, mgl \sin\theta \tag{i}$$

The negative sign has been used because the torque tends to decrease θ.

Since velocity vector $\vec{v}$ (and hence the momentum vector $\vec{p} = m\,\vec{v}$) are tangential to the circle, the moment arm of $\vec{p}$ is always equal to l. Hence the angular momentum about O is given by

$$L = \text{momentum} \times \text{moment arm}$$

or

$$\begin{aligned} L &= p \times l \\ &= mvl = m\,(l\,\omega)\,l \qquad (\because \ v = l\,\omega) \\ &= ml^2\omega \end{aligned}$$

or

$$L = ml^2 \frac{d\,\theta}{dt} \qquad \left(\because \ \omega = \frac{d\theta}{dt}\right)$$

Differentiating with respect to t we get

$$\frac{dL}{dt} = ml^2 \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = ml^2 \frac{d^2\theta}{dt^2} = ml^2\,\alpha \qquad \left(\because \ \alpha = \frac{d^2\theta}{dt^2}\right)$$

From the angular momentum law (Eq. 7.26), we have

$$\frac{dL}{dt} = \tau = ml^2\alpha$$

Using Eq. (i) we get

$$-\, mgl \sin\theta = ml^2\alpha$$

or

$$\alpha = \frac{d^2\,\theta}{dt^2} = -\,\frac{g}{l}\sin\theta$$

This equation governs the oscillation of a pendulum, as the detailed analysis of this problem in Chapter 12 will show.

EXAMPLE 7.3 A car moves at a speed of 36 km h^{-1} on a level road. The coefficient of friction between the tyres and the road is 0.8. The car negotiates a curve of radius 10 m at this speed. Will the car skid (or slip) while negotiating the curve? Take $g = 10\ \text{m s}^{-2}$.

Solution

Speed of the car (v) $= 36 \text{ km h}^{-1} = 10 \text{ m s}^{-1}$

The maximum centripetal force that the friction can provide is

$$f_{\max} = \mu\, mg = \frac{mv^2}{R}$$

$$\therefore \qquad R_{\min} = \frac{v^2}{\mu g} = \frac{10 \times 10}{0.8 \times 10} = 12.5 \text{ m}$$

This is the minimum radius the curve must have for the car to negotiate it at 10 ms^{-1}. But the radius of the curve is only 10 m. Hence the car will skid while negotiating the curve.

EXAMPLE 7.4 A train has to negotiate a curve of radius 300 m. By how much should the outer rails be raised with respect to the inner rails for a speed of 54 km h^{-1}. The distance between the rails is 1.5 m. Take $g = 10 \text{ m s}^{-2}$.

Solution

Speed of train (v) $= 54 \text{ km h}^{-1} = 15 \text{ m s}^{-1}$

Radius of the curve (R) $= 300$ m

Let the outer rails be raised to a height h with respect to the inner rails so that the angle of banking is θ (Fig. 7.21). Then

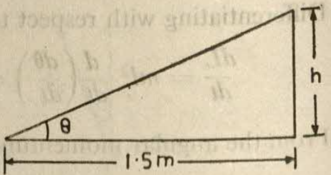


Fig. 7.21

$$\tan\theta = \frac{h}{1.5} = \frac{v^2}{Rg}$$

or

$$h = \frac{1.5 \times (15)^2}{300 \times 10}$$

$$= 0.1125 \text{ m} = 11.25 \text{ cm}$$

EXAMPLE 7.5 A highway has a curved track of radius 100 m which is banked at an angle of 10°. If the coefficient of friction between the wheels of a car and the track is 0.25, what is (a) the optimum speed of the car to avoid wear and tear of the tyres and (b) the maximum speed at which it can negotiate the curve without skidding?

Solution

$$R = 100 \text{ m}$$

$$\theta = 10^\circ$$

$$\mu = 0.25$$

(a) Optimum speed is given by

$$v_0 = (Rg \tan \theta)^{1/2}$$
$$= (100 \times 9.8 \times \tan 10^\circ)^{1/2}$$
$$= 13.1 \text{ ms}^{-1}$$

(b) Maximum permissible speed is given by

$$v_{\max} = \left(Rg \, \frac{\mu + \tan \theta}{1 - \mu \tan \theta} \right)^{1/2}$$
$$= \left(100 \times 9.8 \times \frac{0.25 + \tan 10^\circ}{1 - 0.25 \tan 10^\circ} \right)^{1/2}$$
$$= 20.9 \text{ ms}^{-1}$$

7.8 TORQUE AND ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES

We have so far considered torque and angular momentum of single patricles. We now consider a system consisting of many particles. To calculate the total torque of the system about a fixed reference point (which we choose to be the origin O of a reference frame) we add vectorially the torque vectors of all the individual particles of the system about that point O. The total torque of a system consisting of N particles is then given be

$$\vec{\tau}_{\text{tot}} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_N$$
$$= \sum_{n=1}^{N} \vec{\tau}_n \qquad (7.29)$$

where the vector sum is taken over all the particles of the system. The total torque can be due to two sources.

1. Torques exerted on the particles of the system by mutual *internal* forces between the particles, and
2. Torques exerted on the individual particles of the system by *external* forces.

From Newton's third law the torques due to internal forces do not contribute anything to the total torque of the system. The reason is that the forces between any two particles are not only equal and opposite but also act along the line joining them. Consequently the torque due to each internal action-reaction pair is zero. Hence only external sources contribute and we can write Eq. (7.29) as

$$\vec{\tau}^{\text{tot}}_{\text{ext}} = \sum_{n=1}^{N} \vec{\tau}_n \qquad (7.30)$$

When we have only one particle, there are no internal forces or torques.

The total angular momentum of the system about the reference point O is similarly calculated, it is given by the vectorial sum of the augular momenta of all the individual particles of the system about the same point O. Thus

$$\vec{L}^{\text{tot}} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \ldots + \vec{L}_N$$

$$= \sum_{n=1}^{N} \vec{L}_n \qquad (7.31)$$

The total angular momentum of the system about the fixed reference point may change with time. This change $\frac{d\vec{L}^{\text{tot}}}{dt}$ cannot be due to torques exerted on particles by internal forces, as explained above; it can only be due to torques exerted by external forces which add up to give $\vec{\tau}^{\text{tot}}_{\text{ext}}$. Thus we can write

$$\vec{\tau}^{\text{tot}}_{\text{ext}} = \frac{d\vec{L}^{\text{tot}}}{dt} \qquad (7.32)$$

i.e., *the rate of change of angular momentum of a system of particles about the origin of a reference frame is equal to the total external torque acting on the system about the same origin.*

Equation (7.32) is the generalization of Eq. (7.26) to include many particles. Equation (7.32) holds irrespective of whether the particles constituting the system are in motion relative to each other (i.e. internal motions) or not or whether the particles have fixed positions relative to each other (as in a rigid body) or not. Equation (7.32) holds even when the reference point is chosen to be the centre of mass of the system, in spite of the fact that this point moves in our reference frame. This remarkable property of the centre of mass simplifies the analysis of a complex motion to a large extent, for then we can separate the general motion of a system into the translational motion of its centre of mass (Eq. 7.9) and the rotational motion about its centre of mass (Eq. 7.32).

Conservation of Angular Momentum in the Case of Rigid Bodies

We have stated above that Eq. (7.32) holds for a rigid body rotating about an axis. In the special case when the total torque acting on the body due to external forces is zero, i.e $\vec{\tau}^{\text{tot}}_{\text{ext}} = 0$, then from Eq. (7.32) we have

$$\frac{d\vec{L}^{\text{tot}}}{dt} = 0$$

or

$$\vec{L}^{\text{tot}} = \text{constant}$$

When the total external torque acting on a system of particles (*or on a rigid body*) *is zero, the total angular momentum of the system remains constant.* This is the *principle of conservation of angular momentum.* The angular momenta of individual particles of the system may change but their vector sum $\vec{L}^{\text{tot}}$ remains constant in the absence of a net external torque. The net external torque can be zero due to two reasons: (1) absence of all external forces (*isolated* system) or (2) the external forces producing torques on individual particles adding up to zero (vectorially).

7.9 ROTATIONAL MOTION AND MOMENT OF INERTIA

Let us consider a rigid body rotating with a unifoem angular velocity ω about an axis passing through O perpendicular to the plane of the paper as shown in Fig. 7.22.

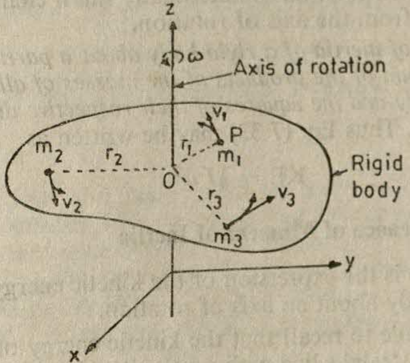


Fig. 7.22 Rotational kinetic energy and moment of inertia of a rigid body

The body is made up of a large number of particles. Consider a particle P of mass m_1 at a distance r_1 from the axis of rotation. Let its linear velocity be v_1. The kinetic energy of this particle is

$$= \tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 r_1^2 \omega^2 \qquad (\because\ v = r\,\omega)$$

Similarly, the kinetic energies of all other particles (of the body) of masses m_2, m_3, etc. are respectively $\frac{1}{2} m_2 r_2^2 \omega^2$, $\frac{1}{2} m_3 r_3^2 \omega^2$, etc. where r_2, r_3, etc. are their respective distances from the axis of rotation. Hence the total kinetic energy (KE) of the body is given by

$$\text{KE} = \tfrac{1}{2} m_1 r_1^2 \omega^2 + \tfrac{1}{2} m_2 r_2^2 \omega^2 + \tfrac{1}{2} m_3 r_3^2 \omega^2 + \ldots$$

$$= \tfrac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots)\,\omega^2 \qquad (7.33)$$

or

$$\text{KE} = \tfrac{1}{2} \omega^2 \textstyle\sum (mr^2)$$

The quantity $\sum mr^2$ is the sum of the products of the masses of the various particles composing the body and the squares of their respective distances from the axis of rotation. This quantity is called the *moment of inertia* of the body and its value depends upon the particular axis about which the body rotates and the way the mass is distributed in the body with respect to the axis of rotation. Moment of inertia (abbreviated as MI) is usually designated by the symbol I.

In the two-dimensional rotation of the body about the z-axis, Eq. (7.33) may be written in terms of (x, y) components of $\vec{r}$ as

$$\text{KE} = \tfrac{1}{2}\,\omega^2 \sum m\,(x^2 + y^2) \tag{7.34}$$

In the case of a body which does not consist of separate, discrete particles but has a continuous and homogeneous distribution of matter in it, the summation in Eq. (7.33) is replaced by integration, so that

$$I = \int r^2\, dm \tag{7.35}$$

where dm is the mass of an infinitesimally small element of the body at a distance r from the axis of rotation.

The moment of inertia of a rigid body about a particular axis may be defined as the sum of the products of the masses of all the particles constituting the body and the squares of their respective distances from the axis of rotation. Thus Eq. (7.33) may be written as

$$\text{KE} = \tfrac{1}{2} I\,\omega^2 \tag{7.36}$$

Physical Significance of Moment of Inertia

Equation (7.36) is the expression of the kinetic energy of the rotational motion of a body about an axis of rotation.

It is worthwhile to recall that the kinetic energy of a body of mass m moving in a straight line with a velocity v is given by

$$\text{KE} = \tfrac{1}{2}\, mv^2 \tag{7.37}$$

We know that linear velocity v is an analogue of the angular velocity ω in rotational motion. Equation (7.36) for the kinetic energy in rotational motion is an analogue of Eq. (7.37) for the kinetic energy in linear (or translational) motion. A comparison of the two equations suggests that the *moment of inertia plays the same role in rotational motion as mass does in translational motion.*

We have learnt in Chapter 4 that the inability of a body to change by itself its state of rest or uniform motion along a straight line is an inherent property of matter and is called inertia (Newton's first law of motion). The greater the mass of a body, the more difficult it is for a force to move it or stop it if it is already moving. Thus, mass can be regarded as a measure of inertia for linear or translational motion.

In exactly the same way, a body free to rotate about an axis has a tendency to oppose any change in its state of rest or of uniform rotation. In other words, it possesses inertia for rotational motion. i.e. it opposes the torque or the moment of the couple applied to change its state of rotation. Since only a couple can oppose another couple, the nature of the inertia must be that of a moment. Hence the name *moment of inertia* is given to it. The higher the moment of inertia of a body about an axis, the more difficult it is for a couple to rotate it or stop its rotation about the axis. Thus *moment of inertia can be regarded as a measure of inertia for rotational motion.*

Unit of Moment of Inertia

It is clear from Eq. (7.35) that the dimensions of moment of inertia I are (length)^2 × (mass), i.e. ML^2. In the SI system the moment of inertia is, therefore, measured in kg m^2. It may be noted that since the moment of inertia of a body about a given axis remains unchanged even when the direction of rotation about that axis is reversed, it is a *scalar quantity*, once its axis of rotation is fixed. Also the dimensions of KE of rotation $= \frac{1}{2} I \omega^2$ are (dimensions of I) × (dimensions of ω^2) $= \mathrm{ML^2T^{-2}} \left(\because \omega = \frac{2\pi}{T}\right)$ which are the same as those of work or energy. Hence, KE $(= \frac{1}{2} I \omega^2)$ is measured in joules.

Radius of Gyration

The radius of gyration of a body about its axis of rotation may be defined as the distance from the axis of rotation at which, if the entire mass of the body were concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass. It is usually denoted by the letter K. If M is the mass of the body. its moment of inertia I in terms of its radius of gyration K can be written

$$I = MK^2 \tag{7.38}$$

We can determine the radius of gyration of a body rotating about a given axis as follows:

Consider a body of mass M consisting of n particles each of mass m, situated at distance $r_1, r_2, r_3, \ldots r_n$ from the axis of rotation (Fig. 7.23).

The moment of inertia about the given axis is given by

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_nr_n^2$$

$$= m\,(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)$$

$(\because \quad m_1 = m_2 = \ldots = m)$

or $$I = mn\frac{(r_1^2 + r_2^2 + r_3^2 \ldots + r_n^2)}{n}$$

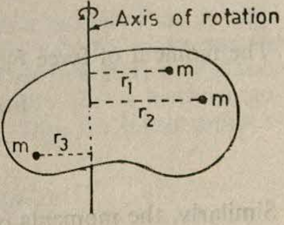


Fig. 7.23 The radius of gyration

The quantity $\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}$ is clearly the mean (average) of the sum of the squares of distances of individual particles. Let us denote it by $\bar{r}^2$ which is called the mean square distance. Thus

$$I = M\bar{r}^2 \qquad (\because \; M = mn)$$

$$\text{or} \qquad MK^2 = M\bar{r}^2$$

$$\text{or} \qquad K^2 = \bar{r}^2$$

$$\text{or} \qquad K = \sqrt{\bar{r}^2}, \text{ i.e. } K = \text{root mean square distance.}$$

Thus K is some kind of an average effective distance of the particles from the axis of rotation. So, the radius of gyration of a body about a particular axis of rotation is equal to the root mean square distance of its particles from the axis of rotation. K has the dimensions of L and is measured in metres.

Torque and Moment of Inertia

When a torque acts on a rigid body that is capable of rotation about an axis, it produces an angular acceleration in the body. If the angular velocity of each particle of the body is ω, the angular acceleration is $\frac{d\omega}{dt}$ which is the same for all particles of the body but the linear acceleration varies with the distance $r_1, r_2, \ldots$ etc. of the particles from the axis of rotation (see Fig. 7.22).

Consider a particle P of mass m_1 rotating about an axis O with an angular velocity ω as shown in the figure. If the distance of this particle from the axis of rotation is r_1, its linear acceleration is $\frac{dv_1}{dt} = \frac{d}{dt}(r_1\,\omega) = r_1\frac{d\omega}{dt}$, where v_1 is its linear velocity.

$$v_1 = r_1\omega$$

The force F_1 acting on this particle is given by

$$F_1 = \text{Mass } m_1 \times \text{ Acceleration of } m_1$$

$$= m_1 r_1 \frac{d\omega}{dt}$$

The moment of force F_1 about the axis of rotation is

$$\text{Force } (F_1) \times \text{ Distance } (r_1)$$

$$= m_1\, r_1^2 \frac{d\omega}{dt}$$

Similarly, the moments of the forces on other particles of masses m_2, $m_3, \ldots, m_n$, etc. are $m_2 r_2^2 \frac{d\omega}{dt}$, $m_3\, r_3^2 \frac{d\omega}{dt}, \ldots, m_n r_n^2 \frac{d\omega}{dt}$, etc. where r_2, r_3,

. . . , r_n are their respective distances from O. The sum of these individual moments is the total torque acting on the body, i.e.

$$\tau = m_1 r_1^2 \frac{d\omega}{dt} + m_2 r_2^2 \frac{d\omega}{dt} + \ldots + m_n r_n^2 \frac{d\omega}{dt}$$

$$= (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2) \frac{d\omega}{dt}$$

$$= (\sum mr^2) \frac{d\omega}{dt}$$

or
$$\tau = I \frac{d\omega}{dt} = I \alpha \tag{7.39}$$

where $I = \sum mr^2$ is the moment of inertia of the body about the given axis and $\alpha = \frac{d\omega}{dt}$ is the angular acceleration produced by torque τ. Thus,

Torque = Moment of inertia × Angular acceleration

Comparing Eq. (7.39) for rotational motion with the equation $F = ma$ in translation motion, we see that the moment of inertia I corresponds to mass m, angular acceleration α corresponds to linear acceleration a and torque τ corresponds to force F. Equation (7.39) gives only the magnitude of $\vec{\tau}$.

From Eq. (7.39) it follows that if $\frac{d\omega}{dt}$ is unity, $\tau = I$. This gives us a new definition of I, i.e. *the moment of inertia of a body about a given axis is the torque that produces in it a unit angular acceleration.*

Angular Momentum and Moment of Inertia

In linear motion, an important property of a moving object is its linear momentum. When an object rotates about an axis, its angular momentum plays an equally important part in its motion. The angular momentum of a body rotating about an axis is defined as the moment of linear momentum about that axis.

Let us again refer to Fig. 7.22 and consider a particle P of a rigid body rotating about an axis perpendicular to the (x, y) plane passing through the point O. Let m_1 be the mass of particle P, r_1 its distance from the axis of rotation and v_1 its linear velocity. Let ω be the angular velocity of the body about the given axis. Then the linear momentum of P has a magnitude p_1 given by

$$p_1 = \text{Mass} \times \text{Linear velocity}$$

$$= m_1 v_1$$

$$= m_1 r_1 \omega \quad (\because \; v_1 = r_1 \omega)$$

The magnitude of the angular momentum of P about O

$$\begin{aligned}
&= \text{Moment of momentum } p_1 \text{ about O}\\
&= \text{Momentum} \times \text{perpendicular distance (moment arm)}\\
&= p_1 \times r_1\\
&= m_1 r_1^2\, \omega
\end{aligned}$$

Similarly, the magnitude of angular momentum of other particles is $m_2 r_2^2\, \omega$, $m_3 r_3^2\, \omega$, $m_n\, r_n^2\, \omega$. Therefore, the magnitude of total angular momentum of the body about the given axis is given by

$$\begin{aligned}
L &= m_1 r_1^2\ \omega + m_2 r_2^2\, \omega + \ldots + m_n r_n^2\, \omega\\
&= (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2)\ \omega\\
&= (\textstyle\sum mr^2)\ \omega
\end{aligned}$$

or
$$L = I\,\omega \qquad (7.40)$$

where $I = \sum mr^2$ is the moment of inertia of the body about the given axis of rotation. Equation (7.40) gives only the magnitude of angular momentum.

Illustrations of Conservation of Angular Momentum

1. A diver jumps from a high diving board keeping his legs and arms outstretched. His body has a certain moment of inertia and a certain angular velocity about the centre of gravity of his body. To make a somersault he curls his body as shown in Fig. 7.24. This decreases the moment of inertia and hence increases his angular velocity. He can then make more somersaults before entering the water below.

2. Dancers on skates can spin faster by folding their arms.

3. Figure. 7.25 shows a person standing on a turn table holding a pair of heavy dumb-bells, one in each hand with his arms outstret-

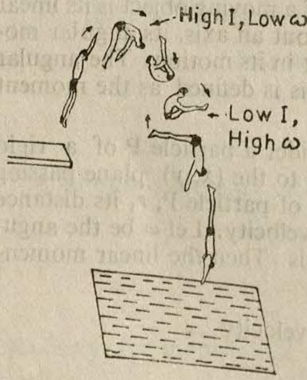


Fig. 7.24 A diver turning a somersault

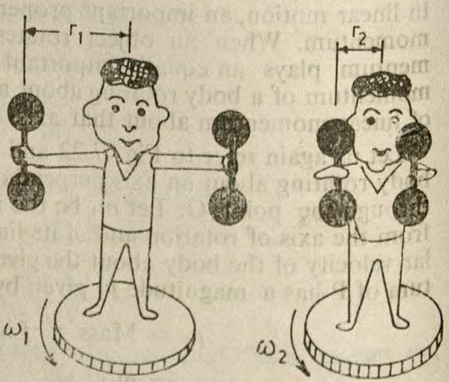


Fig. 7.25 Conservation of angular momentum. If $r_1 > r_2$, then $\omega_2 > \omega_1$.

ched. The table is rotating with a certain angular frequency. The person suddenly pushes the weights towards his chest as shown in Fig. 7.25. The speed of rotation is found to increase considerably (see Example 7.8 on page 241).

Analogy between Linear and Rotational Motion

The following table gives a summary of the analogy between various quantities that describe linear motion and those that describe rotational motion.

Linear Motion	*Rotational Motion*
position x	Angle θ
Linear velocity $v = \frac{dx}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
Linear acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Mass m	Moment of inertia I
Force F	Torque τ
$F = ma$	$\tau = I\alpha$
Translational KE $= \frac{1}{2} mv^2$	Rotational KE $= \frac{1}{2} I \omega^2$
Linear momentum $P = mv$	Angular momentum $L = I\omega$
Linear momentum of a system is conserved in the absence of net external force.	Angular momentum of a system is conserved in the absence of net external torque.
Equations of motion	*Equations of motion*
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$S = v_0 t + \frac{1}{2} at^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 - v_0^2 = 2\,ax$	$\omega^2 - \omega_0^2 = 2\,\alpha\,\theta$

Parallel Axis Theorem

There is a simple and useful relation between the moment of inertia I of a body about any axis and its moment of inertia with respect to a parallel axis through its centre of mass. If M is the total mass of the body and h the distance between the two axes, the relation is

$$I = I_{CM} + Mh^2 \tag{7.41}$$

This relation is known as the *parallel axis theorem.*

Proof Consider a body of any arbitrary shape whose cross-section in the x-y plane is shown in Fig. 7.26. Let C be the centre of mass which we choose to be in the x-y plane and at the origin O. Then coordinates of point C are ($x_{CM} = 0, y_{CM} = 0$). The axis of rotation is the z-axis which is perpendicular to the plane of the paper. Consider an axis through C perpendicular to the plane of the paper and another axis parallel to it through point P at (x_0) and (y_0) which are the (x, y) coordinates of P.

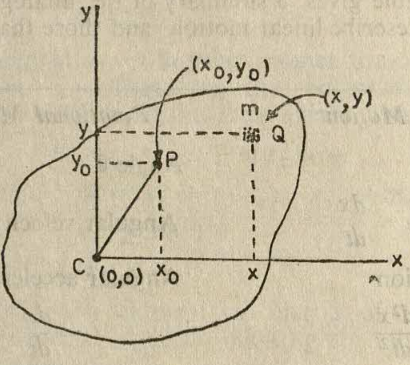


Fig. 7.26 Proof of the parallel axis theorem

The distance PC between the two axes (both perpendicular to the plane of the paper) is

$$h = (x_0^2 + y_0^2)^{1/2} \tag{7.42}$$

Now consider a small element of mass m at Q having position coordinates x, y as shown in the figure. Then the square of the distance of particle Q from the axis through C is $(x^2 + y^2)$. The square of the distance of particle Q from an axis through P is

$$(x - x_0)^2 + (y - y_0)^2$$

Hence the moment of inertia of the body about an axis through P is

$$\begin{aligned} I &= \sum m\,[(x - x_0)^2 + (y - y_0)^2] \\ &= \sum m\,(x^2 + y^2) - 2x_0 \sum mx - 2y_0 \sum my \\ &\qquad + (x_0^2 + y_0^2) \sum m \end{aligned} \tag{7.43}$$

From the definition of centre of mass

$$\sum mx = \sum my = 0 \tag{7.44}$$

The reason is that we have chosen the centre of mass C at the origin of the rectangular coordinate system (see Eq. 7.8); the position vector $\vec{R}_{CM}$ with respect to the origin has a zero length, i.e. $\sum mx = \sum my = 0$. Using Eq. (7.44) in Eq. (7.43) we have

$$I = \sum m\,(x^2 + y^2) + (x_0^2 + y_0^2) \sum m$$

But $\sum m\,(x^2 + y^2) = I_{\text{CM}}$, the moment of inertia of the body about its centre of mass, $x_0^2 + y_0^2 = h^2$ (Eq. 7.42) and $\sum m = M$, the mass of the body.

Hence $$I = I_{\text{CM}} + Mh^2$$

which proves the parallel axis theorem. The theorem holds irrespective of whether the centre of mass C is chosen to coincide with the origin of the reference frame; this has been done only for the sake of simplicity.

Calculation of Moment of Inertia

The calculation of the moment of inertia of a rigid body about an axis of rotation involves the use of integral calculus (which is outside the scope of the present book). We simply quote below the results of such calculations in a few cases.

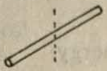
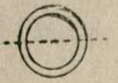
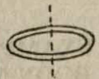
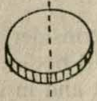
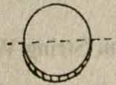
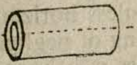
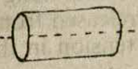
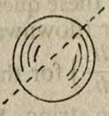
Body	*Axis*	*Figure*	*Moment of Inertia (I)*
1. Thin rod, length L	Perpendicular to rod at midpoint		$\frac{1}{12} ML^2$
2. Thin circular ring, radius R	Diameter		$\frac{1}{2} MR^2$
3. Thin circular ring, radius R	Perpendicular to plane at centre		$\frac{1}{4} MR^2$
4. Circular disc, radius R	Perpendicular to disc at centre		$\frac{1}{2} MR^2$
5. Circular disc, radius R	Diameter		$\frac{1}{4} MR^2$
6. Hollow cylinder radius R	Axis of cylinder		MR^2
7. Solid cylinder, radius R	Axis of cylinder		$\frac{1}{2} MR^2$
8. Sphere, radius R	Diameter		$\frac{2}{5} MR^2$

Fig. 7.27 Moments of inertia of some rigid bodies

EXAMPLE 7.6 A thin metal ring of mass 0.5 kg has a diameter of 0.5 m. It is rotating in a horizontal plane about an axis passing vertically through its centre. If the ring completes 10 revolutions in 5 s, determine (i) its angular velocity and (ii) its rotational kinetic energy.

Solution

$$M = 0.5 \text{ kg}$$

$$R = \tfrac{1}{2}\,(\text{Diameter}) = \tfrac{1}{2} \times 0.5 = 0.25 \text{ m}$$

Moment of inertia $I = MR^2$

$$= 0.5 \times (0.25)^2 = \frac{1}{32} \text{ kg m}^2$$

Time period,

$$T = \frac{\text{Time}}{\text{No. of revolutions}} = \frac{5}{10} = 0.5 \text{ s}$$

(i) Angular velocity $\omega = \dfrac{2\pi}{T} = \dfrac{2 \times 3.14}{0.5} = 12.56 \text{ rad s}^{-1}$

(ii) Kinetic energy $= \tfrac{1}{2} I \omega^2$

$$= \tfrac{1}{2} \times \frac{1}{32} \times (12.56)^2$$

$$= 2.46 \text{ J}$$

7.10 SOME EXAMPLES OF TWO-DIMENSIONAL RIGID BODY MOTION

We will now consider two simple examples of the two-dimensional motion of rigid bodies. In the first example, the centre of mass of the body is at rest and in the second, it moves with an acceleration.

Mass Point on String Wound on a Cylinder

A solid cylinder of radius R and mass M is symmetrically mounted on a frictionless horizontal axle, so that it can freely rotate about its axis. A string of negligible mass is wrapped round the cylinder and a body of mass m is hung from the string as shown in Fig. 7.28. When the mass is released from rest, with what acceleration will it fall and what is the tension in the string?

To answer these questions we will use Newton's second law ($F = ma$) for the linear (downward) motion of m and the angular momentum law $\left(\tau = \dfrac{dL}{dt}\right)$ for the rotational motion of the cylinder. Let T be the tension in the string. Since the suspended body will accelerate downwards, the magnitude of the downward pull of gravity ($= mg$) on it must exceed the upward tension T in the string; the tension being supplied by the falling body. From Newton's second law, the acceleration a of the falling body is given by

$$ma = mg - T \tag{7.45}$$

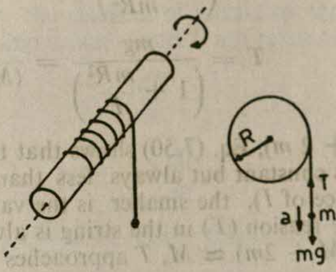


Fig. 7.28 String wound on a cylinder and carrying a mass

The torque on the cylinder about its axis is

$$\tau = \text{force} \times \text{moment arm}$$

or

$$\tau = TR \tag{7.46}$$

The angular momentum of the cylinder about its axis is

$$L = I\,\omega \tag{7.47}$$

where $I = \frac{1}{2} MR^2$ is the moment of inertia of the cylinder about its axis and ω is the angular velocity of its rotation. Now we know that L and τ are related as

$$\tau = \frac{dL}{dt}$$

Substituting for τ and L from Eqs (7.46) and (7.47) in this equation we get

$$TR = \frac{d}{dt}\,(I\,\omega) = I\,\frac{d\omega}{dt} = I\,\alpha \tag{7.48}$$

where α is the angular acceleration of the cylinder. Now, the linear acceleration a and angular acceleration α are related as

$$a = R\,\alpha$$

so that Eq. (7.48) becomes

$$TR = \frac{Ia}{R} \qquad \text{or} \qquad T = \frac{Ia}{R^2} \tag{7.49}$$

Solving Eqs (7.45) and (7.49) simultaneously we get

($\because \quad I = \frac{1}{2} MR^2$)

$$a = \frac{g}{\left(1 + \frac{I}{mR^2}\right)} = \frac{2mg}{(M + 2m)} \qquad (7.50)$$

and

$$T = \frac{mg}{\left(1 + \frac{mR^2}{I}\right)} = \frac{Mmg}{(M + 2m)} \qquad (7.51)$$

Since $2\,m < (M + 2\,m)$, Eq. (7.50) shows that the acceleration (a) of the falling mass is constant but always less than g. The larger the value of M (and hence of I), the smaller is the value of a. Equation (7.51) shows that the tension (T) in the string is always less than mg. If $M \gg m$, so that $(M + 2m) \simeq M$, T approaches mg.

Cylinder Rolling on an Inclined Plane Without Slipping

Consider a solid cylinder of mass M and radius R rolling down an inclined plane at an angle θ to the horizontal (Fig. 7.29). The friction prevents slipping.

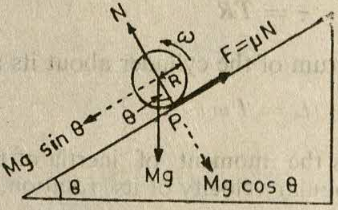


Fig. 7.29 Cylinder rolling without slipping

In this example, the centre of mass moves in a line parallel to the inclined plane. What is its acceleration along the plane ? The external forces on the cylinder are (i) its weight Mg acting vertically downwards through its centre of gravity (or mass) (see Chapter 8), (ii) the normal reaction N at the point of contact P between the cylinder and the plane perpendicular or the plane and (iii) the force of friction F opposing the velocity v of the cylinder.

The cylinder will roll without slipping if the point of contact P is momentarily stationary so that the cylinder can rotate about that axis. If a is the linear acceleration of the translational motion of the cylinder down the inclined plane, then from Newton's second law, the linear motion of the centre of mass is given by (see Fig. 7.29).

$$Ma = Mg \sin\theta - F \qquad (7.52)$$

Let α be the angular acceleration of the rotational motion of the cylinder. Then a and α are related as

$$a = R\,\alpha \qquad (7.53)$$

The torque and the angular momentum about the horizontal axis through the centre of mass are

$$\tau = RF$$

and $$L = I\,\omega$$

where $I = \frac{1}{2} MR^2$, is the moment of inertia of the cylinder about its centre of mass. We know that τ and L are related as

$$\tau = \frac{dL}{dt}$$

or $$RF = I\frac{d\omega}{dt} = I\,\alpha$$

Now from Eq. (7.53) $\alpha = a/R$. Using this in the above equation, we have

$$a = \frac{R^2 F}{I} \tag{7.54}$$

Finally, eliminating F from Eqs (7.52) and (7.54) we get

$$a = \frac{g \sin\theta}{\left(1 + \dfrac{I}{MR^2}\right)} = \frac{2}{3} g \sin\theta \tag{7.55}$$

Since $$I = \tfrac{1}{2} MR^2$$

From Eq. (7.55) we notice that the linear acceleration is constant for a given inclined plane (i.e. given θ) independent of the mass (M) or radius (R) of the solid cylinder. But, if the cylinder were hollow, its $I = MR^2$ (see Fig. 7.27), the value of a would decrease to $\frac{1}{2} g \sin\theta$.

Condition for Rolling Without Slipping Eliminating a from Eqs (7.52) and (7.54) and using $I = \frac{1}{2} MR^2$, we have

$$F = \tfrac{1}{3} Mg \sin\theta$$

From Fig. 7.29, it is clear that the normal reaction is given by

$$N = Mg \cos\theta$$

So that $$\frac{F}{N} = \tfrac{1}{3} \tan\theta$$

Let μ be the coefficient of static friction between the cylinder and the inclined plane. Then the cylinder will roll without slipping if

$$\mu \leqslant \frac{F}{N}$$

or $$\mu \leqslant \tfrac{1}{3} \tan\theta$$

Another very instructive method of analysing such problems is to use energy considerations as in Example 7.7

EXAMPLE 7.7 A solid cylinder of mass 2 kg and radius 0.1 m rolls down an inclined plane of height 3 m. Calculate its rotational energy when it reaches the foot of the plane.

Solution Let M be the mass of the cylinder and R be its radius. When it is at the top of the inclined plane of height h, its potential energy is Mgh (Fig. 7.30). As it rolls down the inclined plane, it moves along the plane and also rotates about an axis passing through its axis perpendicular to the plane of the figure, which shows its section in the plane of the paper. It thus acquires both kinetic energy of translation ($\frac{1}{2} Mv^2$) and kinetic energy of rotation ($\frac{1}{2} I \omega^2$) where v is its linear velocity and ω its angular velocity when it reaches the bottom of the plane. I is its moment of inertia about the axis mentioned above which is given by

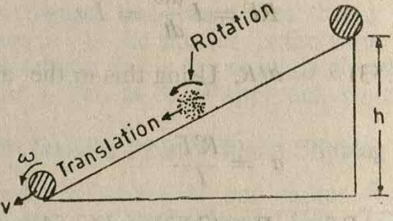


Fig. 7.30

$$I = \frac{1}{2} MR^2$$

From the law of conservation of energy we have

Potential energy = Translational kinetic energy + Rotational kinetic energy

$$\text{or} \qquad Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} I \omega^2$$

$$\text{or} \qquad Mgh = \frac{1}{2} MR^2 \omega^2 + \frac{1}{2} (\frac{1}{2} mR^2) \omega^2$$

$$= \frac{3}{4} MR^2 \omega^2$$

$$\text{or} \qquad \omega^2 = \frac{4gh}{3R^2}$$

Now the rotational kinetic energy $= \frac{1}{2} I \omega^2$

Substituting for ω^2 and I we have

$$\text{Rotational kinetic energy} = \tfrac{1}{2}\left(\tfrac{1}{2} MR^2\right)\frac{4gh}{3R^2}$$

$$= \frac{Mgh}{3}$$

$$= \frac{2\text{ kg} \times 9.8\text{ m s}^{-2} \times 3\text{ m}}{3} = 19.6\text{ J}$$

Notice that the rotational kinetic energy does not depend on the radius of the cylinder.

EXAMPLE 7.8 A lightweight boy holds two heavy dumb-bells of equal weight with outstretched arms while standing on a turn-table as shown in Fig. 7.25(a). He is pushed until he rotates at a rate of one revolution per second. The dumb-bells are at a distance of 60 cm from the axis of rotation. The boy suddenly pulls the dumb-bells towards his chest as shown in Fig. 7.25(b). If the dumb-bells are now at a distance of 10 cm from the axis of rotation, what is the new time period of rotation of the turn-table? Ignore the angular momentum of the boy compared to that of the dumb-bells.

Solution Referring to Fig. 7.25(a), we have

$$r_1 = 60\text{ cm}$$

$$T_1 = 1\text{ s}$$

$$\omega_1 = \frac{2\pi}{T_1} = 2\,\pi\text{ rad s}^{-1}$$

Let the mass of each dumb-bell be m. Then,

$$\text{Total initial angular momentum} = mr_1^2\,\omega_1 + mr_1^2\,\omega_1 = 2mr_1^2\,\omega_1$$

Referring to Fig. 7.25(b), when the boy pulls the dumb-bells towards his chest, let the new value of period be T_2. It is given that

$$r_2 = 10\text{ cm}$$

$$\omega_2 = \frac{2\pi}{T_2}$$

$$\text{Final angular momentum} = 2\,mr_2^2\,\omega_2$$

From the principle of conservation of angular momentum.

$$\text{Initial angular momentum} = \text{Final angular momentum}$$

or

$$2\,mr_1^2\,\omega_1 = 2m\,r_2^2\,\omega_2$$

$$\frac{\omega_1}{\omega_2} = \frac{r_2^2}{r_1^2}$$

Now, since $\omega_1 = \dfrac{2\pi}{T_1}$ and $\omega_2 = \dfrac{2\pi}{T_2}$, we have

$$\frac{T_2}{T_1} = \frac{r_2^2}{r_1^2}$$

or
$$T_2 = T_1 \frac{r_2^2}{r_1^2}$$

$$= 1 \times \left(\frac{10}{60}\right)^2 = \frac{1}{36}\ \text{s}$$

In other words, the boy is now rotating at a rate of 36 revolutions per second.

EXAMPLE 7.9 A cord is wound around the circumference of a bicycle wheel (without a tyre) of diameter 1 m. A mass of 2 kg is attached to the end of the cord and it is allowed to fall from rest. The weight falls 2.0 m in 5 s. The axle of the wheel is horizontal and the wheel rotates with its plane vertical. Find (i) the angular acceleration produced and (ii) the moment of inertia of the wheel about the horizontal axle. Neglect the friction due to air.

Solution Let us first find the linear acceleration a produced using the given data

$$u = 0$$

$$s = 2.0\ \text{m}$$

$$t = 5\ \text{s}$$

$$s = ut + \tfrac{1}{2}at^2$$

or
$$2 = 0 \times 5 + \tfrac{1}{2}a(5)^2$$

or
$$a = \frac{4}{25}\ \text{m s}^{-2}$$

(i) Radius $R = 0.5$ m

The angular acceleration is obtained from

$$\alpha = \frac{a}{R} = \frac{4/25}{0.5} = \frac{8}{25} = 0.32\ \text{rad s}^{-2}$$

(ii) Let I be the moment of inertia of the wheel about the axle. Now since

$$\text{Torque}\quad \tau = I\alpha$$

$$I = \frac{\tau}{\alpha}$$

The torque τ is produced by the force of the falling weight,

$$\begin{aligned}\tau &= mg \times \text{moment arm}\\ &= mgR\\ &= 2\ \text{kg} \times 9.8\ \text{m s}^{-2} \times 0.5\ \text{m}\\ &= 9.8\ \text{N m}\end{aligned}$$

$$\therefore \qquad I = \frac{9.8}{0.32} = 30.625\ \text{kgm}^2$$

EXAMPLE 7.10 You are given two cylinders of the same size, mass and appearance. One of the cylinders is hollow and the other is solid. How will you distinguish between the two?

Solution We know from Eq. (7.55) that the linear acceleration of a cylinder rolling down an inclined plane is given by

$$a = \frac{g \sin \theta}{\left(1 + \frac{I}{MR^2}\right)} \qquad \text{(i)}$$

where I is the moment of inertia of the cylinder of mass M and radius R about its centre of mass, g is the acceleration due to gravity and θ is the angle of inclination of the inclined plane.

Let a_s be the acceleration of the solid cylinder as it rolls down an inclined plane. For a solid sphere $I = \frac{1}{2} MR^2$ (see Fig. 7.27). Equation (i) then gives

$$a_s = \frac{2}{3}\, g \sin \theta \qquad \text{(ii)}$$

For a hollow cylinder $I = MR^2$. Therefore putting $I = MR^2$ in Eq. (i), the linear acceleration of the hollow sphere rolling down the same inclined plane is,

$$a_h = \tfrac{1}{2}\, g \sin \theta \qquad \text{(iii)}$$

From (ii) and (iii) we find that

$$a_s > a_h$$

Thus, if both the solid and the hollow cylinders are rolled down the same inclined plane, the solid cylinder (i.e. the one with a larger acceleration) will reach the bottom first.

EXAMPLE 7.11 A small sphere of radius r rolls without slipping inside a large hemispherical bowl of radius R as shown in Fig. 7.31. The sphere starts from rest at the top left corner of the hemisphere. What fraction of the total energy is translational and rotational when the small sphere is at the bottom of the hemisphere ?

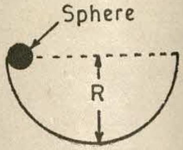


Fig. 7.31

Solution Let v be the linear speed of the small sphere at the bottom of the hemisphere. The translational kinetic energy is

$$K_1 = \tfrac{1}{2}\, mv^2$$

where m is the mass of the small sphere. If ω is the angular speed and I the moment of inertia of the small sphere, the rotational energy is given by

$$K_2 = \tfrac{1}{2}\, I\, \omega^2$$

$$= \frac{1}{2}\left(\frac{2}{5}\, mr^2\right)\omega^2 \quad \left(\because I = \frac{2}{5}\, mr^2\right)$$

Since the small sphere rolls down without slipping, we have

$$v = r\,\omega$$

Therefore $$K_2 = \frac{1}{5}\, mr^2\, \frac{v^2}{r^2} = \frac{1}{5}\, mv^2$$

The total energy (kinetic) of the small sphere at the bottom of the hemisphere is

$$K = K_1 + K_2$$

$$= \frac{1}{2}\, mv^2 + \frac{1}{5}\, mv^2 = \frac{7}{10}\, mv^2$$

Hence, the fractional translational kinetic energy is

$$\frac{K_1}{K} = \frac{5}{7}$$

and the fractional rotational kinetic energy is

$$\frac{K_2}{K} = \frac{2}{7}$$

EXAMPLE 7.12 If the earth were to suddenly contract to half of its present size without any change in its mass, what will be the duration of the new day?

Solution Let M be the mass and R the initial radius of the earth. If ω is the angular velocity of the rotation of the earth, the duration T of the day is

$$T = \frac{2\pi}{\omega}$$

Let R' be the radius of the earth after contraction and ω' its angular velocity. From the conservation of angular momentum, we have

$$I\,\omega = I'\,\omega'$$

where $I\left(=\frac{2}{5}MR^2\right)$ and $I'\left(=\frac{2}{5}MR'^2\right)$ are the moments of inertia of the earth before and after contraction, respectively.

$$\therefore \qquad \frac{2}{5}MR^2\,\omega = \frac{2}{5}MR'^2\,\omega'$$

or $$\omega' = \frac{R^2\omega}{R'^2} = 4\,\omega \quad (\because \quad R' = R/2)$$

The duration T' of the new day will be (since $T = 2\,\pi/\omega$)

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{4\omega} = \frac{T}{4}$$

$$\therefore \qquad T' = \frac{24 \text{ hours}}{4} = 6 \text{ hours}$$

SUPPLEMENT

Vector Product of Two Vectors

Two vectors can be multiplied in such a way as to yield a vector. In this case, the product is called the *vector* (or *cross-product*).

Let the vector product of two vectors $\vec{A}$ and $\vec{B}$ be equal to a vector $\vec{C}$, i.e.

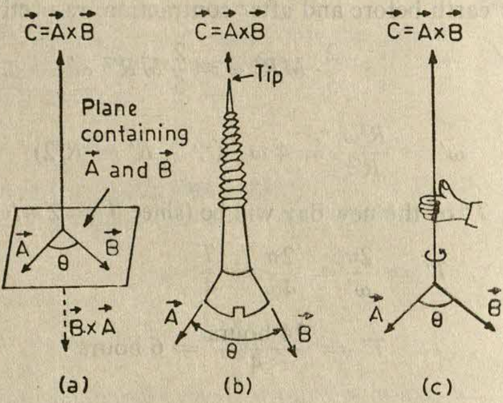
$$\vec{A} \times \vec{B} = \vec{C}$$

which is read as $\vec{A}$ cross $\vec{B}$ is equal to $\vec{C}$. The vector product is represented by putting a cross (×) between the vectors. The magnitude C of the resulting vector $\vec{C}$ is defined by

$$C = AB \sin\theta \tag{S.1}$$

where A and B respectively are the magnitudes of the vectors $\vec{A}$ and $\vec{B}$ and θ is the angle between them.

The direction of $\vec{C}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. If vectors $\vec{A}$ and $\vec{B}$ lie in the plane of the paper, the vector $\vec{C}$ will be perpendicular to this plane, as shown in Fig. 7.32(a). The sense (upward or downward) of the direction of the vector product is given by the direction of the advance of the tip of a right-handed screw when rotated from $\vec{A}$ to $\vec{B}$ through angle θ between them, the screw being placed with its axis perpendicular to the plane containing the two vectors (see Fig. 7.32b). This is known as the *right-handed screw rule.*



$\vec{C}=\vec{A}\times\vec{B}$
Plane containing $\vec{A}$ and $\vec{B}$
$\vec{A}$
$\vec{B}$
θ
$\vec{B}\times\vec{A}$
(a)
$\vec{C}=\vec{A}\times\vec{B}$
Tip
$\vec{A}$
$\vec{B}$
θ
(b)
$\vec{C}=\vec{A}\times\vec{B}$
$\vec{A}$
θ
$\vec{B}$
(c)

Fig. 7.32 Direction of the vector product of two vectors

Another simple method of determining the direction of the vector product is as follows (see Fig. 7.32(c)). Hold your right hand with the thumb erect and the fingers curled. The direction of the thumb is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$. If the direction of the rotation of the vector from $\vec{A}$ to $\vec{B}$ is the same as the direction of the folding of the fingers, then the erect thumb points in the direction of $\vec{A}\times\vec{B}$. We say that the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ in the definition $\vec{C}=\vec{A}\times\vec{B}$ from a right-handed coordinate system.

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is a single vector $\vec{C}$ whose magnitude is equal to the product of the magnitudes of vectors $\vec{A}$ and $\vec{B}$ and the sine of the angle between them and whose direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ in such a way that $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right-handed coordinate system.

This is the geometric way of defining a vector product. The algebraic definition of a vector product gives the components of $(\vec{A}\times\vec{B})$ in terms of those of $\vec{A}$ and $\vec{B}$:

$$(\vec{A}\times\vec{B})_x = A_yB_z - A_zB_y$$

$$(\vec{A} \times \vec{B})_y = A_z B_x - A_x B_z$$

$$(\vec{A} \times \vec{B})_z = A_x B_y - A_y B_x \qquad \text{(S.2)}$$

For a right-handed orthogonal coordinate system, the two definitions of $\vec{A} \times \vec{B}$ give the same result. Which definition we use in a particular problem depends upon convenience.

Properties of a Vector Product

(i) *The vector product is anticommutative,* i.e. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. This can be proved as follows. Let

$$\vec{C} = \vec{A} \times \vec{B}$$

and $$\vec{D} = \vec{B} \times \vec{A}$$

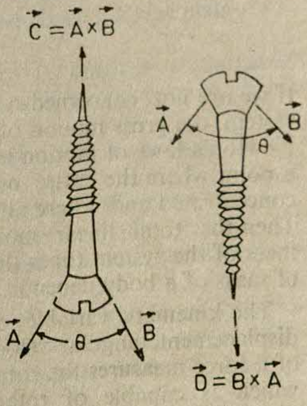
From the definition of the vector product, the magnitudes of C and D are

$$C = AB \sin\theta$$

$$D = BA \sin\theta$$

where A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, and θ is the angle between them. Since A and B are scalars, $AB = BA$. Hence, the magnitude of $\vec{C}$ is equal to that of $\vec{D}$. But their directions are different. To find their directions we use the right-hand screw rule.

Imagine that $\vec{A}$ and $\vec{B}$ lie in the plane of the paper. Place a right-handed screw perpendicular to this plane. Rotate it from $\vec{A}$ to $\vec{B}$ (to get the direction of $\vec{A} \times \vec{B}$) as shown in Fig. 7.33(a). The tip will advance upward (i.e. towards the reader). This is the direction of $\vec{C} = \vec{A} \times \vec{B}$. Now rotate it from $\vec{B}$ to $\vec{A}$ (to get the direction of $\vec{B} \times \vec{A}$) the tip will advance downwards (i.e. into the paper). This is the direction of $\vec{D} = \vec{B} \times \vec{A}$ (Fig. 7.33(b)). Thus, the directions of $\vec{C}$ and $\vec{D}$ are opposite, hence


$\vec{C} = \vec{A} \times \vec{B}$
$\vec{A}$
θ
$\vec{B}$
$\vec{A}$
$\vec{B}$
θ
$\vec{D} = \vec{B} \times \vec{A}$

(a) (b)

Fig. 7.33

$$\vec{C} = -\vec{D}$$

or $$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

This shows that the vector product is not commutative. It is anticommutative.

(ii) $\vec{A} \times \vec{A} = 0$, i.e. the vector product of a vector by itself is zero. This is because, in this case, $\theta = 0$, and hence $\sin\theta = 0$.

Therefore $\vec{A} \times \vec{A} = A\,A \sin\theta^\circ = 0$

Hence, the condition for two vectors to be parallel ($\theta = 0^\circ$) or antiparallel ($\theta = 180^\circ$) is that their vector product should be zero.

If $\vec{A} \times \vec{B} = 0$, it means either (i) $\vec{A}$ is zero or (ii) $\vec{B} = 0$ or (iii) the angle θ between them is 0° or 180°.

(iii) The distributive law holds for both scalar and vector products, i.e.

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iv) $(\lambda \vec{A}) \times \vec{B} = \vec{A} \times (\lambda \vec{B}) = \lambda (\vec{A} \times \vec{B})$; λ a real number

(v) $|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2\, |\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2$

(vi) $\hat{i}, \hat{j}$ and $\hat{k}$ are the three mutually perpendicular unit vectors at the origin O and along OX, OY and OZ respectively, the right-hand rule gives:

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}, \quad \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}, \quad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

SUMMARY

If we are not concerned with the internal motion and structure of a system, the gross motion of the system can be analysed by applying Newton's laws of motion to the centre of mass of the system which is a point where the entire mass of the system may be assumed to be concentrated and where all external forces are assumed to be applied. Then the total linear momentum of the system is equal to the total mass of the system times the velocity of its centre of mass. The centre of mass of a body depends on its shape and mass distribution.

The kinematic variables of rotational motion of a body are angular displacement, angular velocity and angular acceleration. The torque of a force measures the rotatory effect of the force applied to a body which is capable of rotation about an axis. The torque of a force $\vec{F}$ on a particle whose position vector (with respect to some origin

O) is $\vec{r}$ is defined as the vector product of $\vec{r}$ and $\vec{F}$, i.e. $\vec{\tau} = \vec{r} \times \vec{F}$. The angular momentum of the particle about O is defined as the vector product of $\vec{r}$ and $\vec{p}$ (the linear momentum), i.e. $\vec{L} = \vec{r} \times \vec{p}$. The rate of change of the angular momentum is equal to the torque, i.e. $\frac{d\vec{L}}{dt} = \vec{\tau}$. In the absence of the external torques, the total angular momentum of a system is conserved.

The moment of inertia of a rigid body about a particular axis may be defined as the sum of the products of the masses of all the particles of the body and the squares of their respective distances from the axis of rotation, i.e. $I = \sum mr^2$. Rotational kinetic energy $= \frac{1}{2} I \omega^2$. Also $\tau = I \alpha$ and $L = I \omega$.

EXERCISES

1. What do you understand by the term 'centre of mass' of a system? Obtain the expression for the centre of mass of a system consisting of (a) two particles and (b) a large number of particles. What is the physical significance of the centre of mass of a system?
2. Show that the centre of mass of two particles is on the line joining them at a point whose distance from each particle is inversely proportional to the mass of that particle.
3. (a) Should there necessarily be any mass at the centre of mass of a system?
   (b) Does the centre of mass of a solid necessarily lie within the body? If not, give two examples.
4. Show that the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.
5. Give three examples of centre of mass motion.
6. Define a rigid body. State the factors on which the centre of mass of a rigid body depends.
7. Define torque. State its SI units. How is the direction of torque determined? Show that the magnitude of torque = moment arm × magnitude of force.
8. Define angular momentum of a particle. State its SI units. How is its direction determined?
9. Deduce the relation between torque and angular momentum.
10. State the principle of conservation of angular momentum.
11. What is the (a) physical and (b) geometrical meaning of angular momentum?
12. Why are curved race tracks banked? (a) What should be the banking angle so as to minimise the wear and tear of the tyres of a car negotiating a banked curve? (b) What is the maximum speed at which a car can successfully negotiate a banked curve?

13. From angular momentum considerations, deduce the equation that governs the oscillations of a pendulum.
14. Define the moment of inertia of a body rotating about an axis. What is the physical significance of moment of inertia?
15. What do you understand by the term 'radius of gyration'? What is its physical significance?
16. Show that torque = moment of inertia × angular acceleration and hence define moment of inertia.
17. Show that angular momentum = moment of inertia × angular velocity. Give two illustrations of the consequences of this relationship.
18. Give the analogy between various quantities that describe linear and rotational motion.
19. State and prove the parallel axis theorem.
20. (a) Obtain the expression for the linear acceleration of a descending body on a string wound on a cylinder and show that this acceleration is always less than the acceleration due to gravity.
    (b) Obtain the expression for the linear acceleration of a cylinder rolling down an inclined plane and hence find the condition for the cylinder to roll without slipping.
21. Define the vector product of two vectors. State the rule for finding the direction of the product vector. State the properties of vector products.

PROBLEMS

1. Four point masses 1 kg, 1 kg, 2 kg and 2 kg are placed at the corners of a square as shown in Fig. 7.34. Find the centre of mass of the system.
2. A highway has a curved track of radius 100 m which is banked at an angle of 10°. What maximum speed is the track meant for?
3. The combined mass of a motorcycle and its rider is 100 kg. What is the necessary frictional force if the rider is to negotiate a curve of 80 m radius at 72 kmh^{-1}? If the coefficient of friction is 0.6, will the rider negotiate the curve successfully? At what angle should he lean to avoid falling?
4. A train ofmass 10^6 kg negotiates a banked circular track of radius 30 m at a speed of 54 kmh^{-1}. What is the minimum banking angle required to prevent the wearing out of rails ?
5. A dumbell consists of two spherical masses A and B fixed at the ends of a light rigid rod as shown in Fig. 7.35. Calculate the moment of inertia of

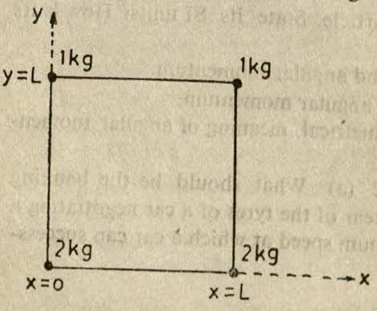


Fig. 7.34

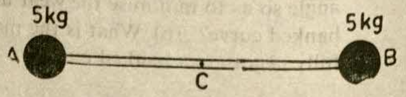


Fig. 7.35

the system about an axis (a) passing through the centre C and perpendicular to the rod, (b) passing through the centre of sphere A and perpendicular to it and (c) passing through the centre of sphere B and perpendicular to it.

6. A uniform disc of radius 10 cm and mass 5 kg is mounted on an axle. A light string is warpped round the rim of the disc and a constant downward force of 20 N is exerted on the string. Find the angular acceleration of the disc and the tangential acceleration of a point on the rim.
7. In Problem 6, instead of exerting at constant force on the string, if we hang a mass of 2 kg from the string and release it, (a) what will be the angular acceleration of the disc, (b) the tangential acceleration of a point on the rim, and (c) the tension in the string ? Take $g = 10 \text{ ms}^{-2}$.
8. A point mass of 1 kg is attached to a light string which passes through a hollow tube clamped vertically on a stand as shown in Fig. 7.36. The mass is set into motion in a horizontal circle of radius 50 cm at a speed of 5 ms^{-1}. The string is suddenly pulled down so that the radius of the circular path is reduced to 10 cm. What is the new speed of the mass ?
9. A particle of mass 2 kg has position coordinates ($x = 3$ m, $y = 4$ m) and is moving with a velocity $\vec{v} = 5\hat{i} - 6\hat{j}$ where $\hat{i}$ and $\hat{j}$ are the unit vectors along x and y axes with respect to origin O. A force of 7 N acts on the particle in the $+x$ direction. (a) What is the angular momentum of the particle about O ? (b) What torque acts on its ? (c) At what rate does the angular momentum change with time?
10. What is the rotational kinetic energy of the earth ? Assume that the earth (mass $= 6.0 \times 10^{24}$ kg, radius $= 6.4 \times 10^{3}$ km) is a sphere of uniform density.

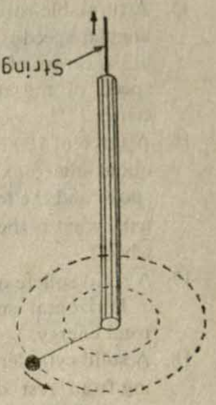


Fig. 7.36

11. An oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis passing through the centre and perpendicular to the line joining the two oxygen atoms. The molecule is moving at a speed of 500 ms^{-1} and its rotational kinetic energy is two-thirds its translational kinetic energy. Calculate its angular velocity.
12. A constant force of 100 N is applied tangentially to the rim of a wheel of radius 0.5 m. The moment of inertia of the wheel is 10^2 kg m^2. Calculate
    (a) the angular acceleration
    (b) the angular speed, 5 s from rest
    (c) the number of revolutions made in 5 s
    (d) the kinetic energy of the wheel.
13. A wheel and axle having a total moment of inertia of 2×10^{-3} kg m^2 is made to rotate about its own axis by means of a mass of 1 kg attached to a cord wrapped round the axle. The axle has a radius of 15 mm. How far

must the weight descend so that the flywheel acquires a speed of 5 revolutions per second?

14. AB and BC are two smooth planes inclined at an angle of 45° as shwon in Fig. 7.37. A solid sphere is rolled on AB with a speed of 30 m s^{-1}. To what height h will it rise on the plane BC?

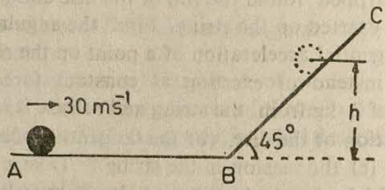


Fig. 7.37

15. A turntable rotates about a fixed vertical axis passing through its centre. Its angular speed is 1.33 π rad s^{-1}. A man weighing 80 kg, initially standing at the centre of the turntable runs along the radius. What is the angular speed of the turntable when the man is at a distance of 0.2 m from the centre ? The moment of inertia of the turntable is 2.8 kg m^2.
16. A force of 100 N is applied to the spoke of a wheel (capable of rotation about the z-axis) at a distance of 50 cm from the axis of rotation. If the spoke and the force make an angle of 45° and 75° respectively with the x-axis, what is the magnitude and the direction of the torque exerted on the wheel?
17. A solid sphere of mass 1 kg and diameter 10 cm rolls (without slipping) on a horizontal smooth surface with a velocity of 20 cm s^{-1}. Calculate its total energy.
18. A solid cylinder (a) rolls without slipping, (b) slips or slides without rolling from rest down an inclined plane. Find the ratio of the velocity in case (a) to that in case (b) when the cylinder reaches the bottom of the plane. Neglect friction.

8

Gravitation

8.1 INTRODUCTION

In earlier chapters we have briefly discussed gravitation. In Chapter 3 we studied the kinematics of a body falling freely under the gravity of the earth. In Chapter 6, we learnt of a very important property of the gravitational force, namely, that it is a conservative force.

The four fundamental forces of nature are the frictional, gravitational, electromagnetic and nuclear forces. The gravitational force happens to be the weakest among them. But at the astronomical level, where masses are enormous, this force plays an important role from initiating the birth of a star, controlling the orbits of planets to determining the structure and evolution of the entire universe.

Scientists before Newton (i.e. before the seventeenth century) believed that things fell on the earth because it was the inherent property of matter, and needed no further explanation. But Newton was not satisfied with this explanation.

The first systematic study of falling bodies was made by Galileo who died in the year Newton was born. Galileo did not know of the basic law governing gravitation. Newton was the first, to realize that the weight of a body on the surface of the earth is due to the force of attraction between the body and the earth. It required his supreme genius to realize that this attraction is only a particular case of a universal attraction between any two bodies situated anywhere in the universe. At the age of twenty-three he discovered a law now known as *Newton's Law of Universal Gravitation* which has had far-reaching consequences in unfolding the mysteries of nature.

In this chapter we will first consider gravitation of bodies on or near the surface of the earth. We will then state Newton's law of gravitation and discuss its applications in relation to heavenly bodies, such as the motion of planets. We shall conclude this chapter by giving a brief outline of the modern theory of gravitation (Einstein's general theory of relativity) to make clear to you our present level of understanding of the mysterious gravitational force.

8.2 ACCELERATION DUE TO GRAVITY NEAR EARTH'S SURFACE

As mentioned earlier, Galileo was the first scientist to make a systematic study of the motion of a body under the gravity of the earth. It is said that he dropped different objects from the Leaning Tower of Pisa and made a quantitative analysis of their motion under gravity. He also made careful observations of balls rolling down inclined planes of different inclinations. On the basis of his experimental investigations he came to the conclusion that '*in the absence of air, all bodies will fall at the same rate*'. It is the air resistance that slows down a flat piece of paper or a parachute falling under gravity. If a parachute and a heavy stone are dropped where there is no air, i.e. in a vacuum, both will fall together at the same rate.

Galileo went a step further and said that during a free fall, the speed of the body does not remain constant. He pointed out that a block of metal allowed to drop from a height of 3 m will drive a big nail much farther into the ground than the same block dropped from a height of half-a-metre. Clearly the block must be moving faster in the first case than in the second.

Experiments have revealed that the velocity of a body falling freely under gravity increases at a constant rate. *In the absence of air resistance, all bodies fall with the same constant acceleration.* For most practical situations, the air resistance has a negligible effect. Only in the special case of an extremely light object with a large area, such as a feather, a flat piece of a paper or a parachute, does the air resistance retard the object significantly. Apart from these exceptional cases, objects fall with the same acceleration. This acceleration is known as the *acceleration due to gravity* (of the earth) and is given the symbol g.

In Galileo's time, the experimental techniques did not permit the measurement of the value of g to any reasonable degree of accuracy. With the help of modern techniques it is now possible to measure g to an accuracy of 1 in 10^8 by observing the free fall of a body in vacuum. These techniques involve the use of lasers and electronic time measurement devices. Thus it is possible to study the small variations of the value of g and its non-dependence on the mass of the body to a high degree of accuracy. Later in this chapter we will examine the variation of g on the earth's surface. For most practical purposes we can take the acceleration due to gravity to be

$$g = 9.8 \text{ m s}^{-2}$$

8.3 NEWTON'S LAW OF UNIVERSAL GRAVITATION

The discovery of the law of universal gravitation by Isaac Newton is one of the greatest landmarks in the history of science. Before Newton's time, scientists believed that the tendency of all objects to fall to the earth and the motion of planets were entirely unrelated phenomena. Newton showed that these two phenomena involved gravitation and that they could be analysed using his equations of motion.

The motion of heavenly bodies, particularly the planets, the moon and the sun, was the subject of active interest among students, including Newton, at Cambridge in 1664. In 1665, the plague broke out and the college was closed. Newton, then just 23 years of age, went home to Woolsthorpe, where he continued to think about the motion of the planets and the moon.

You must have heard the famous story of how a falling apple inspired Newton and led him to the discovery of the law of gravitation. One day Newton was sitting under an apple tree and having tea with his friend and biographer William Stukeley. Newton saw an apple fall to earth. The incident of the falling apple set him thinking about falling bodies. It occurred to him that the same force of gravitation which attracts the apple to the earth might also be responsible for attracting the moon and keeping it in orbit. He thought that the centripetal acceleration of the moon in its orbit and the downward acceleration of a body falling on the earth might have the same origin (Fig. 8.1). This idea that celestial and terrestrial motions might obey the same law was indeed revolutionary at that time.

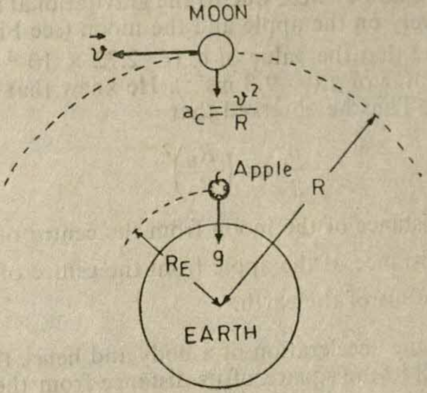


Fig. 8.1 Acceleration due to gravity on earth and centripetal acceleration of moon

Newton compared the acceleration due to gravity of an apple with the centripetal acceleration of the moon which led him to the law of gravitation. We now present his calculations and arguments.

Assuming the moon's orbit to be circular (which is a good approximation) and knowing the distance of the moon from the earth and its period of revolution, Newton calculated the centripetal acceleration as follows,

$$\text{Radius of moon's orbit } (R) = 3.84 \times 10^8 \text{ m}$$

$$\text{Period of moon's revolution } (T) = 27.3 \text{ days}$$

$$= 27.3 \times 24 \times 3600 \text{ s}$$

$$\text{The speed of the moon in orbit } (v) = \frac{2\pi R}{T}$$

$$= \frac{2 \times 3.14 \times 3.84 \times 10^8}{27.3 \times 24 \times 3600}$$

$$= 1.02 \times 10^3 \text{ m s}^{-1}$$

$$\text{Centripetal acceleration } (a_c) = \frac{v^2}{R} = \frac{(1.02 \times 10^3)^2}{3.84 \times 10^8}$$

$$= 2.72 \times 10^{-3} \text{ m s}^{-2}$$

The acceleration of a falling apple is $g = 9.8 \text{ ms}^{-2}$. Newton assumed that both the moon and the apple are accelerated towards the centre of the earth. The difference in their motions arises because the moon has a tangential velocity whereas the apple does not. He assumed that both g and a_c were due to the gravitational force of the earth acting respectively on the apple and the moon (see Fig. 8.1).

Newton found that the value of a_c (= $2.72 \times 10^{-3} \text{ m s}^{-2}$) was about 1/3600 of the value of g (= 9.8 ms^{-2}). He knew that the ratio R_E/R was about $^1/60$. Thus he observed that

$$\frac{a_c}{g} = \left(\frac{R_E}{R}\right)^2$$

where R = distance of the moon from the centre of the earth

and R_E = distance of the apple from the centre of the earth

= radius of the earth.

In other words, the acceleration of a body and hence the force is inversely proportional to the square of its distance from the centre of the earth. This led Newton to postulate that the gravitational force might vary inversely as the square of the distance, i e. the gravitational force obeys the inverse square law.

It is believed that Newton made calculations in 1666. He did not publish his result for 20 years. Only in 1687 did he publish the law in his book *Principia Mathematica.* The reason was that he was not satisfied with his conclusion. In his calculation, the distances are measured from the centre of the earth. This meant that the earth was assumed as a mass point, i.e. the entire mass of the earth was concentrated at its centre for objects outside it. Newton invented calculus during the intervening years. Using calculus he could prove that, for objects outside the earth. it behaved as though its entire mass was concentrated at the centre. In Sec. 8.10 we will prove this without the use of calculus in a special case in which g can be treated as a constant.

Newton went a step further. From his second law of motion ($F = ma$) he knew that the force exerted on an accelerating body depends on its mass. Thus the force exerted by the earth on the apple (or moon) must depend upon the mass of the apple (or moon). From his third law of motion, the apple (or moon) must exert an equal and opposite force on the earth which must depend upon the mass of the earth. Thus he found that the force of attraction between bodies should depend on their masses and vary inversely as the square of the distance between them.

Newton realized that this attraction is only a particular case of universal attraction between any two bodies situated anywhere in the universe. At the age of twentythree he discovered a law, now known as *Newton's law of universal gravitation* which states:

'*Any two particles of matter anywhere in the universe attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them; the direction of the force being along the line joining the particles*; i.e.

$$F \propto \frac{m_1 m_2}{r^2}$$

where F is the magnitude of the force of attraction between two particles of masses m_1 and m_2 separated by a distance r (Fig. 8.2).

In the form of an equation the law is written as

$$F = G\frac{m_1 m_2}{r^2} \qquad (8.1)$$

where G is a constant called the *universal gravitation constant*. The value of this constant is to be determined experimentally and will be the same for any pairs of particles.

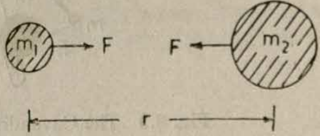


Fig. 8.2 Gravitational force

Some Features of the Law

1. The gravitational forces between two particles are an action-and-reaction pair. The first particle exerts a force of magnitude F (on the second particle) that is directed towards the second particle along the line joining the two particles. Likewise, the second particle exerts an equal and opposite force (on the first particle) that is directed towards the first particle along the line joining the two (Fig. 8.2).

2. The universal gravitation constant G must not be confused with g which is the acceleration due to gravity. The constant G is a scalar having dimensions $M^{-1}L^3T^{-2}$ and is measured in $Nm^2\ kg^{-2}$; but g is a vector, having dimensions LT^{-2} and is measured in ms^{-2}, it is neither universal nor constant.

3. The constant G is to be determined experimentally. It is universal in the sense that, if G is determined for a given pair of bodies, we can use that value in the law of gravitation [Eq. (8.1)] to determine the gravitational force between any other pair of bodies.

The Gravitation Constant

To determine the value of the gravitation constant G it is necessary to measure the force of attraction between two bodies of known masses and known distance separating them. The experiment must have a very high sensitivity since, as mentioned earlier, the gravitational force is extremely weak, i.e. the value of G is very small.

The first accurate measurement of G was made by Lord Cavendish in England in 1798. The apparatus he used is called the *Cavendish torsion balance* which is schematically illustrated in Fig. 8.3.

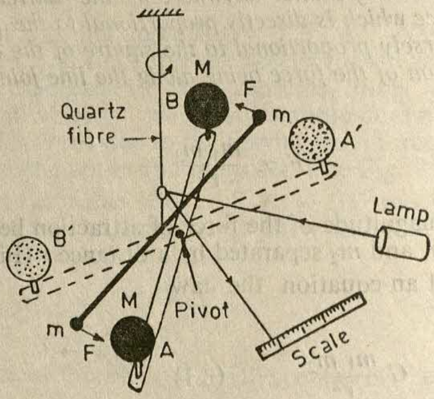


Fig. 8.3 The Cavendish torsion balance for measuring G

Two small lead balls, each of mass m, are attached to the opposite ends of a light rod which is suspended, with its axis horizontal, by a fine quartz fibre. Two large lead balls, each of mass M, are placed on the opposite ends of another rod which is pivoted at the centre, as shown in the figure. When the large balls are at positions A and B, the small masses are attracted by gravitation, and as a result the fibre is twisted in the counter-clockwise sense as seen from above. When the large masses are brought to positions A′ and B′, the fibre is twisted clockwise. The angle θ through which the fibre is twisted when the large balls are moved from one position to the other is measured by observing the deflection of a beam of light reflected from a mirror attached to the fibre.

In a preliminary experiment the deflection of the beam due to the twist of the fibre is calibrated by known forces, so that the force per unit twist is known. Multiplying this by angle θ gives the gravitational force F between masses m and M. Knowing the masses m and M and measuring the separation r between them, the value of G is determined from Eq. (8.1).

Since Cavendish's measurement significant improvements have been made in the determination of G, particularly by Poynting and Boys in the 19th century. The present accepted value of G is

$$G = 6.67 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}$$

Because of the small value of G, the gravitational force between bodies of ordinary sizes is extremely small. The force of attraction between two bodies, each of mass 1 kg separated by a distance of 1 m is only 6.67×10^{-11} N which is too small to be noticed. That is why we are not aware of any such attraction. The only attraction we notice is the attraction of the earth, because the earth is very massive.

'Weighing' the Earth

In fact we can find the mass of the earth from the law of gravitation and the value of G obtained by Cavendish. For this reason Cavendish is said to be the first person to 'weigh' the earth.

The force of gravity of a mass m on the surface of the earth is given by

$$F = G\frac{mM}{R^2}$$

and

$$F = mg$$

where M is the mass of the earth and R its radius which is the separation between the centres of the two bodies involved. Equating the two equations we have

$$M = \frac{gR^2}{G}$$

Substituting the known values of g, R and G we get

$$M = \frac{(9.8\ \text{m s}^{-2}) \times (6.38 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}}$$

$$= 5.98 \times 10^{24}\ \text{kg}$$

EXAMPLE 8.1 In an experiment using the Cavendish torsion balance, the smaller spheres each of mass 1.00×10^{-2} kg are fixed at the ends of a light rod 50.0 cm long. The torsion constant of the fibre is 8.34×10^{-8} Nm rod^{-1}. When two large spheres, each of mass 10.0 kg, are placed near each suspended sphere, the angle of twist is found to be 3.96×10^{-3} radian and the distance between the large and small spheres is 10.0 cm. Calculate the value of G from the given data.

Solution The gravitational force on each small sphere of mass m due to its neighbouring big sphere of mass M is

$$F = G\frac{mM}{r^2} \tag{i}$$

where r is the distance between their centres.

If l is the length of light rod connecting the two small spheres, the moment arm for each force is $\frac{l}{2}$. Then

$$\text{Torque} = \text{force} \times \text{moment arm}$$

or
$$= 2F \times \frac{l}{2} = F \times l$$

This torque equals $\alpha\,\theta$, where α is the torsion constant of the fibre and θ is the angular twist. Thus

$$\alpha\,\theta = F \times l \tag{ii}$$

Using Eq. (ii) in Eq. (i) we get

$$\alpha\,\theta = \frac{G\,mM}{r^2} \times l$$

or
$$G = \frac{\alpha\,\theta\,r^2}{m\,M\,l}$$

$$= \frac{(8.34 \times 10^{-8}) \times (3.96 \times 10^{-3}) \times (0.100)^2}{(1.00 \times 10^{-3}) \times (10.0) \times (0.500)}$$

$$= 6.63 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$$

8.4 INERTIAL AND GRAVITATIONAL MASS

Inertial Mass

Newton's second law of motion ($F = ma$) tells us that the mass of a body can be determined by measuring the acceleration produced in it by a known constant force. Dividing the force by the acceleration gives the mass of the body. The mass so determined is called the *inertial mass* of the body. Hence mass may be regarded as a quantitative measure of inertia.

Suppose a constant (unknown) force F acts on two masses m and m' and produces accelerations a and a' respectively. Then we have

$$F = ma = m'a'$$

which gives
$$\frac{m}{m'} = \frac{a'}{a}$$

i.e. the ratio of the two masses is independent of the constant force used. Thus we can compare the inertial masses of two bodies by measuring the accelerations produced in them by a constant force. If one of the masses, say m', happens to a standard kilogram, the unknown mass m can be determined in kilograms by comparing their accelerations. The mass so determined is called the inertial mass of the body. We shall denote it by m_i.

Gravitational Mass

Newton's law of gravitation (Eq. 8.1) tells us that the gravitational force on a body is proportional to its mass. An important consequence of this proportionality is that we can measure the mass of a body by measuring the gravitational force exerted on it by a massive body, such as the earth. This can be done with the help of a spring balance or by comparing the gravitational force on one mass m with that on a standard mass (1 kg) as in a beam balance. The mass m so determined is called the *gravitational mass* of the body. We will denote it by m_g.

Equivalence of the Two Masses

The question arises whether the above two methods measure the same property, i.e. is the inertial mass of a body really equal to its gravitational mass? Notice that they have been measured in quite different experimental situations. In the measurement of the inertial mass, the gravity does not enter into the picture at all. It wouid require the same force to accelerate the given mass even in gravity-free space. On the other hand, in the measurement of the gravitational mass, the inertia plays no role at all, the property that does play a role is the gravitational attraction between that body and the earth. The force is given by

$$F = G\frac{m_g M_e}{R_e^2}$$

Are the gravitational mass m_g and inertial mass m_i of a body really the same?

Newton devised an experiment to test the equivalence of inertial and gravitational mass. In Chapter 12 we will derive the expression for the time period of a simple pendulum. The expression is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum. In this derivation we have implicitly taken $m_i = m_g$. But if the two masses were taken to be different then we would have obtained the following expression for the period of the pendulum

$$T = 2\pi\sqrt{\frac{m_i\, l}{m_g\, g}}$$

where m_i and m_g respectively are the inertial and gravitational mass of the pendulum bob. Only if we set $m_i = m_g$, do we get the expression for T as written above.

Newton made a pendulum whose bob was a hollow sphere made of tin. He put different substances into the hollow sphere one after another and measured the time period of the pendulum, making sure that the amplitude was the same in each case. He argued that when one substance was replaced by another in the bob, any difference in acceleration could be due to a difference in the inertial mass. This difference would be indicated by a change in the time period. But in all cases, he found the time period to be the same, always given by the expression

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Hence he concluded that $m_i = m_g$, i.e. the inertial and gravitational masses are equivalent.

The degree of accuracy in Newton's measurement was 1 in 10^3. In 1909 Eötvös devised an apparatus which had an accuracy as high as 5 parts in 10^9. He found that equal inertial masses always experienced equal gravitational force to an accuracy of 5 in 10^9. In 1964 R.H. Dicke and his collaborators improved the accuracy by several hundreds and established the equivalence of the two types of masses.

In classical physics, the equivalence of the inertial and gravitational mass was regarded as a surprising conclusion with no deep significance. But in modern physics, this equivalence is of vital significance. It has provided a clue to a deeper understanding of gravitation. In fact this equivalence is the basic fact in the development of the general theory of relativity by Albert Einstein (see Supplement at the end of this chapter).

8.5 VARIATION OF g

We have so far taken the acceleration due to gravity (g) to be a constant. Recent high-precision measurements have shown that, even on the surface of the earth, the value of g varies (though only slightly) from one place to another.

For bodies above the surface of the earth, the value of g must decrease with altitude. This can be seen by calculating g from Newton's second law and the law of gravitation. According to these two laws the force exerted by the earth (of mass M) on an object of mass m is given by

$$F = mg$$

and

$$F = \frac{G\,m\,M}{r^2}$$

where r is the distance between the centre of the body and the centre of the earth, both assumed to be spheres. Equating the two forces, we have

$$mg = \frac{G\,mM}{r^2}$$

or

$$g = \frac{GM}{r^2} \tag{8.2}$$

This equation tells us that g is not a constant; it depends on various factors. First g decreases inversely as the square of the distance of the body from the centre of the earth, i.e. g decreases with altitude (see Table 8.1). Secondly, on the surface of the earth g is given by

$$g = \frac{GM}{R^2} \tag{8.3}$$

where R is the radius of the earth assumed to be spherical. Since the earth is not perfectly spherical and its density also not uniform, the value of g varies slightly with position on earth's surface.

Table 8.1 Variation of g with altitude at 45° latitude

Altitude (*km*)	g (ms^{-2})
0	9.806
1	9.803
4	9.794
8	9.782
16	9.757
32	9.71
100	9.60

Notice the small variation of g with altitude. It may be remarked that the value of g varies slightly from place to place on earth also due to the rotation of the earth about its axis.

EXAMPLE 8.2 A body weighs 63 N on the surface of the earth. How much will it weigh at a height equal to half the radius of the earth?

Solution The acceleration due to gravity at the surface of the earth is given by

$$g = \frac{GM}{R^2}$$

where M is the mass of the earth and R its radius. Therefore, the weight of a body on earth is

$$W_e = mg = \frac{GMm}{R^2} \qquad \text{(i)}$$

At a height h above the surface of the earth, the value of g is

$$g_h = \frac{GM}{(R+h)^2}$$

and the weight of the body at this height is

$$W_h = \frac{G\,Mm}{(R+h)^2} \qquad \text{(ii)}$$

Dividing Eqs (i) and (ii) we have

$$\frac{W_h}{W_e} = \frac{R^2}{(R+h)^2} = \left(\frac{R}{R+h}\right)^2$$

It is given that $h = \frac{R}{2}$ and $W_e = 63$ N. Therefore

$$W_h = W_e\left(\frac{R}{R+h}\right)^2 = 63 \times \left(\frac{R}{R+\frac{R}{2}}\right)^2$$

$$= 63 \times \frac{4}{9}$$

$$= 28 \text{ N}$$

Example 8.3 An astronaut on the moon measures the acceleration due to gravity and finds it to be 1.67 ms^{-2}. He knows that the earth is about 80 times more massive than the moon. What is his estimate of the ratio of radius of the earth to that of the moon?

Solution We know that the acceleration due to gravity on the earth and the moon are given by

$$g_e = \frac{GM_e}{R_e^2}$$

and

$$g_m = \frac{GM_m}{R_m^2}$$

Dividing the two equations we have

$$\frac{R_e}{R_m} = \left(\frac{M_e}{M_m} \times \frac{g_m}{g_e}\right)^{1/2}$$

$$= \left(80 \times \frac{1.67}{9.8}\right)^{1/2} = 3.69$$

8.6 GRAVITATIONAL POTENTIAL ENERGY

In Chapter 6 we discussed the gravitational potential energy between a body and the earth. We considered the special case in which the body was close to the earth so that the gravitational force acting was constant for all positions of the body. In this case, the gravitational potential energy of a body of mass m at a height h ($h \ll R$, where R is the earth's radius) above the earth's surface could be obtained by simply finding the work done against the constant force (mg) in displacing the body through a distance h. Thus

$$\text{PE} = mgh \tag{8.4}$$

In this expression the zero of the potential energy is taken (for convenience) to be a point on the surface of the earth ($h = 0$). The choice of the reference point for the zero of PE is arbitrary and is chosen for convenience. It is immaterial where the reference point is chosen, for eventually we consider only the difference in potential energy. Thus Eq. (8.4) actually represents the increase in PE of a body when it is raised from the earth's surface ($h = 0$) to a height h.

In this section we remove this restriction and consider the general case in which the distance between the body and the earth may be greater than the earth's radius. In this case, the gravitational force is variable and we have to use the method explained in Chapter 6 to compute the work done against the variable force.

Consider a body of mass m at a distance r ($> R$) from the centre of the earth of mass M; R being the radius of the earth. The gravitational force between them is

$$F = \frac{GmM}{r^2} \tag{8.5}$$

It is evident that F is zero when r tends to infinity, i.e. no force is required to hold the body at an infinite distance from the earth. Hence, in this case, it is convenient to choose the reference point for zero of PE to be a point at infinity.

The gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done in bringing the mass m from infinity to a distance r. This can be computed as follows.

The work done in moving m by a small amount dr along the line joining the centres of m and M is

$$dW = F\,dr$$

The total work done in moving m from infinity to a distance r is then given by

$$\begin{aligned} M &= \int_{\infty}^{r} F\,dr \\ &= \int_{\infty}^{r} \frac{GmM}{r^2}\,dr \end{aligned}$$

$$= GmM \int_{\infty}^{r} \frac{1}{r^2}\, dr$$

$$= GmM \left| -\frac{1}{r} \right|_{\infty}^{r}$$

$$= -\frac{GmM}{r}$$

Thus $$\text{PE} = -\frac{GmM}{r} \tag{8.6}$$

The minus sign indicates that the potential energy is *negative* at any finite distance r. The reason is that we have arbitrarily chosen the zero of PE to correspond to infinite separation between the masses m and M. With this choice of the zero potential energy, PE is always *negative* and *increases* to zero at infinity.

The decrease in PE as the mass m moves in from infinity towards the mass M implies that the work done by the gravitational force of mass M on mass m is positive. This follows from the work-energy principle described in Chapter 6. Since the gravitational force is conservative (see Ch. 6), Eq. (8.6) holds no matter what path the mass m follows in moving in from infinity to a position r.

We can use Eq. (8.6) to compute the amount of work done against the gravitational force of mass M to move the mass m from a distance r_1 (position A) to a distance r_2 (position B) as shown in Fig. 8.4.

This work is given by [from Eq. (8.6)]

$$W_{12} = GmM \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{8.7}$$

which represents the increase in PE as m moves ftom A to B. We can apply this general expression to the special case of a mass near the surface of the earth.

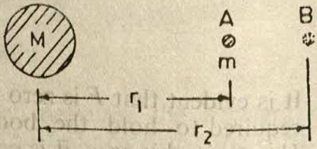


Fig. 8.4

Potential Energy Near the Surface of the Earth

Suppose a body of mass m is raised from the surface of the earth to a height h. In this case, setting $r_1 = R$ and $r_2 = (R + h)$ in Eq. (8.7) we have (R is the earth's radius)

$$W_h = GmM \left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$= \frac{GmM\, h}{R\,(R+h)}$$

For a body near the surface of the earth, $h \ll R$ and $(R + h)$ in the denominator can be replaced by R. Thus we have

$$W_h = \frac{GmM\,h}{R^2}$$

or
$$\text{PE} = mgh \quad \left(\because \quad \frac{GM}{R^2} = g\right)$$

where agrees with Eq. (8.4).

Note: The work done W_{12} given by Eq. (8.7) measures the change in potential energy when the mass m is moved from r_1 to r_2 which may also be written as

$$\text{Change in PE} = W_{12} = \frac{GmM}{r_1\,r_2}\,(r_2 - r_1)$$

(i) If $r_1 < r_2$, W_{12} will be positive. This means that an outside agent has to do *positive* work (force and displacement in the same direction) in separating the two masses against their mutual gravitational attraction.

(ii) On the other hand, if $r_1 > r_2$, W_{12} will be negative. This means that the work has to be done by the gravitational force in bringing the two masses closer. When the two masses are released from a given separation, they accelerate towards each other because of mutual attraction. The potential energy decreases resulting in an increase in kinetic energy. Thus when a stone is dropped, it accelerates downwards towards, the earth at the rate of 9.8 ms^{-2}. The earth must also accelerate upwards towards the stone. This acceleration of the earth is very small due to its enormous mass compared to that of the stone (see Example 8.4 below).

EXAMPLE 8.4 **Calculate the acceleration produced in the earth when a stone of mass 6 kg falls on it. Mass of earth (M) = 6×10^{24} kg and acceleration due to gravity (g) = 9.8 ms^{-2}.**

Solution Let m be the mass of the stone. If F is the gravitational force between the stone and earth, their respective accelerations are given by

$$a_s\,(= g) = \frac{F}{m}$$

and
$$a_e = \frac{F}{M}$$

$\therefore$
$$a_e = \frac{m}{M} \times g$$

$$= \frac{6 \times 9.8}{6 \times 10^{24}}$$

$$= 9.8 \times 10^{-24}\ \text{ms}^{-2}$$

This acceleration is too small to be noticed. That is why we do not see the earth rising towards the falling stone.

However, if there are two objects with comparable masses, ($m \simeq M$), their accelerations are comparable. If the objects are massive, their accelerations are large (because of large F) and they can be seen moving towards each other. In fact, astronomers have actually observed two such objects in the sky. Two stars Sirius A and Sirius B move around each other because their masses are comparable.

Gravitational Potential

Just as we define electrical potential at a point in the electric field around a charge, we define gravitational potential at a point in the gravitational field around a mass.

The gravitational potential V, at a point in the gravitational field of a mass is defined as the work done in moving a unit mass from infinity to that point. Thus if we set $m = 1$ in Eq. (8.6) we get the potential V due to mass M at a distance r from it, i.e.

$$V = -\frac{GM}{r} \tag{8.8}$$

The minus sign indicates that the potential at infinity (which is taken to be zero by convention) is *higher* than the potential close to mass M.

8.7 ESCAPE VELOCITY

If we throw a ball upwards, it reaches a certain height and then falls back due to the attraction of the earth. If it is thrown with a greater initial velocity, it rises to a greater height. We shall show just now that if, somehow, the body is projected with a velocity of about 11.2 km s^{-1}, it will break loose from the earth and never come back. In other words, it will escape the gravitational pull of the earth and go into outer space. *The escape velocity is the minimum velocity with which a projectile must be projected in order that it may escape the earth's gravitational pull.* The magnitude of the escape velocity can be deduced as follows.

Consider of a body of mass m at a distance r from the centre of the earth of mass M. Clearly, the gravitational force acting on the body is

$$F = \frac{GmM}{r^2}$$

Therefore, the work dW done by the body against the gravitational pull of the earth in moving upward through a further small distance dr is given by

$$dW = Fdr = \frac{GmM}{r^2}\,dr$$

Hence, the total work W done by the body in escaping from the surface of the earth, i.e. in moving to an infinite distance away from it, is given by

$$W = \int_R^\infty dW = GmM \int_R^\infty \frac{1}{r^2}\, dr = GmM \left| -\frac{1}{r} \right|_R^\infty$$

$$= \frac{GmM}{R}$$

where R is the radius of the earth.

Let v_e be the escape velocity. Then the initial kinetic energy of the body is $\frac{1}{2} mv_e^2$. This must at least be equal to the work done by the body in escaping from the earth. We, therefore, have

$$\tfrac{1}{2} mv_e^2 = \frac{GmM}{R}$$

or

$$v_e = \sqrt{\frac{2MG}{R}} \tag{8.9}$$

The expression can also be written in terms of g, the acceleration due to gravity at the earth's surface.

Now

$$g = \frac{GM}{R^2}$$

or

$$\frac{GM}{R} = gR$$

Substituting in Eq. (8.9), we get

$$v_e = \sqrt{2gR} \tag{8.10}$$

Putting $g = 9.8\ \text{ms}^{-2}$ and $R = 6.38 \times 10^6$ m, we have

$$v_e = (2 \times 9.8 \times 6.38 \times 10^6)^{\frac{1}{2}} = 11.2 \times 10^3\ \text{ms}^{-1}$$

$$= 11.2\ \text{km s}^{-1}$$

For planet Mercury the escape velocity is about 4.2 km s^{-1} and for Jupiter it is 61 km s^{-1}

An Interesting Consequence of Escape Velocity

We notice from Eq. (8.9) that the escape velocity is independent of the mass of the body. Thus, very massive rockets and extremely tiny particles (such as the molecules of a gas) will require the same initial velocity to escape from the earth or from any other planet or star.

In Chapter 9, we shall learn that the molecules of a gas move around with a certain average velocity that depends on the nature and temperature of the gas. For oxygen, nitrogen, carbon dioxide and water vapour, the average velocity at moderate temperatures is of the order of (0.5 to 1.0) × 10^3 $\mathrm{ms^{-1}}$ or (0.5 to 1.0) km $\mathrm{s^{-1}}$, whereas for lighter gases, such as hydrogen and helium it is (2-3) km $\mathrm{s^{-1}}$. Now, we know that the moon has no atmosphere. The velocity of escape from the moon is about 2.5 km $\mathrm{s^{-1}}$. It is clear that lighter gases, such as hydrogen and helium whose average molecular velocities are of the order of the escape velocity, will escape from the moon. The gravitational pull of the moon is too weak to hold these gases. Thus, in course of time, the gases that were supposed to be present in the atmosphere of the moon, have escaped from it. The presence of the lighter gases in relative abundance in the atmosphere of the sun should not surprise us in view of the much stronger gravitational attraction of the sun and consequently much higher escape velocity which, for the sun, is about 620 km $\mathrm{s^{-1}}$.

EXAMPLE 8.5 Estimate the mass of the sun, assuming the orbit of the earth around the sun to be a circle. The distance between the sun and the earth is 1.49×10^{11} m and $G = 6.67 \times 10^{-11}$ N $\mathrm{m^2\,kg^{-2}}$.

Solution The gravitational force of attraction between the sun and the earth is $\frac{GmM}{r^2}$, where m is the mass of the earth, M that of the sun and r the distance between them. This, then, is the centripetal force $\frac{mv^2}{r}$ acting on the earth as it goes round the sun with a uniform speed v. Therefore

$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

or

$$M = \frac{v^2 r}{G}$$

Here

$$r = 1.49 \times 10^{11}\ \mathrm{m}$$

$$G = 6.67 \times 10^{-11}\ \mathrm{Nm^2\,kg^{-2}}$$

and $v = \frac{2\pi r}{T}$ where T is the time taken by the earth to make one complete revolution around the sun, i.e. 365 days or 365 × 24 × 3600 s.

Substituting for r, G and v, we have

$$M = \left(\frac{2\pi r}{T}\right)^2 \frac{r}{G} = \frac{4\pi^2 r^3}{T^2 G}$$

$$= \frac{4 \times (3.14)^2 \times (1.49 \times 10^{11})^3}{(365 \times 24 \times 3600)^2 \times (6.67 \times 10^{-11})}$$

$$= 1.97 \times 10^{30}\ \mathrm{kg}$$

EXAMPLE 8.6 Calculate the escape velocity on the surface of the moon. Given the radius of the moon = 1.7 × 10^6 m, mass of the moon = 7.35 × 10^{22} kg and universal gravitational constant $G = 6.67 \times 10^{-11}$ Nm² kg^{-2}.

Solution The escape velocity is given by

$$
\begin{aligned}
v_e &= \sqrt{\frac{2Gm}{R}} \\
&= \left(\frac{2 \times 6.67 \times 10^{-11} \times 7.35 \times 10^{22}}{1.7 \times 10^6}\right)^{\frac{1}{2}} \\
&= 2.4 \times 10^3 \text{ ms}^{-1} \\
&= 2.4 \text{ km s}^{-1}
\end{aligned}
$$

EXAMPLE 8.7 A rocket is launched vertically from the surface of the earth with an initial speed of 10 km s^{-1}. How far above the surface of the earth would it go? Radius of earth = 6400 km. Ignore the atmospheric resistance.

Solution On the surface of the earth, the total energy is

$$\text{KE} + \text{PE} = \frac{1}{2} mv^2 - G\frac{mM}{R}$$

At the highest point

$$v = 0$$

$$\text{PE} = -G\frac{mM}{(R+h)}$$

where h is the maximum height attained.
From the law of conservation of energy we have

$$
\begin{aligned}
\frac{1}{2} mv^2 - G\frac{mM}{R} &= -G\frac{mM}{(R+h)} \\
\frac{1}{2} v^2 &= G\frac{M}{R}\left\{1 - \frac{R}{(R+h)}\right\} \\
&= \left(G\frac{M}{R^2}\right)\frac{hR}{(R+h)} \\
&= g\frac{Rh}{(R+h)}
\end{aligned}
$$

where g is the acceleration due to gravity on the surface of the earth.

Therefore $$\frac{R+h}{h} = \frac{2gR}{v^2}$$

or
$$h = R\left(\frac{2gR}{v^2} - 1\right)^{-1}$$
$$= R\left(\frac{2 \times 9.8 \times 6.4 \times 10^6}{10^8} - 1\right)^{-1}$$

or
$$h = 3.93\ R$$
$$= 3.93 \times 6.4 \times 10^3 \text{ km}$$

$\therefore$
$$h = 2.5 \times 10^4 \text{ km}$$

EXAMPLE 8.8 The escape velocity of a projectile on the earth's surface is 11.2 km s^{-1}. A body is thrown up with three times this speed. What will be the speed of the body far away from the earth (i.e at infinity) assuming that the sun and planets do not influence its motion?

Solution If v_i and v_f are respectively the initial and final speeds of the body, we have, from the law of conservation of energy

$$\frac{1}{2}\,mv_i^2 - \frac{GmM}{R} = \frac{1}{2}\,mv_f^2 \tag{i}$$

where m is the mass of the body and M is the mass of the earth and R its radius. The escape velocity is given by

$$\frac{1}{2}\,mv_e^2 = \frac{GmM}{R} \tag{ii}$$

Using (ii) in (i) gives

$$\tfrac{1}{2}\,mv_i^2 - \tfrac{1}{2}\,mv_e^2 = \tfrac{1}{2}\,mv_f^2$$

or
$$v_f = (v_i^2 - v_e^2)^{\frac{1}{2}} \tag{iii}$$

Now $v_e = 11.2$ kms^{-1} and $v_i = 3\,v_e$. Eq. (iii) then gives

$$v_f = \sqrt{8}\ v_e = \sqrt{8} \times 11.2 = 31.7 \text{ kms}^{-1}$$

EXAMPLE 8.9 A hole is drilled half-way to the centre of the earth. A body is dropped into the hole. How much will it weigh half-way down to the centre if it weighs 250 N on the surface of the earth? Assume the earth to be a sphere of uniform density.

Solution As illustrated in Fig. 8.5, the gravitational force on mass m arises only due to the mass of the sphere of radius r. The mass of this sphere is

$$M' = \frac{4}{3}\pi r^3 \rho \tag{i}$$

where ρ is the density of the earth. Hence the gravitational force on mass m is

$$F' = \frac{GmM'}{r^2} \qquad \text{(ii)}$$

From Eqs (i) and (ii) we have

$$F' = m\left(\frac{GM}{R^2}\right)\frac{M'}{M}\,\frac{R^2}{r^2} \qquad \text{(iii)}$$

where M is the mass of the whole earth and R is its radius. But

$$\frac{GM}{R^2} = g,$$

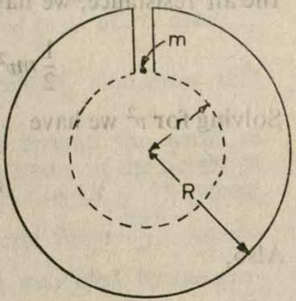


Fig. 8.5

the acceleration due to gravity at the surface of the earth. Also

$$\frac{M'}{M} = \frac{(4/3)\,\pi\, r^3\, \rho}{(4/3)\,\pi\, R^3\, \rho} = \frac{r^3}{R^3}$$

Therefore Eq. (iii) becomes

$$F' = mg\frac{r}{R} = F\frac{r}{R}$$

where $F = mg$ is the weight on the surface of the earth. It is given that $F = 250$ N and $r = R/2$. Hence

$$F = \frac{250 \times R}{2\,R} = 125 \text{ N}$$

EXAMPLE 8.10 A body is released from above at a distance r from the centre of the earth. Compute the velocity of the body when it strikes the surface of the earth.

Solution Since the initial velocity of the body is zero, its total energy is

$$E = -\,G\,\frac{Mm}{r}$$

where m is the mass of the body and M is the mass of the earth. When the body reaches the earth's surface, its velocity is v and its distance from the centre of the earth is the earth's radius R. Therefore,

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{R}$$

Equating the values of E in the above two equations and neglecting the air resistance, we have

$$\frac{1}{2}mv^2 - G\frac{Mm}{R} = -G\frac{mM}{r}$$

Solving for v^2 we have

$$v^2 = 2\,GM\left(\frac{1}{R} - \frac{1}{r}\right)$$

Also, $$g = \frac{GM}{R^2}$$

$$\therefore \quad v^2 = 2R^2 g\left(\frac{1}{R} - \frac{1}{r}\right) \quad \text{or} \quad v = R\left[2g\left(\frac{1}{R} - \frac{1}{r}\right)\right]^{1/2}$$

This expression can also be used for finding the distance r reached by a body thrown vertically with a velocity v from the surface of the earth.

If the body is released at a great height so that $1/r$ is negligible compared with $1/R$, we get

$$v = \sqrt{2\,Rg}$$

which is the expression for the escape velocity.

8.8 GEOSTATIONARY SATELLITES

A body moving in an orbit around the earth is called the earth's satellite. The moon is the natural satellite of the earth. It goes around the earth once in 27.3 days in a nearly circular orbit of mean radius 3.85×10^5 km. The first artificial (i.e. man-made) satellite was launched in 1956. Since then many artificial satellites have been launched and put into orbit a few hundred kilometres above the surface of the earth. India launched its first satellite, named *Aryabhatta*, on April 19, 1975.

Orbital Velocity

Artificial satellites are put into orbit at an altitude of a few hundred kilometres. At this height, the air is extremely rarefied and the friction due to it is negligible. The satellite is carried by a rocket which, on achieving the desired height, releases the satellite horizontally by imparting to it a very high velocity so that it remains moving in a nearly circular orbit. We now ask if there is any prescribed value of the velocity of the satellite for it to be able to continue to move in an orbit? The answer is, yes. The reason is that for a body to move in a circular path, a centripetal force (dependent on the velocity) must be provided.

The velocity which is imparted to an artificial satellite a few hundred kilometers above the earth's surface so that it may start orbiting the earth is called the *orbital velocity*.

We shall now calculate the value of orbital velocity and show that it is about 8 km s^{-1}.

Let us assume that a satellite of mass m goes around the earth in a circular orbit of radius r with a uniform speed v. If the height of the satellite above the earth's surface is h, then $r = (R + h)$, where R is the mean radius of the earth. The centripetal force $\frac{mv^2}{r}$ necessary to keep the satellite in its circular orbit is provided by the gravitational force $\frac{GmM}{r^2}$ between the earth and the satellite. This means that

$$\frac{mv^2}{r} = G\frac{mM}{r^2}$$

where M is the mass of the earth. Thus

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R+h)}} \tag{8.11}$$

Now the acceleration due to gravity on earth's surface is given by

$$g = \frac{GM}{R^2}$$

or

$$GM = gR^2$$

Substituting for GM in Eq. (8.11) we get

$$v = R\sqrt{\frac{g}{(R+h)}} \tag{8.12}$$

If the satellite is a few hundred kilometres above the earth's surface (say 100 to 300 km), we can replace $(R + h)$ by R. The error involved in this approximation is negligible since the radius of the earth $(R) = 6.4 \times 10^6$ m = 6400 km.

Thus, we may write

$$v = \sqrt{gR} = \sqrt{9.8 \times 6.4 \times 10^6}$$
$$= 7.9 \times 10^3 \text{ m s}^{-1} \simeq 8 \text{ km s}^{-1}$$

Geostationary Satellites

A geostationary satellite is a particular type used in communication. A number of communication satellites are launched which remain in fixed positions at a specified height above the equator. They are called *geostationary* or *synchronous satellites* used in international communication. Some television programmes or events occurring in distant countries are often relayed 'live' via these satellites.

For a satellite to appear fixed at a position above a certain place on the earth, it must corotate with the earth so that its orbital period around the earth is exactly equal to the rotational period of the earth about its axis of rotation. Can this condition be satisfied at any height above the earth? The answer is yes, as we shall see.

Suppose a satellite of mass m is in a circular orbit round the earth at a distance r from the centre of the earth. For synchronisation, its period of revolution round the earth must equal the period of rotation (T) of the earth which is 1 day = 86400 seconds. Then the orbital speed is given by

$$v = \frac{\text{circumference of orbit}}{\text{period}}$$

or

$$v = \frac{2\pi r}{T} \tag{8.13}$$

The centripetal acceleration is [using Eq. (8.13)]

$$a_c = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2} \tag{8.14}$$

The gravitational force on the satellite due to the earth is

$$F = \frac{GmM}{r^2} \tag{8.15}$$

where M is the mass of the earth. From Newton's second law of motion, we have

$$F = ma_c \tag{8.16}$$

Using Eq. (8.15) in Eq. (8.16) we get

$$a_c = \frac{GM}{r^2} \tag{8.17}$$

Now we know that

$$g = \frac{GM}{R^2} \tag{8.18}$$

where R is the radius of the earth. Using Eqs (8.14), (8.17) and (8.18) we get

$$r^3 = \frac{T^2}{4\,\pi^2} \times gR^2$$

or

$$r = \left(\frac{T^2 g R^2}{4\pi^2}\right)^{1/3} \tag{8.19}$$

Equation (8.19) gives the orbital radius of a geostationary satellite. Substituting $T = 86400$ s, $R = 6.38 \times 10^6$ m and $g = 9.8$ m s^{-2} in Eq. (8.19) gives

$$r = 42{,}250 \text{ km}$$

Therefore, the height of satellite above the surface of the earth is

$$h = r - R = 35{,}870 \text{ km}$$

Thus, for a satellite to appear stationary, it must be placed in orbit round the earth at a height of 35,870 km from the earth's surface. This is often called the *parking orbit* of the satellite. Relay satellites are placed in parking orbits to transmit TV programmes from one part of the world to another.

8.9 PLANETARY MOTION

Ptolemy, a Greek astronomer who lived in the second century AD, was one of the earliest to make a detailed study of the motion of planets and their relative positions in space. He advanced a theory called the *Geocentric theory* which considered the earth as the centre of the universe, around which all the planets, the moon and the stars revolved in various orbits. The orbits proposed by the theory were very complex. Despite its complexities, the theory was universally accepted for more than fourteen centuries.

In the year 1543, a Polish astronomer Copernicus proposed a new theory called the *Heliocentric theory*, according to which the sun was at the centre with planets revolving around it in various orbits. Copernicus also believed that the earth rotated on its axis once each day. The great Indian mathematician and astronomer Aryabhat who lived in the fifth century AD had made astronomical observations and said that the earth moved about an axis.

The two scientists whose contributions made the most profound impact on the understanding of the universe were a Danish astronomer Tycho Brahe (1546-1601) and a German theorist Johannes Kepler (1571-1630). Tycho Brahe made very accurate observations of the motion of planets and Kepler analysed these observations carefully and systematically and modified the heliocentric theory. Kepler was able to unify the observations into three empirical laws relating to planetary motion. These laws formed the basis of the famous Newton's law of universal gravitation.

Kepler's Laws of Planetary Motion

Kepler (1571-1630) made a careful study of Tycho Brahe's records of observations on planetary motion and proposed the following three laws, now known after his name, as Kepler's laws.

1. *Each planet revolves round the sun in an elliptical orbit with the sun at one focus of the ellipse* (the law of orbits).

2. *The line joining the sun and the planet sweeps out equal areas in equal times* (the law of areas).

3. *The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun* (the law of periods).

Figure 8.6 shows the elliptical orbit of a planet round the sun at one focus of the ellipse. The shaded portions are the areas swept out by the line joining the sun and the planet in equal intervals of time. If r is the mean distance of the planet from the sun and T the time period of its revolution, then Kepler's third law states that

$$T^2 \propto r^3$$

or

$$\frac{T^2}{r^3} = \text{constant for all planets}$$

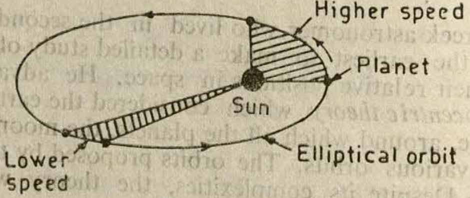


Fig. 8.6 Kepler's laws of planetary motion

The speed of the planet in its orbit varies but the variation is such that the areas swept out (shown shaded) in equal times are equal. The gravitational force between the sun and the planet provides the necessary inward force to keep it in orbit.

Kepler's laws were empirical. Kepler could not give a theoretical proof of his laws. He obtained them by inductive reasoning. It is inded remarkable that the predictions of the positions of planets based on Kepler's laws agree very well with observation.

Derivation of Kepler's Law of Periods from the Law of Gravitation

Newton was able to give complete derivations of Kepler's laws. His derivations, which use calculus, are beyond the mathematical level of this book. We will give here a simpler derivation of Kepler's third law without using calculus. Let us assume that the planet of mass m moves around the sun of mass M in a circular orbit of radius r. Hence, from Newton's law of gravitation, the force of attraction of the sun on the planet is given by

$$F = G\frac{mM}{r^2}$$

If this force is the centripetal force keeping the planet in orbit, then (see Fig. 8.7)

$$F = \frac{mv^2}{r}$$

where v is the speed of the planet in the circular orbit. Therefore, equating the two, we have

$$\frac{mv^2}{r} = G\frac{mM}{r^2}$$

or
$$v^2 = \frac{GM}{r} \qquad \text{(i)}$$

If T is the period of revolution of the planet around the circle,

$$v = \frac{2\pi r}{T} \qquad \text{(ii)}$$

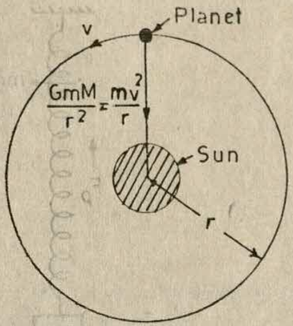


Fig. 8.7

Substituting (ii) in (i), we get

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

or
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Since GM is constant for any planet, it follows that $\frac{r^3}{T^2}$ is constant, which is Kepler's third law.

EXAMPLE 8.11 If the earth were a perfect sphere of radius 6.37×10^6 m, rotating about its axis with a period of 1 day ($= 8.64 \times 10^4$ s), how much would the acceleration due to gravity (g) differ from the poles to the equator?

Solution Let us take an object of mass m and measure its weight at the poles and at the equator with a spring balance. At the north or south pole, the object has no rotation about the polar axis. The restoring force F_p in the spring measures the weight mg_p of the object, where g_p is the acceleration due to gravity at the poles (Fig. 8.8a).

When the same object is taken to the equator, it rotates with the earth at an angular velocity ω given by

$$\omega = \frac{2\pi}{T}$$

where T is the rotational period of the earth. The object, therefore, has a centripetal acceleration $R\omega^2$ which is directed to the centre of the earth. Here R is the radius of the earth (Fig. 8.8 (b)). The restoring force F_e on the spring acting upwards is, therefore, given by

$$F_e = mg_p - mR\omega^2$$

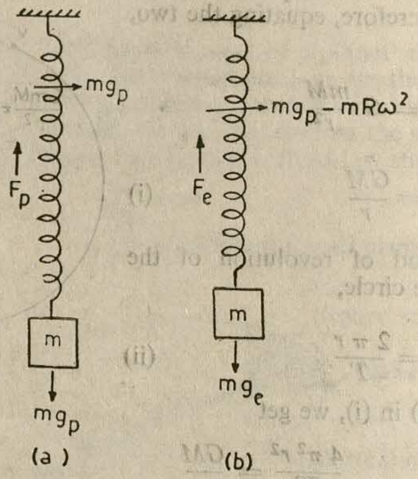


Fig. 8.8

But $F_e = mg_e$, where g_e is the acceleration due to gravity at the equator. Thus we have

$$mg_e = mg_p - mR\,\omega^2$$

or

$$g_p - g_e = R\,\omega^2$$

$$= R\left(\frac{2\,\pi}{T}\right)^2$$

$$= \frac{(6.37 \times 10^6) \times (2 \times 3.14)^2}{(8.64 \times 10^4)^2}$$

$$= 0.0337 \text{ m s}^{-2}$$

Example 8.12 Calculate the minimum energy required to launch a 250 kg satellite from earth's surface in a circular orbit at an altitude of 2 R, where R is the radius of the earth and is equal to 6400 km. Mass of the earth = 6 × 10^{24} kg.

Solution Consider a satellite of mass m moving with a speed v at an altitude r (measured from the centre of the earth). Then

$$\text{Kinetic energy (KE)} = \tfrac{1}{2}\, mv^2$$

$$\text{Gravitational potential energy (PE)} = -\frac{GmM}{r}$$

where M is the mass of the earth.

For a satellite in circular orbit, we have

$$\frac{mv^2}{r} = \frac{g\, mM}{r^2} \quad \text{or} \quad v^2 = \frac{GM}{r}$$

or $$\frac{1}{2}\, mv^2 = \frac{GmM}{2\,r}$$

i.e. $$\text{KE} = \frac{GmM}{2r}$$

Thus the KE of a satellite in a circular orbit is numerically half its PE but opposite in sign. The total energy of the satellite in orbit is·

$$E = \text{KE} + \text{PE}$$

$$= \frac{GmM}{2r} - \frac{GmM}{r} = -\frac{GmM}{2r}$$

It is given that $r = 2\,R + R = 3\,R$, where R is the radius of the earth.

$$\therefore \qquad E = -\frac{GmM}{6\,R}$$

Now PE on the surface of the earth $= -\dfrac{GmM}{R}$

$$\therefore \quad \text{Minimum energy required } (E_{\text{min}}) = -\frac{GmM}{6\,R} - \left(-\frac{GmM}{R}\right)$$

$$= \frac{5\,GmM}{6\,R}$$

Substituting the given values of m, M and R and $G = 6.67 \times 10^{-11}$ $\text{Nm}^2\ \text{kg}^{-2}$, we get

$$E_{\text{min}} = 1.3 \times 10^{10}\ \text{J}$$

EXAMPLE 8.13 A high jumper can jump 2.0 m on earth. With the same effort how high will he be able to jump on a planet whose density is one-third and radius one-fourth those of the earth?

Solution Suppose the person of mass m jumps a height h_e on earth and a height h_p on the planet. If he gives himself the same kinetic energy on the earth and on the planet, the potential energy gained at maximum height will be the same, i.e.

$$mg_p h_p = mg_e h_e$$

where g_p and g_e are the accelerations due to gravity on the planet and on earth respectively. So

$$h_p = \frac{g_e}{g_p} \times h_e \qquad \text{(i)}$$

Now for the earth,

$$g_e = \frac{GM_e}{R_e^2} = \frac{G}{R_e^2} \times \left(\frac{4\pi}{3} R_e^3 \rho_e\right) = \frac{4\pi}{3} G R_e \rho_e \qquad \text{(ii)}$$

where R_e and ρ_e are respectively the radius and density of the earth.

Similarly for the planet we have

$$g_p = \frac{4\pi}{3} GR_p \rho_p \qquad \text{(iii)}$$

From (ii) and (iii), we get

$$\frac{g_e}{g_p} = \frac{R_e\rho_e}{R_p\rho_p} = 4 \times 3 = 12$$

From (i) we have

$$h_p = 12 \times 2 = 24 \text{ m}$$

8.10 TORQUE DUE TO GRAVITY: CENTRE OF GRAVITY

While determining the gravitational potential energy of a spherical body at a certain distance from the earth, we had assumed that the body and the earth behave as if their entire mass were concentrated at their centres. We will now show that this assumption is true. For simplicity, we will consider the case in which the acceleration due to gravity (g) is constant.

We have seen in Chapter 7 that, while dealing with extended bodies, an important quantity we must consider is the torque about any point in the body due to forces acting at all points in the body. Here we consider only the gravitational forces.

Consider a body in a constant gravitational field ($\vec{g}$ = constant at all points of the body). We regard the body as being made up of a large number of point masses. Consider one such point mass m_i located at position vector $\vec{r_i}$ with respect to an arbitrary origin O, as shown in Fig. 8.9. We will find the total torque about O. Since the force acting on m_i is $m_i \vec{g}$, the total torque about O is given by

$$\vec{\tau} = \sum \vec{\tau_i} = \sum \vec{r_i} \times m_i \vec{g} = \sum m_i \vec{r_i} \times \vec{g}$$

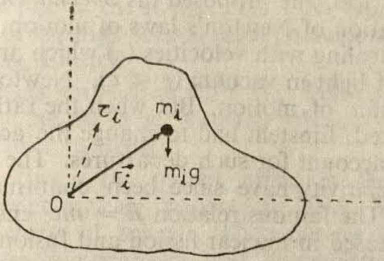


Fig. 8.9 Torque due to gravity about a point O in a body

where the summation is taken over all the mass points of the body. Since $\vec{g}$ is constant, we can write

$$\vec{\tau} = (\sum m_i \ \vec{r_i}) \times g$$

Now from the definition of the centre of mass we have

$$\sum m_i \vec{r_i} = \vec{R}_{\mathrm{CM}} \sum m_i = \vec{R}_{\mathrm{CM}} M$$

where $M = \sum m_i$ is the total mass of the body and $\vec{R}_{\mathrm{CM}}$ is the radius vector drawn from O to the centre of mass of the body.

Thus, $$\vec{\tau} = \vec{R}_{\mathrm{CM}} \times (M \vec{g})$$

Thus we find that the torque acts as if the entire mass M of the body were concentrated at its centre of mass at which the force $(M \vec{g})$ appears to act.

If the point O itself happens to be the centre of mass then $\vec{R}_{\mathrm{CM}} = 0$ because the radius vector $\vec{R}_{\mathrm{CM}}$ is now drawn from O to itself which is obviously zero. Hence the torque about the centre of mass of the body is zero, i.e. the body is in equilibrium under gravity. That is why the centre of mass is often called the *centre of gravity*. It is evident that the centre of gravity of a sphere (with a uniform mass distribution) is located at its centre. Thus a uniform spherical body behaves as if its entire mass were concentrated at its centre.

SUPPLEMENT

The General Theory of Relativity

Albert Einstein, one of the greatest human minds ever, was born on March 14, 1879 in the German city of Ulm, near München. His revolutionary ideas about space and time have become part and parcel of modern physics. In 1905, he proposed his *Special Theory of Relativity* which is a modification of Newton's laws of motion. It turns out that as long as we are dealing with velocities (v) which are small compared to c, the velocity of light in vacuum ($v \ll c$), Newton's laws give an adequate description of motion. But when the ratio $v/c \to 1$, departures become marked. Einstein had to change the accepted notions of space and time to account for such departures. The conclusions of his special theory of relativity have since been confirmed experimentally in the laboratory. The famous relation $E = mc^2$ enables us to calculate the energy released in nuclear fission and fusion reactions.

Einstein's special theory of relativity considered how natural phenomena appear when viewed from different inertial (i.e. non-accelerating) frames with his new notion about space and time which are considered on an equal footing.

Einstein wanted to incorporate gravitation into his spacetime picturce. Because gravitation produces acceleration in material bodies, he had to consider phenomena in *accelerated frames* (i.e. non-inertial frames). He got the clue from the equivalence of gravitational and inertial mass of a body. He argued that by a suitable selection of an *accelerated frame* with respect to an inertial frame, gravitation could be increased, decreased or even annulled. This was a stroke of superhuman genius. We can understand it by an example.

Imagine that you are in lift (or elevator) and the cable is cut. The lift falls freely. You and all objects in the lift will fall with the same acceleration (g) and will be in a state of 'weightlessness'. You and all objects around you will float around. The gravity is absent, i.e. it has been annulled. Your freely falling (i.e. accelerating) frame has cancelled gravity due to its acceleration. Now imagine that you are in a space-ship far away from the earth where gravity is negligible. If the space-ship is accelerated upwards (by shooting rockets downwards), you will begin to feel gravity as the floor of the ship presses your feet. Thus you have 'created' gravity using an upward accelerated frame.

To include gravitation, therefore, Einstein had to generalize his special theory of relativity by considering accelerated frames. He accomplished this formidable task in 1915 and developed the *General Theory of Relativity* which revolutionized our idea of gravitation. The predictions of his new theory such as bending of light rays by gravitation and the discrepancy in the precession of planet Mercury have been confirmed.

A Biographical Note on Einstein

Albert Einstein, was born on 14th March, 1879, in the German city of Ulm, near München. As a child he was very slow in learning to speak. This created an apprehension that he might be mentally retarded. After his parents had moved to Italy, Einstein tried to enter the Federal Polytechnic School in Zurich. He failed in his first attempt. Although he did remarkably well in mathematics, he failed in modern languages, zoology, and botany. After managing to gain entrance, he successfully graduated in the summer of 1900. This lack of interest in studies made one of his high school teachers remark, "You will never amount to anything, Einstein".

Two years after his graduation with the influence of his friend's father, Einstein managed to get a steady job as a patent examiner in the Swiss Patent Office in Berne.

It was while working at the patent office in Berne that Einstein published four epoch-making papers in 1905. This was an event unique in the history of science. These papers were to change our view of the Universe in a profound and far-reaching manner.

Einstein's first paper contained an explanation of the photoelectric effect—the phenomenon of emission of electrons from a metallic surface when light falls on it. His theory postulated that light consists of particles (which were later named *photons* by G.N. Lewis). These ideas were experimentally confirmed by Millikan, nine years later. For this work, Einstein was awarded the Nobel Prize in 1921. His second paper provided an explanation of Brownian motion. This consists of the random motion of microscopic particles suspended in a viscous medium, like pollen grains suspended in water, which was first observed, under the microscope, by the Scottish botanist, Robert Brown in 1827. The experimental confirmation of Einstein's theory by Perrin, Svedberg, Westgren, and others, was the first direct experimental evidence for the existence of atoms and molecules.

Einstein's third paper developed the theory of special relativity. In a footnote to this paper, Einstein derived the famous $E = mc^2$ formula. This is the basis of energy generation in nuclear fission (in an atom bomb or a reactor) and fusion (in stellar sources, the hydrogen bomb, etc.).

After ten years of effort, Einstein published his monumental work, "Foundations of the General Theory of Relativity", in 1916. This was a reformulation of the laws of physics so as to make them invariant with respect to non-inertial observers. Einstein's general theory is a theory of gravitation which is based on the geometry of space-time. All the predictions of this theory have been confirmed by experiment. It is the starting point of all cosmological models of the Universe.

After the end of the First World War the rise of militarism in Germany horrified Einstein. In 1933 the Nazi leader Adolf Hitler became Chancellor of the German Reich. When the systematic persecution of Jews started, Einstein left Germany in 1933, never to return. He settled down in the United States at Princeton, New Jersey, and took up a post at the Institute for Advanced Studies. This last phase of Einstein's life was not very productive in terms of scientific achievements. He attempted to unify gravitation and electromagnetism but was not successful.

On April 18, 1955, Einstein died in his sleep at the Princeton Hospital.

SUMMARY

Newton's law of universal gravitation states that any two particles of matter anywhere in the universe attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. i.e.

$$F = \frac{G\, m_1\, m_2}{r^2}$$

where $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} is the universal gravitation constant.

The acceleration due to gravity on the surface of a planet of mass M and radius R (assumed spherical and of uniform mass density) is given by

$$g = \frac{GM}{R^2}$$

The gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done in bringing the mass m from infinity to a distance r and is given by

$$\text{PE} = -\frac{GmM}{r}$$

PE is always negative and increases to zero at infinity. The gravitational potential is given by

$$V = -\frac{GM}{r}$$

The escape velocity for a planet of mass M and radius R is

$$v_e = \sqrt{\frac{2\,MG}{R}}$$

$$= \sqrt{2\,gR}$$

Kepler's laws of planetary motion are:

1. Each planet revolves round the sun in an elliptical orbit with the sun at one focus of the ellipse.

2. The line joining the sun and the planet sweeps out equal areas in equal times.

3. The square of the period of revolution of the planet is proportional to the cube of its mean distance from the sun.

EXERCISES

1. State Newton's law of universal gravitation. How did Newton arrive at the law?
2. What is the universal gravitation constant? In what sense is it a universal constant? State its value with units.
3. Briefly describe the Cavendish torsion balance. How is it used to measure *G*?
4. How did Cavendish 'weigh' the earth?
5. Distinguish clearly between the inertial and gravitational mass of a body. Comment on their equivalence.
6. Discuss the variation of *g* with altitude. Why are the values of *g* different at the poles and the equator?
7. Define gravitational potential energy. Deduce the expression for it for a mass in the gravitational field of the earth.
8. Define gravitational potential at a point in a gravitational field.
9. What is escape velocity? Obtain an expression for the escape velocity for a planet.
10. The moon has no atmosphere. Why?
11. What are geostationary satellites? Deduce the expression for the orbital radius of a geostationary satellite.
12. State Kepler's laws of planetary motion. How will you deduce the law of periods from the law of gravitation?
13. Obtain an expression for the torque acting on a body due to gravity and hence show that the entire mass of an extended body may be treated to be concentrated at its centre of mass.
14. Write a brief essay on Einstein's general theory of relativity.

PROBLEMS

(Take $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} wherever necessary.)

1. The acceleration due to gravity on the surface of the moon is 1.7 m s^{-2}. If the average radius of the moon is 1.7×10^6 m, compute its mass.
2. The mass of the planet Jupiter is 1.9×10^{27} kg and that of the sun is 1.99×10^{30} kg. The mean distance of the sun from Jupiter is 7.8×10^{11} m. Calculate the gravitational force which the sun exerts on Jupiter. Assuming that Jupiter moves in a circular orbit around the sun, calculate the speed of Jupiter.
3. Calculate the value of the escape velocity for a particle of a gas 1000 km above the surface of (i) the earth, (ii) the moon and (iii) the sun. Given mass of the earth = 5.98×10^{24} kg, its radius = 6.37×10^6 m; mass of the moon = 7.34×10^{22} kg, its radius = 1.74×10^6 m; mass of the sun = 1.99×10^{30} kg, its radius = 6.96×10^8 m.
4. Two isolated equal masses of 6.40 kg are separated by a distance of 0.16 m. A small mass is released from a point P equidistant from the two heavy masses and at a height 0.06 m from the line joining them (see Fig. 8.10). Calculate the velocity of this mass when it passes point Q.

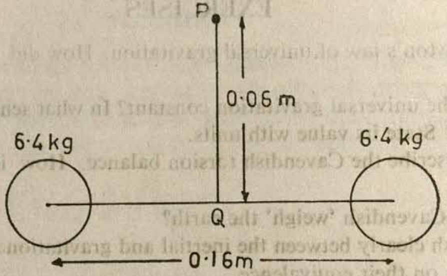


Fig. 8.10

5. If the earth were rotating about its axis at an angular velocity of 1.237 $\times 10^{-3}$ rad s^{-1}, what would be the weight of a body of mass 5 kg at the equator?
6. An artificial satellite moving in a circular orbit around the earth has a period of revolution equal to that of its rotation about its own axis. Find its distance from the earth's surface. Radius of earth = 6400 km.
7. Find the velocity acquired by a body in falling to the surface of the earth from

   (a) an extremely large distance

   (b) a point distant twice the earth's radius from its surface.
   R = 6400 km.
8. A research satellite of mass 200 kg circles the earth in an orbit of average radius $3R/2$ where R is the radius of the earth. Assuming the gravitational pull on a mass 1 kg at the earth's surface to be 10 N, calculate the pull on the satellite.
9. Taking the earth to be a uniform sphere of radius 6400 km, and the value of g at the surface to be 10 m s^{-2}, calculate the total energy needed to raise a satellite of mass 2000 kg to a height of 800 km above the ground and to set it into a circular orbit at that altitude.
10. A satellite revolving in a circular equatorial orbit of radius 2×10^4 km from west to east appears over a certain point at the equator every 11.6 hours. Calculate the mass of the earth.
11. What is the acceleration of a satellite moving in a circular orbit of radius $2R$ round the earth of radius R.
12. The maximum height up to which a fully-dressed astronaut can jump on the earth is 0.5 m. With the same effort, to what height can he jump on the moon? What is the ratio of the time duration of his jump on the moon to that on the earth? The mean density of the moon is 2/3 that of the earth and the radius of the moon is 1/4 that of the earth.

9

Molecules

9.1 INTRODUCTION

A significant development in science took place when man began to consider the properties of matter, rather than the behaviour of particular objects. It is one thing to know the properties of various iron objects of all sizes and shapes, but it is surely of greater importance to study the nature of the substance iron itself than to limit ourselves to specific iron objects. Every substance, such as iron or aluminium, has its own characteristic properties, namely, heat conductivity, electrical conductivity, specific gravity and specific heat, to mention only a few. Can we go a step further and consider what, if anything, is common to both iron and aluminium? The study of this aspect deals with the structure of matter.

9.2 THE ATOMIC HYPOTHESIS: ATOMS AND MOLECULES

What would happen if you cut a piece of iron in two halves, then cut the halves in half and continue to subdivide the substance into tinier and tinier pieces? Would you ever obtain the tiniest piece of iron which cannot be further subdivided? Or is matter continuous? Man has asked these questions about the ultimate nature of matter since the time of the ancient Greeks. Democritus, the Greek philosopher who lived in the fifth century BC believed that there is a limit to how far matter can be subdivided. These tiny particles of matter beyond which no further subdivision was possible were called 'atoms' which in Greek means 'indivisible'. However, many ancient philosophers, including Aristotle, believed that matter was continuous, consisting of four basic elements earth, air, water and fire.

The continuous theory and the atomic theory of matter had a number of followers. However, towards the end of the 16th century, philosophers noticed that certain aspects of the behaviour of matter were easily understood in terms of the atomic theory. But the atomic theory was not universally accepted until chemical experiments were performed in the 18th and early 19th centuries.

Elements and Compounds

Chemists have long been concerned with how different materials could be combined to form other materials or with questions like whether a given substance could be broken down into simpler substances. They found that certain substances such as iron, gold and silver could not be broken down by any chemical means. They called them *elements.* Other substances were found to be made up of combinations of elements and were called *compounds.* For example, water is a compound because it can be broken down into the elements hydrogen and oxygen. In the year 1800 chemists discovered the *law of definite proportions*, which says that when two or more elements combine to form a compound, they always do so in the same proportion by mass. For example, water is always formed from one part of hydrogen and eight parts of oxygen. Similarly common salt (sodium chloride) is always formed by combining 23 parts by mass of sodium with 35 parts by mass of chlorine.

It was the British scientist John Dalton who, in the year 1808, recognised that the law of definite proportions could be explained only by the theory that elements consist of atoms. He argued that the weight proportions required to make a compound correspond to the relative masses of the combining atoms. For example, to explain why 23 parts by mass of sodium always combine with 35 parts by mass of chlorine to form the compound sodium chloride, Dalton reasoned as follows: each atom of sodium combines with a single atom of chlorine and one atom of sodium have a mass 23/35 times the mass of one atom of chlorine. The combination of one sodium atom and one chlorine atom was called a *molecule* by Dalton. Thus a molecule is a fixed combination of atoms.

The atomic theory neatly describes the difference between elements and compounds. *Substances that contain only one kind of atoms, such as iron and aluminium, are elements. Substances that are made up of one kind of molecules, such as water and common salt, are compounds.* A molecule comprises atoms; it may consists of atoms of the same kind or different kinds. For example a hydrogen molecule consists of atoms of the same kind whereas a water molecule contains atoms of different kinds. Molecules, in general, may contain any number of atoms. Those having just one atom are called *monoatomic* molecules, those with two atoms are called *diatomic* molecules. Often molecules may be *polyatomic* (see Fig. 9.1).

9.3 BROWNIAN MOTION

Perhaps the most direct visual evidence for molecular structure of matter was a phenomenon discovered accidentally by botanist Robert Brown in the year 1827. One day he happened to examine the tiny pollen grains suspended in water with the help of a microscope. He

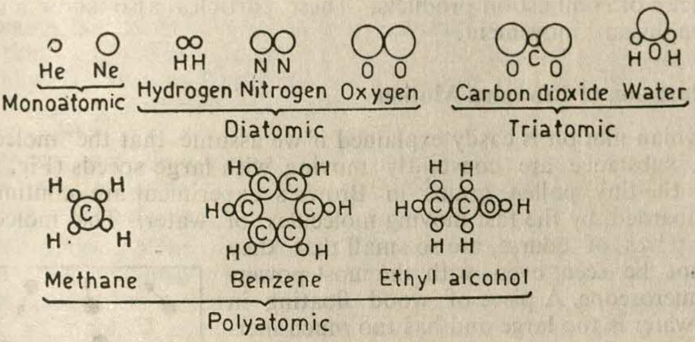


Fig. 9.1 Monoatomic, diatomic and polyatomic molecules

was surprised to see that the tiny grains were in a state of continuous vigorous haphazard motion even though the water was perfectly still. This phenomenon is called *Brownian motion*. The haphazard path of a typical grain is shown in Fig. 9.2.

Experimental Demonstration of Brownian Motion

The experiment to observe Brownian motion is illustrated in Fig. 9.3. The apparatus consists of a glass cell containing water in which very fine particles of graphite are suspended. The particles are illuminated by focussing light from a bright source by means of a convex lens as shown. When viewed through a microscope, these particles appear as bright points moving about in haphazard fashion as pollen grains observed by Brown (Fig. 9.2).

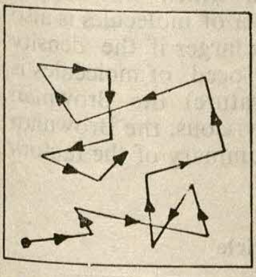


Fig. 9.2 Brownian motion

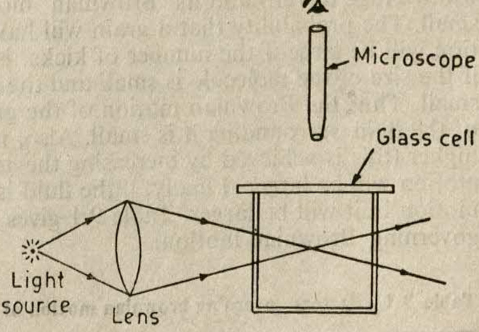


Fig. 9.3 Experiment to observe Brownian motion

This experiment shows Brownian motion in a liquid. To see this movement in a gas, such as air, the glass cell is filled with air containing a little smoke from a burning candle. The smoke consists of tiny particles of combustion products. These particles also show a similar haphazard movement.

Explanation of Brownian Motion

Brownian motion is easily explained if we assume that the molecules in a substance are constantly moving with large speeds (Fig. 9.4). Thus the tiny pollen grains in Brown's experiment are continually 'bombarded' by the fast moving molecules of water. The molecules themselves, of course, are so small that they cannot be seen even with the most powerful microscope. A piece of wood floating in still water is too large and has too much inertia to be influenced by the bombardment by the molecules. But a pollen grain (although much larger than a water molecule) is so small that if several molecules strike its imultaneously, it rebounds. The random bombardment of water molecules from all directions thus gives rise to the observed zig-zag motion of Fig. 9.2.

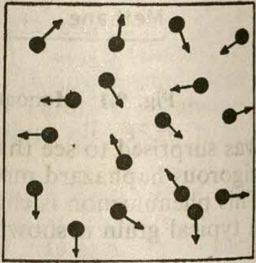


Fig. 9.4

The fact that the molecules are able to kick a pollen grain shows that the molecules must be moving with very high speeds.

Factors Governing Brownian Motion

How large is the Brownian motion of a particle such as a pollen grain? If the molecules kicking the grain are moving about randomly and if their number is very large, the net impulse imparted to the grain will average to zero and its Brownian movement will be negligibly small. The probability that a grain will have a sizeable Brownian motion will be large if the number of kicks is small which will happen if the size of the molecule is small and the number of molecules is also small. Thus the Brownian motion of the grain is larger if the density of the fluid surrounding it is small. Also, if the speed of molecules is higher (this is achieved by increasing the temperature) the Brownian motion will be larger. Finally, if the fluid is less viscous, the Brownian motion in it will be larger. Table 9.1 gives the summary of the factors governing Brownian motion.

Table 9.1 **Factors governing brownian motion of a particle**

Factor	*Effect*
1. Decrease in size of particle	1. Increase in Brownian motion
2. Decrease in density of medium	2. Increase in Brownian motion
3. Increase in temperature of medium	3. Increase in Brownian motion
4. Increase in viscosity of medium	4. Decrease in Brownian motion

Atomic and Molecular Sizes

It was Albert Einstein who, in the year 1905, first explained Brownian motion using the atomic point of view. The theoretical investigations of physcists such as Rudolf Clausius, Einstein and Jean Perrin in the 19th century predicted reasonably accurately the atomic diameters. It was found that atoms do not vary a great deal in size even though their masses cover a wide range (if the atomic mass of hydrogen is 1 unit, the atomic mass of uranium will be 238 units). Most atoms have a diameter in the range 1.0×10^{-8} cm to 2.0×10^{-8} cm.

Molecules, on the other hand, have a wide range of sizes. The size of a molecule depends upon the number of atoms that it consists of. Most molecules are small because they comprise only a few atoms. For example, a water molecule has a size of about 3.0×10^{-8} cm. At the other extreme are the long *macromolecules* found in plastics and living organisms. For example, if one DNA molecule in a human cell were stretched out, it would be about 10 cm long.

Brownian Motion of Larger Objects

Particles, such as pollen grains, are not the only ones that undergo Brownian motion. The molecules of a gas themselves are in Brownian

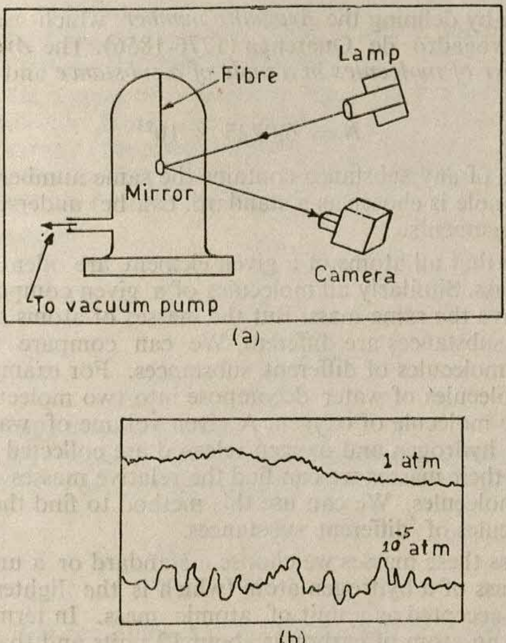


Fig. 9.5 (a) Brownian motion of a suspended mirror
(b) Photographic record of the random twisting of the suspended mirror at two pressures

movement owing to collisions with other molecules. Brownian movement of bigger objects has also been observed. Figure 9.5 (a) illustrates a famous experiment for observing the Brownian movement of a suspended mirror.

A small mirror is suspended by a fine quartz fibre in a glass chamber which is connected to a vacuum pump so that the air pressure in the chamber can be varied. The mirror will be hit by the air molecules and should show Brownian motion. The small twisting of the mirror is detected by the movement of a spot of light incident on it and reflected on to a movable photographic film. Figure 9.5 (b) shows the photographic record of the motion of the mirror at two pressures of air in the chamber. Decrease in air pressure increases Brownian motion of the air.

9.4 AVOGADRO NUMBER

Can you guess how many molecules of water are there in a glass of water? There are about 10^{25} molecules in about 250 cc of water. This is an enormous number. Such large numbers are not found by actual counting; it would take millions of years to count so many molecules if we could see them. How then do we find the number of molecules in a given substance?

We begin by defining the *Avogadro number* which is named after Amadeo Avogadro de Quarenga (1776-1856). The *Avogadro number is the number of molecules in a mole of a substance* and its value is

$$N = 6.0225 \times 10^{23}$$

Why a mole of any substance contains the same number of molecules and why a mole is chosen as a standard, can be understood from the following arguments.

We know that all atoms of a given element are identical and have the same mass. Similarly all molecules of a given compound are identical and have the same mass. But the masses of atoms and molecules of different substances are different. We can compare the masses of atoms and molecules of different substances. For example, we know that two molecules of water decompose into two molecules of hydrogen and one molecule of oxygen. A given volume of water is decomposed and hydrogen and oxygen released are collected and weighed. Comparing their masses we can find the relative masses of oxygen and hydrogen molecules. We can use this method to find the relative masses of molecules of different substances.

To express these masses we choose a standard or a unit. For a long time, the mass of a hydrogen atom (which is the lightest of all elements) was accepted as a unit of atomic mass. In terms of this unit, the mass of an atom of carbon is about 12 units and that of oxygen is 16 units. One could also express the masses of molecules in terms of this unit. For example a molecule of water (H_2O) has a mass $1 + 1 + 16 = 18$ units. This number is called the *relative molecular mass* (*or* *simply molecular mass*). For example the molecular mass of methyl alcohol (CH_3OH) is $12 + (1 + 1 + 1) + 16 + 1 = 32$.

The Mole of a Substance

Isotopes of many elements were discovered in the first quarter of the 19th century. The isotopes of an element have the same chemical properties (and cannot be distinguished by chemical means) but the atoms of one isotope of an element have a different mass from those of another isotope of the same element. For example $^{11}_{6}C$, $^{12}_{6}C$, $^{13}_{6}C$, $^{14}_{6}C$, etc. are all isotopes of carbon. The nucleus of carbon in each isotope has the same number (6) of protons but a different number of neutrons. For example $^{13}_{6}C$ stands for carbon, the nucleus of each atom of which has 6 protons and $13 - 6 = 7$ neutrons.

To be precise, therefore, it is necessary to assign a standard mass to one isotope, and find the relative atomic masses of the other. For this purpose, it has been internationally agreed to assign a mass of 12.000 units to a particular isotope, the carbon-12 isotope ($^{12}_{6}C$). This then is used to fix the relative atomic and molecular masses.

Consider a molecule of a substance A, let its relative mass on the C-12 scale be M_A. Then a mole of substance A is defined as the mass M_A in grams. Thus *a mole is defined as the amount of the substance which containts as many elementary units as there are in 12.000 g of C-12.* Thus the mass of 1 mole of C-12 is 12.000 g. Similarly a mole of water has mass of 18 g (approximately).

It is clear that a mole of a substance has a mass proportional to the mass of its molecule. Hence the mass of a mole of a substance can be expressed in terms of the absolute mass of its molecule. For example, suppose m_A is the absolute mass of a molecule of substance A and m_C is the absolute mass of an atom of C-12.

Then it follows that

$$\frac{m_A}{M_A} = \frac{m_C}{12}$$

or

$$M_A = \left(\frac{12}{m_C}\right) m_A \tag{9.1}$$

where M_A is expressed in grams.

The ratio M_A/m_A is the number of molecules of substance A in one mole of it. This is called the Avogadro number N. From Eq. (9.1) it follows that

$$N = \frac{M_A}{m_A} = \frac{12}{m_C} \tag{9.2}$$

which shows that the number N is independent of A. In other words, *Avogadro number is the same for all substances.*

Determination and Significance of Avogadro Number

If we can measure the absolute mass m_A of an atom or a molecule of a substance A and the mass M_A of a mole of that substance, then according to our definition (9.2) the ratio M_A/m_A determines the Avogadro number N. Absolute masses can be measured to a high degree of accuracy by using an instrument called Aston's mass spectrometer. For example, the mass of atom of C-12 is found to be 1.9925 × 10^{-23} g. Hence the Avogadro number is

$$N = \frac{12.000}{1.9925 \times 10^{-23}} = 6.0225 \times 10^{23}$$

A very accurate value of N can also be obtained from the study of crystals. Crystals have a regular arrangement of atoms (see Chapter 10). The spacing between the atoms can be computed from the x-ray diffraction experiments. Once the arrangement of atoms in a crystal is known, it is possible to compute the number of atoms in a given volume of the crystal. From the knowledge of the density of the crystal, one can find the volume of a mole of the crystal and hence the Avogadro number.

The following example illustrates the significance of the value of Avogadro number.

EXAMPLE 9.1 A mole of aluminium has a mass of 27g and the relative density of aluminium is 2.7. Estimate the radius of an aluminium atom. Avogadro number $N = 6.02 \times 10^{23}$.

Solution Mass of a mole of aluminium (M_A) = 27 g = 2.7×10^{-2} kg. By definition, Avogadro number N is the number of atoms in a mole of any substance. Hence the mass of an atom of aluminium is

$$m_A = \frac{M_A}{N}$$

Now the density of aluminium (ρ_A) = 2.7×10^3 kg m^{-3}. Therefore, the volume of an atom of aluminium is

$$V_A = \frac{m_A}{\rho_A}$$

$$= \frac{M_A}{\rho_A N} = \frac{2.7 \times 10^{-2}}{2.7 \times 10^3 \times 6.02 \times 10^{23}}$$

$$= 1.66 \times 10^{-29} \text{ m}^3$$

Assuming that the atom is a sphere of radius r_A we have

$$V_A = \frac{4\pi}{3} r_A^3 = 1.66 \times 10^{-29}$$

which gives $r_A = 1.58 \times 10^{-10} \text{ m} = 1.58 \text{ Å}$

A more accurate analysis gives $r_A = 1.43$ Å. Thus we find that the atoms are very small in size; the atomic radius being of the order of 10^{-10} m or just 1Å.

9.5 AVOGADRO'S HYPOTHESIS

In the year 1811, the Italian physicist Amadeo Avogadro proposed a strikingly simple hypothesis, now known as the *Avogadro's hypothesis* which states that:

'*Equal volumes of all gases at the same temperature and pressure contain an equal number of molecules*'.

Avogadro proposed this hypothesis much before it was verified by experiments. The fact that this hypothesis is true is a remarkable expression of a simplicity in nature. In the next chapter (Chapter 10) we will see how Avogadro's hypothesis follows from the properties of an ideal gas. Most real gases, under ordinary pressure and temperature, obey the ideal gas laws.

Here is an interesting consequence of Avogadro's hypothesis. A mole of any substance has the same number of molecules (6.02×10^{23}). If we take a mole of different gases, then it follows from Avogadro's hypothesis, that they must all occupy the same volume at the same temperature and pressure. Experiments show that this indeed is true. At 0 °C and atmospheric pressure, all gases are found to occupy a volume of 22.4 litres or 2.24×10^{-2} m³.

This verifies Avogadro's hypothesis.

EXAMPLE 9.2 Estimate the fraction of molecular volume to the actual volume occupied by oxygen at STP. Take the diameter of an oxygen molecule to be roughly 3Å.

Solution Let us consider one mole of oxygen at STP. It contains $N = 6 \times 10^{23}$ molecules.

Diameter of one oxygen molecule $= 3$ Å $= 3 \times 10^{-10}$ m

∴ Volume of an oxygen molecule (assumed spherical) is

$$V_0 = \frac{4\pi}{3} r_0^3 = \frac{4\pi}{3} \times (1.5 \times 10^{-10})^3 = 1.4 \times 10^{-29} \text{ m}^3$$

Volume of N molecules $(V_N) = 6 \times 10^{23} \times 1.4 \times 10^{-29} = 8.4 \times 10^{-6}$ m³.

Now, 1 mole of a gas at STP occupies a volume of $V = 2.24 \times 10^{-2}$ m³. Therefore the fraction of molecular volume to actual volume is

$$\frac{V_N}{V} = \frac{8.4 \times 10^{-6}}{2.2 \times 10^{-2}} = 3.75 \times 10^{-4} \simeq 4 \times 10^{-4}$$

EXAMPLE 9.3 A drop of olive oil of 1 mm diameter is transferred lightly over water dusted with lycopodium powder. It spreads out into a circular thin film of diameter 14.1. cm. Estimate the size of a molecule of the oil assuming that the film is only one molecule thick.

Solution

$$\text{Volume } (V) \text{ of the oil drop} = \frac{4\pi}{3} r^3 \qquad \text{(i)}$$

where $r = 0.5$ mm $= 0.5 \times 10^{-3}$ m is the radius of the drop. The drop spreads out into a circular film of radius $R = 14.05$ cm $=$ 0.1405 m. The thickness of this film = diameter (d) of a molecule. Hence the volume of the circular oil film is

$$V = \pi R^2 d \qquad \text{(ii)}$$

Equating (i) and (ii) we get

$$d = \frac{4r^3}{3R^2} = \frac{4 \times (0.5 \times 10^{-3})^3}{3 \times (0.1405)^2}$$

$$= 8.4 \times 10^{-9} \text{ m}$$

This is a very rough estimate of the molecular size since the film is never one molecule thick.

9.6 INTERATOMIC AND INTERMOLECULAR FORCES

It is clear from Fig. 9.1 that a molecule may consist of atoms of the same kind or of different kinds. How are the atoms held together in a molecule? Why do some atoms cling together in pairs while others do not? Newton's law of universal gravitation does not give the right answers to these questions. This does not mean that gravitational attraction does not exist in molecules, but there are other forces which are much stronger than the gravitational force. These forces are electrical in nature, which can be attractive or repulsive. A proper understanding of these forces is related to the structure of the atom itself.

A Brief Look Inside the Atom

In the early part of the 20th century, physicists found that an atom consists of a tiny positively charged nucleus (size $\simeq 10^{-15}$ m) consisting of protons and neutrons with negatively charged electrons revolving around it in various elliptical orbits which extend over a distance of order 10^{-10} m (atomic size) (Fig. 9.6).

Interatomic Potential Energy and Force

The potential energy of an atom is due to interactions with the surrounding atoms. When two atoms are brought closer together, then at distances comparable to atomic size (10^{-10} m) the positively and negatively charged parts of the two atoms are at substantially different distances from each other. Consequently, there is a net electrostatic force between them which is responsible for the interatomic potential energy.

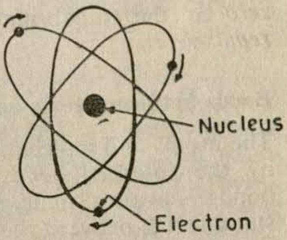


Fig. 9.6 Model of an atom

Detailed mathematical calculations and experiments show that the potential energy V of a pair of identical atoms varies with R (where R is the separation between their nuclei) as shown in Fig. 9.7(a). Along the part AB of the curve, the potential energy is positive and along the part BCD of the curve the potential energy is negative. The energy becomes more and more negative as R decreases from a large value up to a distance R_0 where the energy has a maximum negative value. As R is decreased further, the potential energy begins to increase.

The force between the atoms is given by $F = -\dfrac{dV}{dR}$; the *negative potential gradient*. Thus the force F at any given separation R can be found by drawing a tangent to the potential energy curve at that value of R and finding the slope. The negative of the slope gives the magnitude of the force F. When this is done for various values of R to the curve in Fig. 9.7(a), we get the magnitude of force at various values of R. Figure 9.7(b) shows how the magnitude of the interatomic force varies

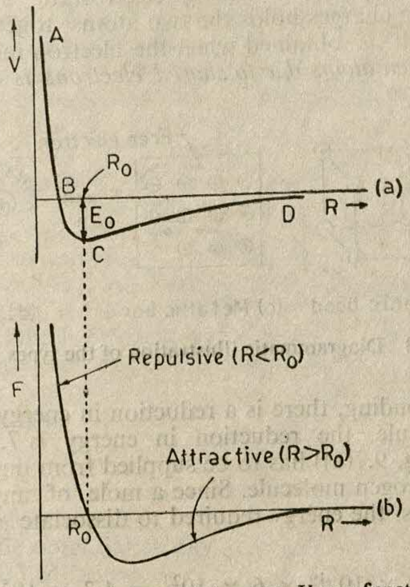


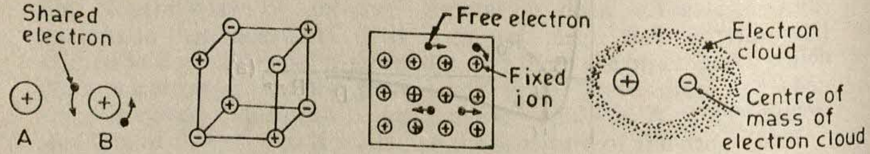
Fig. 9.7 (a) Interatomic potential energy V as a function of R, the distance between the nuclei of two identical atoms
(b) Variation of inter atomic force F as a function of R

with R. Notice that for $R > R_0$, the force is *attractive*. The force is *zero* at the equilibrium value $R = R_0$ and for $R < R_0$, the force is *repulsive*.

Bonds between Atoms and Molecules

The atoms and molecules in solids, liquids and gases are held together by the so-called *bonds* between them. There are different kinds of bonds—covalent, ionic, metallic, etc. These bonds are due to electrostatic forces between the positive charge of the nucleus and the negative charge of the electrons.

Covalent bond The value of $R = R_0$ when the potential energy V is a minimum corresponding to a *stable* or *equilibrium* spacing between two atoms. Detailed calculations show that for two hydrogen atoms the value of $R_0 = 0.74$ Å. At this spacing the repulsive and attractive forces balance so that $F = 0$ and no net force acts and the two hydrogen atoms will be in equilibrium with minimum potential energy. This stable or equilibrium arrangement is called the hydrogen molecule. The radius of a hydrogen atom is 1.2 Å. Since R_0 is smaller than the atomic radius, the two atoms in the hydrogen molecule cannot be treated as independent and the two electrons are *shared* between them. The electron in one atom, say atom A, wanders to the other atom B. Thus at any instant, the atom A from which the electron has departed is left with a positive charge and the atom B to which the electron moves has a negative charge. The resulting electrostatic attraction between the two opposite charges holds the two atoms together [Fig. 9.8 (a)]. The same result is obtained when the electron moves from B to A. *The bond between atoms due to shared electrons is called a covalent bond.*



a) Covalent bond (b) Ionic bond (c) Metallic bond (d) Vander Waals' bond

Fig. 9.8 Diagrammatic illustration of the types of bonds

In covalent bonding, there is a reduction in energy. In the case of a hydrogen molecule, the reduction in energy is 7.2×10^{-19} J. This energy (E_0 in Fig. 9.7(a)) has to be supplied from outside in order to dissociate a hydrogen molecule. Since a mole of any substance has 6×10^{23} molecules, the energy required to dissociate a mole of hydrogen is

$$7.2 \times 10^{-19} \times 6 \times 10^{23} = 4.3 \times 10^5 \text{ J}$$

which is quite large. The stability of a hydrogen molecule indicates that the covalent bond is a strong bond.

Ionic bond Sodium chloride, in the solid state, consists of positive sodium ions (Na^+) and negative chlorine ions (Cl^-). The ions are formed because a sodium atom readily gives up an electron to a chlorine atom (Fig. 9.8(b)). The ions in the crystal are held together by the electrostatic attraction between Na^+ and Cl^- ions. This is called an '*ionic*' bond. Notice that in an ionic bond, there is a transfer of electrons from one atom to another; but in a covalent bond the electrons are shared between the two atoms. Like the covalent bond, the ionic bond is a strong bond.

Metallic bond In solid metals, such as sodium or copper, one or more electrons in the outermost part of the atom may leave the atom and occupy the orbit of another. These 'free' electrons wander through the structure of the metal which consists of fixed positive ions (Fig. 9.8(c)). The electrons are not attached to any particular atom as in the case of a covalent bond. The metallic bond is not as strong as the covalent or ionic bonds.

It may be remarked that the ionic bond is '*non-saturated*'; the force of attraction between two ions is not influenced by the surrounding ions. But covalent bonds can be *saturated.* The metallic bond is an unsaturated covalent bond. This explains why the atoms are closely packed in metals.

Van der Waals' bond From Fig. 9.7(b) we notice that for $R > R_0$, the force between atoms or molecules is attractive. The potential energy varies as $V = \text{constant} \times R^{-6}$ and the attraction between neutral (or uncharged) atoms or molecules can be explained by assuming that the shape of the atom undergoes a fluctuating distortion due to its interaction with other atoms. At any instant, more electrons may appear on one side of the nucleus than on the other. Thus the centre of mass of the electron cloud (i.e. the negative charge) is slightly displaced form the positive charge of the nucleus (Fig. 9.8d). The two charges form what is called an *electric dipole*. A dipole will attract electrons from the neighbouring atoms, thus forming other dipoles. The forces between electric dipoles are weak; they are called van der Waals' forces in honour of the Dutch physicist C.F. van der Waals who had predicted similar attractive forces between molecules of gases.

Intermolecular Forces

The variation of potential energy (V) and force (F) between two molecules with separation (R) between their centres of mass has the same general form as shown in Figs. 9.7 (a) and (b), which shows the variation of interatomic potential energy and force. But there are important differences as detailed below.

1. Since molecules are not generally spherical in shape, the intermolecular potential energy and force depend not only on the separation R but also on their relative orientation.

2. The intermolecular forces are fifty or a hundred times weaker than interatomic forces. The minimum force between two molecules occurs at larger separations than between two atoms. In the stable configuration of an oxygen molecule, the oxygen atoms are 1.2 Å apart while the distance of minimum potential energy for two oxygen molecules is 2.9 Å. The energy required to break the bond between two oxygen atoms is 4.2×10^5 J per mole, whereas that required to separate two oxygen molecules is 8.4×10^3 J per mole, which is fifty times less.

The larger minimum energy distance between molecules implies that molecular bonds are *unsaturated*, i.e. a molecule can attract not just one but many molecules to itself. This is in contrast to a saturated covalent bond between atoms, for instance if two hydrogen atoms form a molecule then they cannot take in (or bind) a third atom. But a molecule can attract neighbouring molecules without any appreciable loss in its attractive force. Thus neighbouring molecules continue approaching each other as long as there is room to accommodate.

The forces between atoms at short distances are due to the sharing of electrons (covalent bonding), which leads to their rearrangement. Consequently, when one or more atoms are added to a molecule, two possibilities exist: (i) a new stable arrangement of electrons is obtained, i.e. a new molecule is formed or (ii) the new arrangement is energetically unfavourable. For example, when an atom of oxygen is added to a water molecule, a new molecule, hydrogen peroxide is formed ($O + H_2O \rightarrow H_2O_2$). But when a hydrogen molecule is added to a water molecule no new molecule is formed; they remain separate because forces between molecules are much weaker and cannot bring about a rearrangement of electrons to correspond to a new molecule.

EXAMPLE 9.4 The potential energy curve for two hydrogen atoms is the same as that shown in Fig. 9.7(a) with $R = R_0 = 0.74$ Å. Is the interatomic force repulsive or attractive at $R = 0.5$ Å, 0.74 Å and 1.9 Å? At which of these points is the magnitude of force the greatest?

Solution At $R = R_0 = 0.74$ Å, the force $F = 0$ since the potential energy is minimum at this point. At $R = 1.9$ Å, the force is *attaractive*, since the slope of the potential energy curve is *positive* at this point. At $R = 0.5$ Å, the slope of the potential energy curve is *negative*; hence the force is *repulsive*. Since the curve is not symmetric around $R = R_0$, it is clear from Fig. 9.7(a) that the magnitude of the slope at $R = 0.5$ Å is greater than that at $R = 1.9$ Å. Therefore, the repulsive force at $R = 0.5$ Å is stronger than the attractive force at $R = 1.9$ Å.

EXAMPLE 9.5 The binding energy of a hydrogen molecule is 4.75 eV. How much energy is needed to dissociate 0.05% of hydrogen gas occupying a volume of 5.6 litres at STP?

Solution The electron volt (eV) is a unit of energy and is equal to the kinetic energy acquired by an electron in falling through a potential difference of 1 volt. $1 \text{ eV} = 1.6 \times 10^{-19}$ J. Therefore

$$\text{Binding energy of a hydrogen molecule} = 4.75 \times 1.6 \times 10^{-19}$$
$$= 7.6 \times 10^{-19}\ \text{J}$$

This is the energy required to dissociate one molecule of hydrogen. Now, the volume of hydrogen to be dissociated is

$$v = \frac{0.05}{100} \times 5.6\ \text{litre}$$

We know that one mole of any gas at STP contains 6.02×10^{23} molecules and occupies a volume of 22.4 litres. Therefore, the number of molecules of hydrogen to be dissociated is

$$\frac{6.02 \times 10^{23}}{22.4} \times \frac{0.05 \times 5.6}{100} = 7.525 \times 10^{19}$$

$$\therefore \quad \text{Total energy needed} = 7.525 \times 10^{19} \times 7.6 \times 10^{-19}$$
$$= 57.2\ \text{J}$$

9.7 STATES OF MATTER

The forces which exist between molecules can not only explain the existence of the three states of matter (solid, liquid and gas) but can also explain many properties of matter, such as change of state, elasticity, thermal expansion, evaporation, diffusion, etc. These intermolecular forces arise from two main causes.

1. The *potential energy* of the molecules, which is due to interactions with surrounding molecules and is electrical in nature, as described in Section 9.6.
2. The *thermal energy* of the molecules which is also their *kinetic energy*, and which depends on temperature (see Chapter 10).

The particular state in which matter appears, i.e. solid, liquid or gas, and the properties of matter in a given state are determined by the relative magnitudes of these two energies.

Distinction between the Three States of Matter

The difference between solids, liquids and gases can be attributed to the strength of the forces between molecules.

The Solid State In a solid, the intermolecular distances are small and hence the forces between the molecules are too strong for the molecules to undergo translation. They can only vibrate about their mean positions. Because of these strong forces, a solid maintains a fixed shape and size.

The Liquid State In a liquid, the forces between the molecules are not sufficiently strong to hold them in fixed positions. Consequently they move freely. It is this freedom of movement which enables a liquid to take up the shape of any vessel in which it is placed. However the forces are sufficiently strong to prevent the molecules from going too far apart, and leave the body of the liquid. Therefore, liquids have a definite size (volume) but no definite shape.

The Gaseous State In a gas, the molecules move rapidly because the forces between them are extremely weak. They do not even stay close together. They are much farther apart in a gas than in a liquid or a solid. They move everywhere, filling any container they are placed in. Therefore, gases have no definite size or shape.

Transition from One State to Another

Suppose we have oxygen gas at 0 °C and at normal pressure. The average spacing between molecules is about ten times their diameter. Therefore, the intermolecular forces are very weak. The molecules are constantly moving about because they have kinetic energy which is proportional to absolute temperature (see Chapter 10). Because the intermolecular attractive forces are very weak, the molecules wander about as essentially *free particles* colliding occasionally with each other (Fig. 9.9 a).

Now suppose the gas is cooled (so that the thermal motion of the molecule slows down) and the pressure is increased (so that the intermolecular spacing is decreased). The intermolecular potential energy, and hence the attractive intermolecular force, increase. A stage is reached when the average kinetic energy of a molecule becomes less than the average attractive potential energy. At this stage, the gas condenses into a liquid. The molecules no longer wander about as free molecules (as in a gas) because their freedom of movement is now restricted by the increased attractive intermolecular potential energy or force.

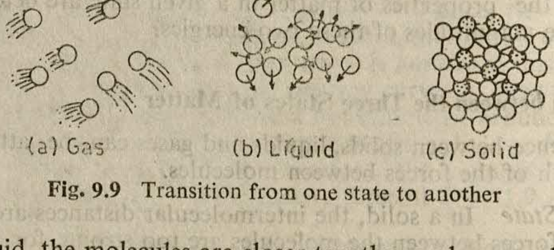

(a) Gas (b) Liquid (c) Solid

Fig. 9.9 Transition from one state to another

In a liquid, the molecules are closer together and move about much more slowly than in a gas (Fig. 9.9 (b)). Suppose the liquid is cooled further. The thermal motions become negligibly slow and intermolecular spacing decreases. A stage is reached when the average spacing between molecules becomes comparable to that for minimum potential energy, i.e. $R \simeq R_0$. At this stage, the molecules begin to reorient themselves and tend to attain the stable equilibrium configuration corresponding to minimum potential energy. Molecules for which the spacing R is slightly greater than the equilibrium value R_0, will experience an attractive force which will restore them to their equilibrium position. Molecules, for which R is slightly less than R_0, will experience a repulsive force which will restore them to their equilibrium position. At this stage the liquid condenses to the solid state. So the molecules of a solid can only vibrate about their equilibrium positions (Fig. 9.9 (c)). The molecules are still in motion; but the amplitude of motion is very small.

Theoretical and experimental investigations carried out during the last 30 years have shown that the picture of the solid state presented above is not entirely correct. Computer simulation experiments involving the study of the motion and collisions of about a hundred tiny rigid spheres have shown that the attractive part of the interatomic and intermolecular potential energy curve is not needed for the formation of a solid; only the repulsive part (for $R < R_0$) is necessary. Following the motion of a group of such spheres on a computer, one finds that the centres of the spheres trace a zig-zag path (like the path of a molecule in a fluid) until a certain density is reached. At densities higher than this density (which describes a fluid), the spheres tend to arrange themselves in a regular pattern and the centre of each sphere oscillates about a fixed equilibrium position. These mean positions are arranged regularly in space. This arrangement describes a crystalline solid.

Thus the computer experiments show that the formation and existence of the liquid state requires attraction between atoms or molecules; but the solid state requires a strong repulsion. It is difficult to believe that it is the strong repulsion between atoms which is responsible for the crystalline structure of solids.

EXAMPLE 9.6 Estimate the average intermolecular binding energy per molecule of water. The molecular mass of water is 18 and its latent heat of vaporisation is $2.26 \times 10^6\ \text{J kg}^{-1}$.

Solution Let ϵ be the binding energy per molecule of water. It is the energy needed to separate a molecule of water from other molecules. Now in a mole of water there are N molecules where N is the Avogadro number. So the energy required to break the bonds of all the molecules at the boiling point is $N\epsilon$. This is the latent heat of vaporisation of water *per mole* (L). Thus

$$L = N\epsilon$$

or

$$\epsilon = \frac{L}{N} \qquad \text{(i)}$$

The latent heat of vaporisation of water is $2.26 \times 10^6\ \text{J kg}^{-1}$ or $2.26 \times 10^3\ \text{J g}^{-1}$. Now a mole of water has a mass of 18 g. Therefore, L the latent heat of vaporisation per mole is

$$L = 2.26 \times 10^3 \times 18 = 4.068 \times 10^4\ \text{J mol}^{-1}$$

Also Avogadro number $N = 6.02 \times 10^{23}$. Substituting for L and N in Eq. (i) we get

$$\epsilon = 0.676 \times 10^{-19}\ \text{J}$$

$$= 0.42\ \text{e V} \qquad (\because\ 1\ \text{eV} = 1.6 \times 10^{-19}\ \text{J})$$

SUMMARY

The phenomena of diffusion, evaporation and Brownian motion provide an evidence of the fact that matter consists of atoms or molecules and that the molecules are in ceaseless random motion.

The number of molecules in a mole of any substance is called the Avogadro number and its value is 6.0225×10^{23}. A mole of a substance is the amount of substance which contains as many elementary units as there are in 12.000 g of C-12.

Avogadro's hypothesis states that equal volumes of all gases at the same temperature and pressure contain equal number of molecules. All gases at STP occupy a volume of 22.4 litres or $2.24 \times 10^{-2}\ \text{m}^3$.

The atoms and molecules in solids, liquids and gases are held together by various types of bonds, such as the covalent bond, ionic bond, metallic bond and van der Waals bond.

There are three states of matter, namely, solid, liquid and gas. The particular state in which matter appears under given conditions of temperature and pressure and the properties it has in that state are determined by the relative magnitudes of the potential energy of the molecules (which is due to molecular interactions) and the thermal (kinetic) energy of the molecules (which depends upon temperature).

EXERCISES

1. Distinguish between elements and compounds. Give four examples of each.
2. State the law of definite proportions. Give two examples of the law. How is the law explained by the atomic hypothesis? Explain with the help of an example.
3. (a) What is Brownian motion? How is it experimentally observed?
   (b) How is Brownian motion accounted for?
   (c) Explain qualitatively how the extent of Brownian motion is affected by the size of the Brownian particle, density, temperature and viscosity of the medium.
4. (a) Define a mole of a substance.
   (b) What is the Avogadro number? Why does it have the same value for all substances?
   (c) How is the Avogadro number determined? What is its significance?

5. State and explain Avogadro's hypothesis.
6. Graphically show the variation of the interatomic potential energy as a function of the separation between the nuclei of two identical atoms. How is the graph used to find the interatomic force at any separation? Which points on the graph correspond to (a) repulsive force, (b) zero force and (c) attractive force?
7. Describe and explain the different types of bonds between atoms in a substance.
8. In what respects are intermolecular potential energy and force different from interatomic potential energy and force? Illustrate your answer with examples. In which respects are they similar?
9. How will you distinguish between the three states of matter on the basis of the strength of intermolecular forces?
10. Explain the transition from gaseous to liquid and solid states on the basis of the molecular structure of matter.
11. 'The formation and the existence of the liquid state requires attraction between atoms or molecules but the solid state requires a strong repulsion'. Comment.

PROBLEMS

Take Avogadro number = 6.02×10^{23} wherever necessary.

1. A mole of mercury has a mass of 200.6 g. Estimate the size of a mercury atom. The relative density of mercury is 13.6.
2. Estimate the fraction of the volume of all the molecules to the actual volume occupied by a gas at STP. Take the diameter of a molecule to be roughly 3 Å.
3. Estimate the fraction of molecular volume to the volume occupied by water vapour at 100°C and 1 atm pressure. The density of water vapour under these conditions of temperature and pressure is 0.6 kg m^{-3}.
4. Oxygen at STP occupies a volume which is 2500 times the volume occupied by its molecules. Estimate the diameter of an oxygen molecule.
5. Figure 9.7 (a) in the text shows the potential energy curve for two oxygen atoms. The value of $R_0 = 1\ 2$ Å. Is the interatomic force repulsive or attractive at (i) $R = 0.5$ Å, (ii) $R = 1.5$ Å and (iii) $R = \propto$? At which of these points is the magnitude of force the greatest. (iv) What is the magnitude of the force at $R = 1.2$ Å?
6. The binding energy of an oxygen molecule is 4.4 eV. How much energy is needed to dissociate the atoms of 0.01% of oxygen gas occupying a volume of 11.2 litres as STP?
7. 57.2 J of energy is needed to dissociate the atoms of 0.05% of hydrogen gas occupying a volume of 5.6 litres at STP. Find the minimum potential energy of a hydrogen molecule.
8. The intermolecular binding energy per molecule of water is 0.42 eV. Estimate the latent heat of vaporisation per mole of water.

9. Estimate the average intermolecular binding energy per molecule of mercury. The atomic mass of mercury is 200.6 and its latent heat of vaporisation is 2.72×10^5 J kg^{-1}.
10. The intermolecular binding energy per molecule of mercury is 57 eV and the latent heat of vaporisation is 2.72×10^5 J kg^{-1}. Estimate the mass of a mole of mercury.

10

Properties of Matter

10.1 INTRODUCTION

All elements and compounds generally exist in one of the three states of aggregation, namely, solid, liquid or gas. In Chapter 9 we have learnt the basic differences between the three states of matter in terms of inter-molecular forces. In this chapter we will discuss a few characteristic properties of solids, liquids and gases.

This chapter is divided into four sections. Section A deals with solids. We will describe the most characteristic property of solids, namely, elasticity or elastic deformation under external forces. The electrical and magnetic properties of solids are outside the scope of the present study. In Section B, devoted to fluids, we will discuss properties which are common to both liquids and gases, such as transmission of pressure, buoyancy, viscosity and fluid flow. In Section C we will discuss a property called surface tension characteristic only of liquids. In Section D, we will discuss the relationship between pressure, volume and temperature of a gas. We will conclude this section by describing the postulates and implications of the kinetic theory of gases.

SECTION A—SOLIDS

10.2 INTRODUCTION

We know that each molecule or atom in a solid is at a fixed average location with respect to other molecules which make up the solid. The average locations of the molecules do not change with time. Since the molecules are almost completely lacking in mobility, their kinetic energy is very small; almost negligible. It is this lack of mobility which endows a solid with a definite shape and appreciable stiffness or rigidity. This rigidity is the cause of elastic behaviour (called elasticity) in solids—which we will discuss in Section 10.4. Let us first look at two basic structures in solids.

10.3 CRYSTALLINE AND AMORPHOUS SOLIDS

Solids may be classified into two main groups – *crystalline* and *amorphous* (or glassy) solids. In a solid, each atom or molecule is confined to a definite small space between neighbouring atoms or molecules. How an atom is arranged in relation to its neighbours distinguishes a crystalline solid from an amorphous one.

Crystalline Solids

A collection of atoms in a symmetrical arrangement is called a *crystal.* Within each crystal, the atoms or molecules are arranged in regular geometrical patterns. The crystalline structure can be noticed if you examine the grains of solids, such as common salt, sugar or copper sulphate with a good microscope. The main feature of crystalline solids is their *long-range order*. The crystal structures of crystalline solids are different in both the spacing and the arrangement of atoms in them. The simplest example is that of a crystal of copper which can be described as follows: the basic structural unit in the case of a copper crystal is the cube. We consider cubes of equal size and construct the regular geometrical pattern by stacking together a large number of cubes. They are stacked in three dimensions, i.e. next to each other and on top of each other. Then we imagine copper atoms placed at the corners of the cubes and at the centres of the square faces, as shown in Fig. 10.1 (a). Each atom at a corner of a cube is, therefore, shared by eight cubes and the atom at the centre of a face (called the face-centre atom) is shared by two cubes. In a copper crystal, the side of each cube is 3.61 Å (1 A = 10^{-10} m). This arrangement is called *face centred-cubic* arrangement.

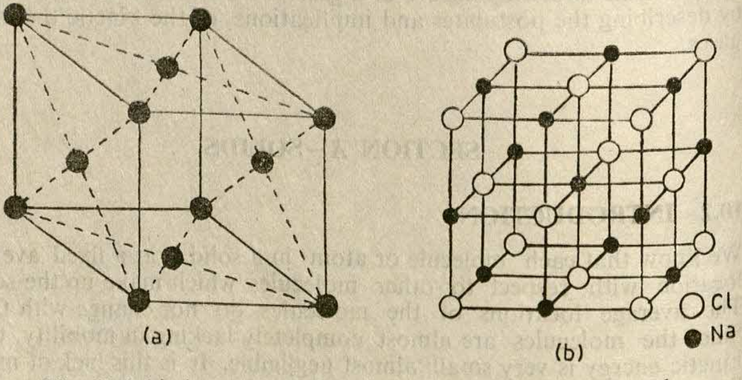


Fig. 10.1 (a) Arrangement of copper atoms in a copper crystal; atoms at front and back face centres are not shown.
(b) Arrangement of sodium and chlorine ions in a sodium chloride crystal

Figure 10.1 (b) shows the arrangement of sodium and chlorine ions in a sodium chloride crystal which is ionic in nature. The small spheres represent Na^+ ions and the large spheres Cl^- ions. They are held together by the electrostatic forces of attraction between ions of opposite charge. The crystal consists of two face-centred cubic structures, one occupied by Na^+ ions and the other by Cl^- ions. Each Na^+ ion is surrounded by six Cl^- ions and vice versa. The nearest neighbour separation is 2.82 Å and the side of each cube is 5.63 Å.

The arrangement of atoms in a crystalline solid can be complicated but a basic feature is preserved, i.e. all crystals can be obtained by a periodic (or regular) repetition of a basic pattern in three dimensions. How do we know that atoms in a crystal are indeed arranged in this pattern? The fact that naturally occurring and artificially prepared crystals have flat faces suggests this kind of face-centred structure. But with modern techniques called *x*-ray diffraction and scanning tunneling microscopy, it is possible to determine the various geometric parameters, like spacing between atoms, bond angles, etc. in different crystals.

Amorphous or Glassy Solids

The word 'amorphous' means 'without form'. Many solids are not crystalline; they are amorphous such as glass, wheat flour, rubber, cellulose, fused mica etc. In amorphous solids the molecules have no definite orderly arrangement; they exhibit only a *short-range order*. In such solids, the force of attraction between molecules varies from molecule to molecule. When such a solid is heated the weakest bonds are the first to snap followed by stronger ones at higher temperatures. As a consequence of this, the melting process in amorphous solids is rather gradual. Unlike crystalline solids, amorphous solids do not have a definite melting point. The difference in the behaviour between crystalline and amorphous solids is beautifully summed up by Azaroff in the following words:

"In a crystal, the atoms are arranged like soldiers on a parade ground in well-defined columns and rows. When the command 'Dismissed' is shouted, the soldiers dash off in every direction and the orderly arrangement disappears instantaneously, it may be said that the ranks have melted. In glass, the atoms never have a chance to organise themselves in an orderly arrangement in the first place."

When a liquid is cooled far below its freezing point without crystallizing, it solidifies to form a glass. Hence, glasses are sometimes called supercooled liquids. These are characterized by fantastically high values of viscosity. Liquids which have a simple structure crystallize too quickly to form glasses at ordinary rates of cooling. Among the substances which form glasses easily are complex molecules, many oxides (like SiO_2, B_2O_3 etc.), the elements Se, Te, etc.

In addition to crystalline and amorphous solids, there is another class of solids called *polymers*. These consist of long-chain of organic molecules.

10.4 ELASTICITY

Newton's first law of motion is concerned with the most fundamental property of matter, namely, inertia. Inertia is the tendency of a body to preserve its state of rest or of uniform motion. Newton's second law tells us exactly what effect an applied force has on the state of motion of a body. Newton's third law is related to another fundamental property of matter, namely, elasticity, which is responsible for the reaction force. When we discuss this property, we are concerned with the effect the applied force has on the shape or size of a body rather than on its state of motion.

It is common knowledge that all material bodies get deformed, i.e. undergo a change in shape, when subjected to a suitable force. Take a rubber ball and press it; it gets deformed. When the deforming force is removed, it regains its shape. Even a steel wire, when loaded, is deformed. It gets elongated. When the load is removed, it regains its original length. *The ability of a body to regain its original shape and size, when the deforming force is withdrawn, is known as elasticity.* A body which completely regains its original shape and size after the removal of the deforming force is said to be *perfectly elastic*. A body which retains its deformed shape and shows no tendency at all to regain its original shape, after the removal of the external force, is said to be *perfectly plastic*. In fact, no material body can be perfectly elastic or perfectly plastic. A body is more elastic or plastic in relation to another body. Examples of nearly perfectly plastic bodies are mud, modelling clay (plasticine) and flour dough (a paste of flour and water). After the removal of the force, they do not recover their original size or shape.

Load-Extension Curve

Let us consider what happens when a deforming force is applied to a metal such as copper. If the metal is in the form of a wire, it can be stretched by applying a force. The force may be applied by fixing one end of the wire rigidly to a support and loading the other end. If the metal is in the form of a block or a rod, it can be compressed by applying a force. The force may be applied by holding one end of the rod against a rigid wall and hammering the other end. The portion OABC of the load-extension curve (Fig. 10.2) shows how the extension changes as the load is gradually increased. The portion OA′ B′ of the curve shows how the compression changes as the applied force is gradually increased.

We will now discuss the mechanical properties of a solid subjected to a deforming force.

Elasticity When the deforming force is small, the solid completely regains its original size after the deforming force is removed. The solid is then said to be elastic. The straight-line portion AA′ of load-extension curve corresponds to the elastic beleviour of the solid. Between A and A′ the deformation is proportional to the applied force.

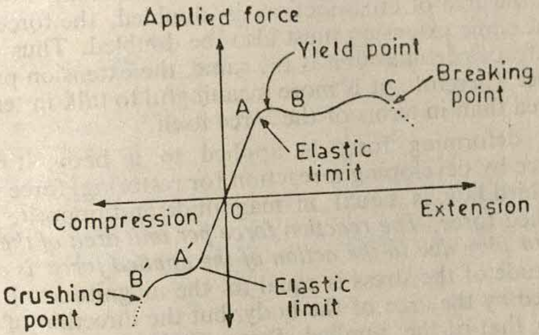


Fig. 10.2 Load-extension curve for a metal

Plasticity When the deforming force is increased a limit is reached beyond which, if the applied force is removed, the solid does not come back to its original shape or size. It remains deformed. This limit is called the *elastic limit.* The points A and A′ correspond to this limit. Beyond the elastic limit the solid behaves like a plastic body. Thus the same body behaves like an elastic body (for small applied forces) and like a plastic body (for large applied forces). Beyond the elastic limit (point A) the deformation of the body may continue under the influence of its own weight. The body is then said to flow. For example, a paste of flour and water flows under its own weight.

Ductility The point B beyond which a solid flows is called the *yield point*. Metals beyond the yield point are said to be ductile. Under such circumstances, rods of metals can be drawn into wires.

Brittleness Different bodies behave differently when subjected to a deforming force. Make a ball of clay and let it dry. Press the ball. It breaks into pieces. Try to bend a piece of glass; it breaks. It is said to be brittle. Thus, if a body beyond the elastic limit flows, it is said to be plastic; but if it breaks into piecs, it is called brittle.

Malleability When a solid is compressed, a stage is reached beyond which it cannot recover its original shape after the deforming force is removed. This is the elastic limit (point A′) for compression. The solid behaves like a plastic hody. The yield point (B′) obtained under compression is called the *crushing point*. After this stage, metals are said to be *malleable*. In other words, they can be hammered or rolled into sheets.

Stress

Take two steel wires of equal lengths but different diameters. It is clearly much easier to extend the thinner wire than the thicker one. In other words, the extension produced by a force depends on the area of the cross-section of the wire. It can be shown that, within the elastic limit, if the area of cross-section is doubled, the force required to produce the same extension must also be doubled. Thus we conclude that if the force per unit area is the same, the extension produced will be the same. Therefore, it is more meaningful to talk in terms of force per unit area than in terms of the force itself.

When a deforming force is applied to a body, it reacts to the applied force by developing a reaction (or restoring) force which, from Newton's third law, is equal in magnitude and opposite in direction to the applied force, *The reaction force per unit area of the body which is called into play due to the action of the applied force is called stress.* The magnitude of the stress is equal to the magnitude of the applied force divided by the area of the body, but the direction of the stress is opposite to that of the applied force. Stress is measured in units of force per unit area, i.e. Nm^{-2}.

Strain

When a deforming force is applied to a body, it may suffer a change in length, volume or shape. For simplicity, first we shall consider changes only in length, i.e. longitudinal deformations. If we take two steel wires of the same thickness, one of length 50 cm and the other of length 100 cm, and apply the same load, we shall observe that the extension produced in the 100 cm wire is twice that produced in the 50 cm wire as long as the elastic limit is not exceeded. Thus we conclude that, within the elastic limit, the ratio of the extension in length to the original length, for the same load, remains constant. This ratio is called *longitudinal strain* which is defined as the *ratio of change in length to the original length*. Strain resulting from a change in volume is called *volume strain* and that due to a change of shape is called *shearing strain*. Strain, being a ratio of two similar quantities, has no unit. It is just a number.

Hooke's Law

The load-extensive curve shown in Fig. 10.2 also represents the stress versus strain curve. Robert Hooke of England obtained the stress-strain curves experimentally for a number of solid materials. He studied these curves and observed that, as long as the elastic limit was not exceeded, the curves were straight lines. This portion AA′ of the curve (Fig. 10.2) is called the linear region. From this observation he came to the conclusion that, *within the elastic limit, the stress developed in a solid is proportional to the strain produced in it.* This statement is known as *Hooke's law*. Refer to Supplement S1 at the end of this chapter for a derivation of Hooke's law.

Young's Modulus

Suppose that a rod of length L and a uniform cross-sectional area A is subjected to a longitudinal pull. In other words, two equal and opposite forces are applied at its ends as shown in Fig. 10.3 (a).

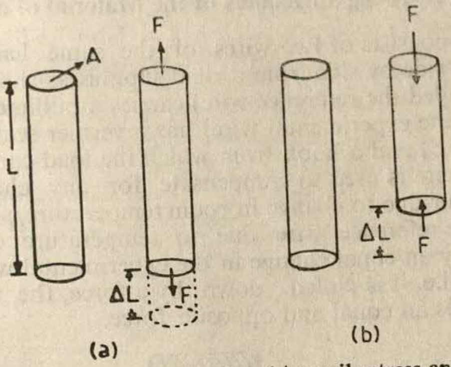


Fig. 10.3 A rod subjected to (a) tensile stress and (b) compressional stress

As a result of applying the deforming forces, there is an extension in length which we denote ΔL. The strain is given by

$$\text{Strain} = \frac{\Delta L}{L}$$

Since the deformation involves a change in length, the strain is called *longitudinal strain* or *linear strain.* Since the length increases ($\Delta L > 0$) it is also called *extensional strain* or *tensile strain.* The stress is given by

$$\text{Stress} = \frac{F}{A}$$

The stress in the present case is called linear stress, tensile stress, or extensional stress. If the direction of the forces is reversed (Fig. 10.3b) so that ΔL is negative, we speak of *compressional strain* and *compressional stress*. If the elastic limit is not exceeded, then from Hooke's law

$$\text{Stress} \propto \text{strain}$$

or

$$\text{Stress} = Y \times \text{sirain}$$

or

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F}{A} \cdot \frac{L}{\Delta L} \tag{10.1}$$

where Y, the constant of proportionality, is called the *Young's modulus* of the material of the rod and may be defined as *the ratio of the linear stress to linear strain, provided the elastic limit is not exceeded.* Since strain has no units, the unit of Y is the same as that of stress, namely, Nm^{-2}.

Determination of Young's Modulus of the Material of a Wire

The apparatus consists of two wires of the same length and equal thickness hung side by side from a rigid support as shown in Fig. 10.4. The wire A (called the reference wire) carries a millimetre scale S and wire B (called the experimental wire) has a vernier scale V (which can slide over scale S) and a hook from which the load can be suspended. The reference wire is used to compensate for any change in length which may occur due to change in room temperature, since any change in length of the reference wire due to temperature change will be accompanied by an equal change in the experimental wire. When the wire is loaded, i.e. it is pulled down by a force, the reaction of the support provides an equal and opposite force.

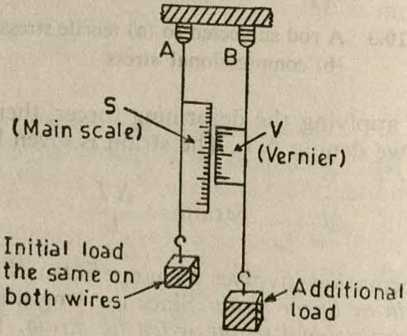


Fig. 10.4

The experimental wire is initially loaded with some weight so as to make it straight and the reading of the vernier is recorded. The load is increased by, say, 250 g and the vernier reading is recorded again. The difference between two vernier readings gives the extension produced by the load.

If L is the initial length of the wire and M is the load required to produce an extension L then

$$\text{Stress} = \frac{F}{A} = \frac{Mg}{\pi r^2}$$

where

r = radius of the wire,

and

$$\text{Strain} = \frac{\Delta L}{L}$$

so that

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{Mg}{\pi r^2} \cdot \frac{L}{\Delta L}$$

Knowing M, L, ΔL and r, the value of Y is determined. Table 10.1 gives the measured value of Y for some common substances.

Table 10.1 Young's modulus of some common substances

Substance	*Young's Modulus* (Nm^{-2})
Steel	20×10^{10}
Copper	10×10^{10}
Aluminium	7.0×10^{10}
Lead	1.5×10^{10}
Brass	9×10^{10}
Glass	5.8×10^{10}

The value of the Young's modulus is a measure of the elasticity of the substance. Notice from Table 10.1 that steel is more elastic than copper and aluminium is more elastic than lead. This means that steel can be subjected to much higher stresses than lead. It is for this reason that heavy-duty machines and bridges are made of steel.

Stress-Versus Strain in Different Solids

The linear relation, stress proportional to strain, is found to hold for small values of strain, i.e. the graph of stress versus strain is linear for small values of strain, especially for metals. Typically the strain must not exceed 0.1% for metals for Hooke's law to hold. The range of stress and strain values for which Hooke's law holds is called the *elastic region*,

If the load in the experiment illustrated in Fig. 10.4 is increased beyond this range, the graph of stress versus strain is no longer linear. Figure 10.5 shows a typical graph for a metal (aluminium).

Hooke's law holds for the linear portion OA of the graph. The point A corresponds to the *elastic limit* for aluminium. Within this limit, if the stress is removed, aluminium regains its original state. But at a point beyond A (such as point B), if the stress is reduced to zero, the metal does not recover its unstrained state; since some strain remains. This is known as *elastic hysterisis*. As the strain is increased (by increasing stress) to a value indicated by point C, the metal *breaks* or *flows*, as it were. The stress at this point is called the *tensile strength* of the metal.

Bulk Modulus (or Volume Elasticity)

A solid may also be deformed by subjecting it to a uniform normal pressure P in all directions as shown in Fig. 10.6.

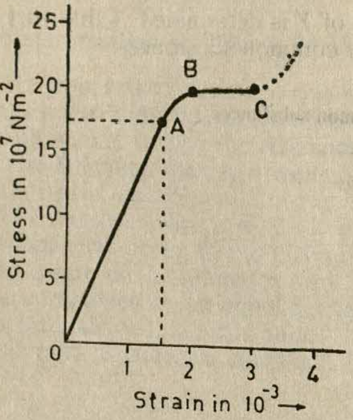


Fig. 10.5 Stsess versus strain graph for aluminium

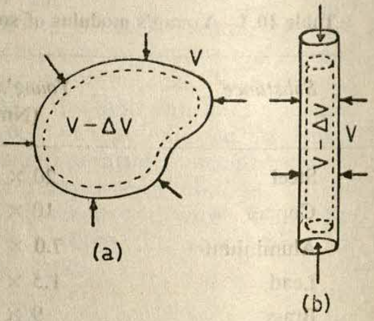


Fig. 10.6 (a) Pressure on a general surface, (b) Pressure on a rod or wire

Stress and pressure have the same dimension (force per unit area), but pressure is not the same thing as stress. Pressure is the force per unit area acting on the surface of a system, *the force being everywhere perpendicular to the surface* so that, for a uniform pressure, the force per unit area is the same. Stress is also force per unit area, but the force may not be the same in magnitude on different surfaces and its direction may also not be perpendicular to the surface. In the case of the rod in Fig. 10.3 or wire in Fig. 10.4, the force is perpendicular to the circular cross-section but there is no force on the cylindrical face. In this case, the rod or the wire is not said to be subjected to a pressure.

Pressure is a particular kind of stress which changes only the volume of the substance and not its shape as shown in Fig. 10.6. The substance may be a solid, liquid or gas. A small increase in pressure ΔP applied to a substance decreases its volume from, say, V to $V - \Delta V$ so that ΔV is the small decrease in volume. The volume strain is given by

$$\text{Volume strain} = -\frac{\Delta V}{V}$$

The bulk modulus is defined as the ratio of the excess pressure and the corresponding volume strain, i.e.

$$B = \frac{\Delta P}{\left(-\frac{\Delta V}{V}\right)} = -\frac{\Delta P V}{\Delta V} \tag{10.2}$$

If ΔP is positive, ΔV will be negative and vice versa. The negative sign in our definition of bulk modulus B (Eq. 10.2) ensures that B is always positive. The SI unit of B is Nm^{-2}. The reciprocal of B is known as *campressibility*. Table 10.2 gives the measured values of B for some substances.

Table 10.2 Bulk moduli of some substances

Substance	*Bulk modulus* (Nm^{-2})
Solids	
Aluminium	7.0×10^{10}
Copper	1.2×10^{11}
Glass	3.6×10^{10}
Iron	8.0×10^{10}
Steel	1.6×10^{11}
Liquids	
Water	2.2×10^{9}
Mercury	2.5×10^{10}
Gases	
Air (or any gas at atmospheric pressure)	1.4×10^{5}

A study of this table shows that the bulk modulus of solids is about 50 times that of water. This means that solids are very difficult of compress; their compressibility ($1/B$) is very small. On the other hand, the bulk modulus of gases is about one million times less than for solids. This means that gases are easily compressed.

Poisson's Ratio

Let us go back to the experiment shown in Fig. 10.4 where the wire is under tension but not under pressure (in the true sense of the term pressure) since no force acts perpendicular to the cylindrical surface of the wire. Careful experiments show that, apart from an increase in length, there is a slight decrease in the diameter of the wire, i.e. both shape and volume change under the longitudinal stress. The ratio of the decreases ΔD in diameter to the original diameter D is called *lateral strain*, i.e. strain at right angles to the deforming force. Thus

$$\text{Lateral strain} = \frac{\text{change in diameter}}{\text{original diameter}} = \frac{\Delta D}{D}$$

Also $$\text{Longitudinal strain} = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta L}{L}$$

It is found experimentally that within the elastic limit, the lateral strain is proportional to the longitudinal strain. The ratio of the two is called Poisson's ratio and is denoted by σ. Hence,

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\Delta D/D}{\Delta L/L}$$

$$= \frac{\Delta D}{D} \cdot \frac{L}{\Delta L}$$

Since it is a ratio between two types of strain, σ is dimensionless. Theoretically, one can show that it must be less than 0.5. For most solids it lies between $\frac{1}{4}$ and $\frac{1}{3}$, and for rubber it is very nearly 0.5.

Shear Modulus (or Modulus of Rigidity)

Shear is a particular kind of stress which only solids can withstand. The solid is deformed by changing its shape without changing its size. The body does not move or rotate as a whole: there is a relative displacement of its contiguous layers. One can visualize this by imagining a pack of cards pressed with the hand and pushed horizontally.

Consider a solid in the form of a rectangular cube as in Fig. 10.7. Suppose the lower face PQRS is held fixed and a force F is applied parallel to the upper face MNUV. If we do this to a liquid, it will begin to flow; it will not remain in equilibrium. In other words, liquids cannot withstand this kind of shear force. But when it is applied to a solid, it will remain in equilibrium because no net force and no net torque act on it. There is a couple produced by this force and an equal and opposite force coming into play on the lower face. As a result of this, the lines joining the two faces turn through an angle θ. We say that the face MNRS is sheared through an angle θ (measured in radians). The angle θ is called the *shear strain* or the *angle of shear* and is a measure of the degree of deformation.

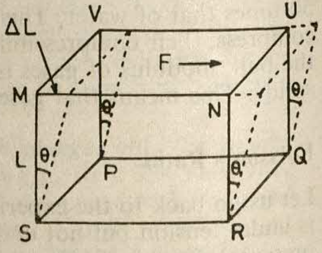


Fig. 10.7 A solid subjected to a shear stress

If A is the area of the face MNUV, the ratio F/A is the shearing stress. It is found that for small deformation, the shearing stress is proportional to the shear strain, i.e.

$$\frac{F}{A} \propto \theta$$

or
$$\frac{F}{A} = G\,\theta$$

or
$$G = \frac{F}{A\,\theta} \tag{10.3}$$

The quantity G is called the *shear modulus* or the modulus of rigidity.

Referring to Fig. 10.7, if θ is small
$$\theta \simeq \tan\theta = \frac{\Delta L}{L}$$

so that
$$G = \left(\frac{F}{A}\right) \cdot \frac{L}{\Delta L} \tag{10.4}$$

This equation looks similar to Eq. (10.1) for Young's modulus with the difference that F/A here is the *tangential stress* and not longitudinal stress.

A common method of finding the shear modulus is to fix a wire or rod at one end and apply a known force at the other and measure the twist produced in the wire (see Fig. 10.8).

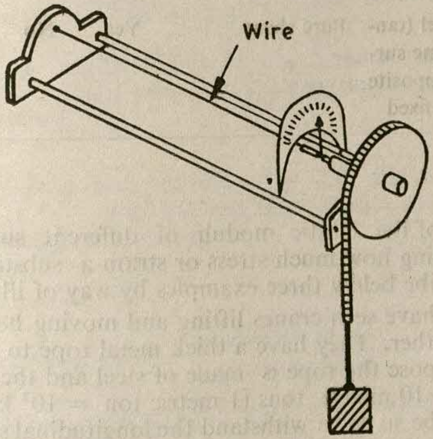


Fig. 10.8 Schematic diagram of an apparatus for measuring the shear modulus

Table 10.3 Shear moduli of some common solids

Solid	*Shear modulus* (Nm^{-2})
Aluminium	2.5×10^{10}
Copper	4.0×10^{10}
Glass	3.0×10^{10}
Iron	5.0×10^{10}
Steel	8.0×10^{10}
Wood	1.0×10^{10}

The stress and strain in the three elastic moduli can be summarised as follows.

Stress	*Strain*	*Change in*		*Name of modulus*	*State*
		Shape	*Volume*		
1. Forces perpendicular to two parallel surfaces, no force on other surfaces	Extension or compression parallel to force and compression or expansion perpendicular to force (longitudinal and lateral strain)	Yes	Yes	Young's	solid
2. Forces perpendicular everywhere to the surface, pressure (= force ÷ area) constant everywhere	Compression or expansion (volume strain)	No	Yes	Bulk	Solid, liquid, gas
3. Forces parallel (tangential) to one surface, the opposite surface being fixed	Pure shear	Yes	No	Shear	Solid

Applications

The knowledge of the elastic moduli of different substances is very useful in estimating how much stress or strain a substance can withstand. We describe below three examples by way of illustration.

1. You must have seen cranes lifting and moving heavy loads from one place to another. They have a thick metal rope to which the load is attached. Suppose the rope is made of steel and the crane is to lift a load of, say, 10 metric tons (1 metric ton = 10^3 kg). How thick should the rope be so as to withstand the longitudinal strain produced by the load? The elastic limit for steel is about 3×10^8 Nm^{-2} and the load to be lifted is $M = 10 \times 10^3$ kg. Therefore the area of cross-section of the rope should at least be

$$A = \frac{Mg}{\text{Elastic limit}} = \frac{10 \times 10^3 \times 9.8}{3.0 \times 10^8} \simeq 3.3 \times 10^{-4}\ \text{m}^2$$

which means that the diameter of the rope should at least be equal to 2 cm.

2. What should be the dimensions and the kind of material to be used in a bridge so that it can withstand the load of the traffic, its own load and the force of winds, so that it does not bend or brake? For a given load, the bending of a bridge depends on its length, its cross-sectional area and the Young's modulus of the material of which it is made.

Having fixed the length and cross-sectional area of the bridge, the bending under a given load is inversely proportional to the Young's modulus of the material. The higher the value of Y, the smaller is the bending of the bridge. Hence steel (having the highest Y) is used in the construction of bridges.

3. The maximum height of a mountain on earth depends upon the shear modulus of rock. At the base of the mountain, the stress due to all the rock on top should be less than the critical shear stress at which the rock begins to flow. This can be estimated as follows. Suppose the height of a mountain is h and the density of its rock is ρ. Then the force per unit area (due to the weight of the mountain) at the base is $h\rho g$. The material at the base experiences this force per unit area in the vertical direction; but the sides of the mountain are free. Hence there must be a tangential shear which is approximately of the order of $h\rho g$. Now the elastic limit of a typical rock is about 3×10^8 $\mathrm{Nm^{-2}}$ and the density is about 3×10^3 $\mathrm{kg\,m^{-3}}$. Hence

$$h_{\max}\, \rho\, g = 3 \times 10^8$$

or

$$h_{\max} = \frac{3 \times 10^8}{3 \times 10^3 \times 9.8} \simeq 10{,}000 \text{ m} = 10 \text{ km}$$

which is about the height of the Mount Everest. A height greater than this will not be able to withstand the shearing stress due to the weight of the mountain.

EXAMPLE 10.1 A steel wire of diameter 2.0 mm can withstand a maximum strain of 10^{-3}. Find the maximum load it can hold. Young's modulus of steel is 2.0×10^{11} $\mathrm{Nm^{-2}}$.

Solution

$$\text{Young's modulus} = \frac{\text{maximum stress}}{\text{maximum strain}}$$

$\therefore$ $$\text{Maximum stress} = 2.0 \times 10^{11} \times 10^{-3} = 2.0 \times 10^{8} \text{ Nm}^{-2}$$

Now, $$\text{Area of cross-section} = \frac{\pi d^2}{4}$$

$$d = 2.0 \text{ mm} = 2.0 \times 10^{-3} \text{ m}$$

and $$\text{Stress} = \frac{\text{Load } F}{\text{area}}$$

$\therefore$ $$F = \text{stress} \times \text{area}$$

$$= 2.0 \times 10^8 \times \frac{\pi \times (2.0 \times 10^{-3})^2}{4}$$

$$= 628.4 \text{ N}$$

EXAMPLE 10.2 A steel wire of cross-sectional area 4.0×10^{-5} m^2 stretches by the same amount as a copper wire of cross-sectional area 3.0×10^{-5} m^2 under the same load. What is the ratio of the original length of the steel wire to that of the copper wire ? Young's modulus of steel is 2.0×10^{11} Nm^{-2} and that of copper is 1.0×10^{11} Nm^{-2}.

Solution Let L_s and L_c be the original lengths of the steel and copper wires respectively and A_s and A_c their respective cross-sectional areas. The stretchings produced in them by a load F are given by

$$(\Delta L)_s = \frac{F}{A_s} \cdot \frac{L_s}{Y_s}$$

and

$$(\Delta L)_c = \frac{F}{A_c} \frac{L_c}{Y_c}$$

where Y_s and Y_c are the Young's moduli of steel and copper respectively. It is given that $(\Delta L)_s = (\Delta L)_c$. Hence

$$\frac{F}{A_s} \cdot \frac{L_s}{Y_s} = \frac{F}{A_c} \frac{L_c}{Y_c}$$

$$\therefore \quad \frac{L_s}{L_c} = \left(\frac{A_s}{A_c}\right) \cdot \left(\frac{Y_s}{Y_c}\right)$$

$$= \left(\frac{3 \times 10^{-5}}{4 \times 10^{-5}}\right) \cdot \left(\frac{2.0 \times 10^{11}}{1.0 \times 10^{11}}\right)$$

$$= \frac{3}{2}$$

EXAMPLE 10.3 A steel wire (original length 1.5 m, original diameter 0.10 cm) and a brass wire (original length 1.0 m, original diameter 0.20 cm) are loaded as shown in Fig. 10.9. Compute the extension of the steel and brass wire. Young's modulus of steel is 2.0×10^{11} Nm^{-2} and that of brass is 0.91×10^{11} Nm^{-2}.

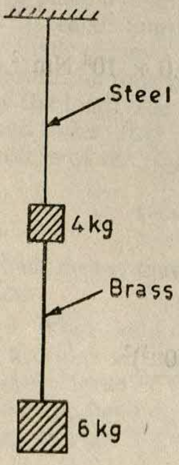


Fig. 10.9

Solution

Force on steel wire $(F_s) = (6 \text{ kg} + 4 \text{ kg})$ wt
$= 10$ kg wt $= 10 \times 9.8$ N
$= 98$ N

Force on brass wire $(F_b) = 6$ kg wt $= 6 \times 9.8$ N
$= 58.8$ N

The extension of the steel wire is given by

$$(\Delta L)_s = \frac{F_s}{A_s} \cdot \frac{L_s}{Y_s} = \frac{4\,F_s}{\pi\, d_s^2} \cdot \frac{L_s}{Y_s}$$

Substituting the numerical values $F_s = 98$ N,

$d_s = 0.10$ cm
$= 0.1 \times 10^{-2}$ m^2

$L_s = 1.5$ m and $Y_s = 2.0 \times 10^{11}$ Nm^{-2}

we get $(\Delta L)_s = 9.36 \times 10^{-4}$ m

Similarly the extension of the brass wire is

$$(\Delta L)_b = 2.06 \times 10^{-4} \text{ m}$$

EXAMPLE 10.4 A 10 m long cable consists of 40 strands of steel each 5×10^{-6} m^2 in cross-section. By how much will the cable stretch when it is used to lift a crate weighing 400 kg? Young's modulus of steel $= 2.0 \times 10^{11}$ Nm^{-2}.

Solution

$$\text{Stress} = \frac{\text{force}}{\text{area}} = \frac{400 \times 9.8}{5 \times 10^{-6}} = 7.84 \times 10^{8} \text{ Nm}^{-2}$$

This stress must distribute equally over the entire cross-section of the cable consisting of 40 strands. Hence stress on each strand is

$$\frac{7.84 \times 10^{8}}{40} = 1.96 \times 10^{7} \text{ Nm}^{-2}$$

$$\therefore \text{ Strain in each strand} = \frac{\text{Stress}}{\text{Young's modulus}}$$

$$= \frac{1.96 \times 10^{7}}{2 \times 10^{11}} = 0.98 \times 10^{-4}$$

$$\therefore \text{ Extension of each strand} = 0.98 \times 10^{-4} \times 10 = 0.98 \times 10^{-3} \text{ m}$$

$$= 0.98 \text{ mm}$$

Since each strand stretches by 0.98 mm, the cable is stretched by 0.98 mm.

EXAMPLE 10.5 A composite wire of a uniform cross-section of 2.5×10^{-5} m^2 consists of a steel wire of length 1.5 m and a copper wire of length 2.0 m. By how much will it stretch when it is loaded with a mass of 200 kg? Young's modulus of steel is 2.0×10^{11} Nm^{-2} and that of coppor is 1.0×10^{11} Nm^{-2}.

Solution Since the steel and copper parts of the wire are of the same cross-section (A) and are loaded with the same weight ($F = Mg$) they are under the same stress (F/A). Therefore

$$\frac{Y_s (\Delta L)_s}{L_s} = \frac{Y_c (\Delta L)_c}{L_c} = \frac{F}{A} = \frac{Mg}{A}$$

where subscripts s and c refer to steel and copper respectively.

$$\therefore \quad (\Delta L)_s = \frac{Mg}{A} \cdot \frac{L_s}{Y_s}$$

and

$$(\Delta L)_c = \frac{Mg}{A} \cdot \frac{L_c}{Y_c}$$

Substituting the given values of the various quantities, we get

$$(\Delta L)_s = 5.88 \times 10^{-4}\ \text{m} = 0.588\ \text{mm}$$

and

$$(\Delta L)_c = 15.68 \times 10^{-4}\ \text{m} = 1.568\ \text{mm}$$

$\therefore$ Extension of the composite wire $= 0.588 + 1.568 = 2.156$ mm
$= 2.16$ mm

EXAMPLE 10.6 Artificial diamonds are made by subjecting graphite to a pressure of 1.5×10^5 atmosphere at a high temperature. Assuming that natural diamonds were formed at such high pressure within the earth, what must have been the original volume of a diamond whose mass before cutting was 175 g? The density of diamond $= 3500$ kg m^{-3} and its bulk modulus $B = 6.2 \times 10^{11}$ Nm^{-2}.

Solution

Mass of the diamond $= 175$ g $= 0.175$ kg

Density $= 3500$ kg m^{-3}

$$\therefore \quad \text{Volume} = \frac{0.175}{3500} = 50 \times 10^{-6}\ \text{m}^3 = 50\ \text{cm}^3$$

The decrease in volume $(-\Delta V)$ under the pressure p during formation is given by

$$-\Delta V = \frac{PV}{B}$$

where $P = 1.5 \times 10^5$ atm $= 1.5 \times 10^5 \times 1.01 \times 10^5$
$= 1.515 \times 10^{10}$ Nm^{-2}, since 1 atm $= 1.01 \times 10^5$ Nm^{-2}. Hence

$$-\Delta V = \frac{1.515 \times 10^{10} \times 50 \times 10^{-6}}{6.2 \times 10^{11}} = 1.22 \times 10^{-6}\ \text{m}^3$$
$$= 1.22\ \text{cm}^3$$

$\therefore$ Original volume of the diamond $= 50 + 1.22 = 51.22$ cm^3

EXAMPLE 10.7 A rubber eraser 3.0 cm × 1.0 cm × 8.0 cm is clamped at one end with the 8.0 cm edge vertical. A horizontal force of

2.4 N is applied at the free end (the top face). Calculate the horizontal displacement of the top face if the shear modulus of rubber is 1.4×10^5 Nm^{-2}.

Solution Figure 10.10 shows a rubber earser ABCD clamped at the end AB and a horizontal force F applied at the free end DC resulting in a displacement DD′ = CC′ which is to be evaluated. The shear strain θ is given by

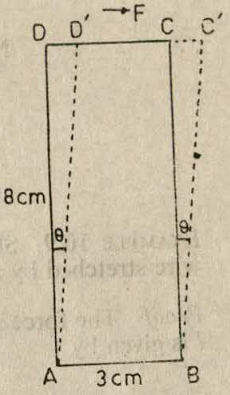


Fig. 10.10

$$\tan\theta = \frac{DD'}{AD}$$

Since θ is small $\tan\theta \simeq \theta$, so that

$$\text{Shear strain } \theta = \frac{DD'}{DA}$$

Now shearing stress is applied on the top face whose area is

$$A = 3.0 \text{ cm} \times 1.0 \text{ cm}$$

$$= 3 \text{ cm}^2$$

$$= 3 \times 10^{-4} \text{ m}^2$$

$$\therefore \text{ Shearing stress} = \frac{F}{A} = \frac{2.4}{3 \times 10^{-4}} = 0.8 \times 10^4 \text{ Nm}^{-2}$$

$$\text{Now shear modulus } (G) = \frac{\text{shearing stress}}{\text{shear angle}}$$

$$\therefore \quad \text{Shear angle} = \frac{\text{shearing stress}}{G} = \frac{0.8 \times 10^4}{1.4 \times 10^5} = 0.057 \text{ rad}$$

or

$$\frac{DD'}{AD} = 0.057$$

But

$$AD = 8.0 \text{ cm}$$

Therefore

$$DD' = 0.057 \times 8.0 = 0.46 \text{ cm} = 4.6 \text{ mm}$$

EXAMPLE 10.8 A piece of lead is subjected to a constant pressure of 2×10^8 Nm^{-2}. What will be the density of lead under this pressure? Normal density of lead is 11400 kgm^{-3} and its bulk modulus is 0.8×10^{10} Nm^{-2}.

Solution Let V m^3 be the original volume of the lead piece. Its mass is

$$M = \text{density} \times \text{volume} = 11400\, V \text{ kg}$$

The decrease in volume under pressure is given by

$$-\Delta V = \frac{\Delta P V}{B} = \frac{2 \times 10^8 \times V}{0.8 \times 10^{10}} = \frac{V}{40} \text{ m}^3$$

$$\therefore \text{ New volume} = V - \Delta V = V - \frac{V}{40} = \frac{39V}{40} \text{ m}^3$$

$$\therefore \qquad \text{New density} = \frac{\text{mass}}{\text{new volume}}$$

$$= \frac{11400 \; V \text{ kg}}{39V/40 \text{ m}^3}$$

$$= 11692 \text{ kg m}^{-3}$$

EXAMPLE 10.9 Show that the potential energy per unit volume of a wire stretched by a force within elastic limit is ½ stress × strain.

Proof The force F needed to stretch a wire of length L by an amount l is given by

$$F = \frac{YAl}{L} \tag{i}$$

where A is the cross-sectional area of the wire and Y its Young's modulus.

The work done by this force in stretching the wire is stored in it as potential energy. This work is calculated as follows.

Work done for a small increase dl in the length of the wire

$$= F \cdot dl$$

Therefore, the total work done for an increase l in the length of the wire is

$$W = \int_0^l F \, dl \tag{ii}$$

Using Eq. (i) in Eq. (ii) we get

$$\text{PE} = W = \int_0^l \frac{YAl}{L} \, dl$$

$$= \frac{YA}{L} \int_0^l l \, dl$$

($\because$ Y, A and L are constants)

$$= \frac{YA}{L} \cdot \frac{l^2}{2}$$

Now, volume of the wire = AL. Therefore, the potential energy per unit volume

$$= \frac{YA}{L} \cdot \frac{l^2}{2} \cdot \frac{1}{AL}$$

$$= \frac{1}{2} \frac{Yl}{L} \cdot \frac{l}{L}$$

$$= \frac{1}{2} \frac{F}{A} \cdot \frac{l}{L} \qquad \left(\because \; F = \frac{YAl}{L}\right)$$

$$= \tfrac{1}{2} \text{ stress} \times \text{strain}$$

SECTION B: FLUIDS

10.5 INTRODUCTION

The general term 'fluid' refers to any substance that has no definite shape and has the ability to flow. Liquids and gases do not have any definite shape and they flow. Thus liquids and gases are fluids. Many areas of science and industry are concerned with fluids. Engineers and scientists who are engaged in the design of submarines, aeroplanes, missiles, rockets, ships and satellites are concerned with fluids. Devices used in science and industry, such as barometers, hydraulic presses, hydraulic brakes, etc. are manufactured by taking advantage of the properties of fluids.

We will study some basic properties of a fluid at rest and in motion.

10.6 FLUID PRESSURE

Unlike solids, fluids do not have a definite shape; they take the shape of the vessel in which they are contained. We have seen that solids can withstand a tangential force and can be sheared. But fluids cannot withstand any force tangential to their surface. This can be understood as follows.

Suppose you are sitting in a boat floating on water in a lake and the boat and the water are still, i.e. they are in a state of static equilibrium. The forces acting vertically downwards on the system are (i) the weight of the atmosphere, (ii) your weight, (iii) the weight of the boat and (iv) the weight of the water in lake. All these forces add up to a total force W exerted at the bottom of the lake. From Newton's third law the bottom of the lake exerts an upward force equal to W so that the system (i.e. the boat, you and water) is in equilibrium (see Fig. 10.11(a).

Now suppose a small horizontal force (parallel to the water surface is applied to the boat. This force can be applied by a person on the shore by pulling at the rope tied to the boat. You can also exert such a force sitting in the boat by throwing something (say fish) in a direction parallel to the water surface. The fish will, in turn, exert an equal and opposite force on you and the boat. You will notice that a small force F (see Fig. 10.11(b)) is enough to move the boat. If you look at the water near the boat, you will notice that water also moves in the direction of the boat. The system is no longer in equilibrium. Thus if a tangential force is applied to a liquid, it begins to flow, since it cannot withstand any tangential force. Thus we conclude that *the forces acting on a fluid in equilibrium are always perpendicular to its surface.* It is for this reason that the free surface of a fluid at rest in a container is always horizontal.

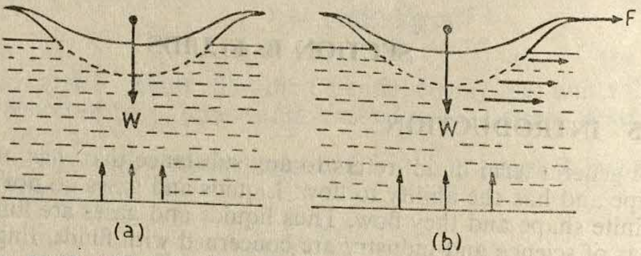


Fig. 10.11 (a) Boat and water in equilibrium (b) Equilibrium disturbed by a small force F applied parallel to the water surface

Since the direction of the force acting on a fluid in equilibrium is always specified (being normal to its surface), the force is completely described by its magnitude. Hence, while dealing with fluids, it is more meaningful and convenient to use the concept of *pressure* rather than force. The total force acting on a surface is called *thrust*. *Pressure is defined as the thrust exerted normally on a unit area of the surface.* In the SI system, the unit of pressure is *newton per square metre* (Nm^{-2}) also called *pascal* (symbol Pa).

Since pressure is force per unit area, it follows that a given force will exert a different pressure if it acts over a different area. To clarify

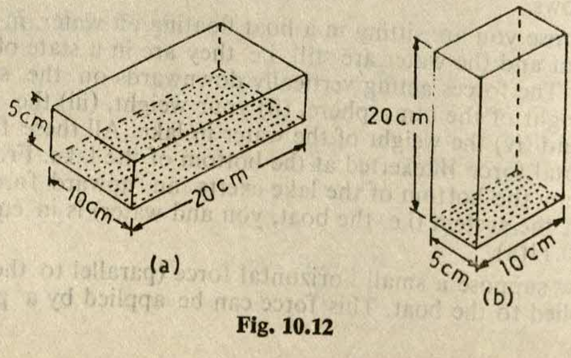
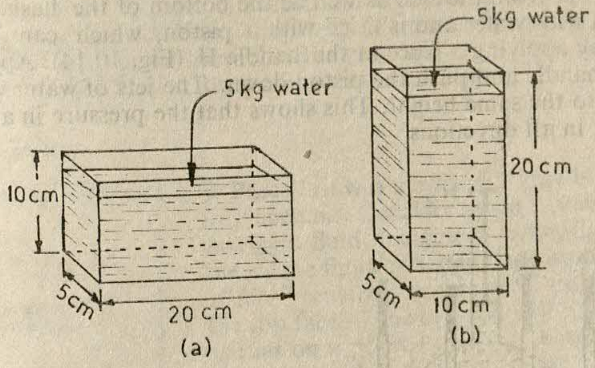


Fig. 10.12

further, let us first consider the example of the pressure exerted by a solid, for example, a brick. Figure 10.12 shows a brick of dimensions 20 cm × 10 cm × 5 cm lying on a table. In Fig. 10.12(a) the brick is lying on its side and in Fig. 10.12(b) it is standing on end. If the mass of the brick is 5 kg, it exerts a force of 5 kg × $9.8\ \mathrm{ms^{-2}} = 49$ N on the surface. Here $9.8\ \mathrm{ms^{-2}}$ is the acceleration due to gravity. In case of Fig. 10.12(a) the force of 49 N acts over an area = 20 cm × 10 cm $= 200\ \mathrm{cm^2} = 0.02\ \mathrm{m^2}$. Hence it exerts a pressure of $\frac{49\ \mathrm{N}}{0.02\ \mathrm{m^2}} = 2450$ $\mathrm{Nm^{-2}}$. In case of Fig. 10.12(b) the same force of 49 N now acts over an area $= 10\ \mathrm{cm} \times 5\ \mathrm{cm} = 50\ \mathrm{cm^2} = 0.005\ \mathrm{m^2}$. Hence it exerts a pressure of $\frac{49\ \mathrm{N}}{0.005\ \mathrm{m^2}} = 9800\ \mathrm{Nm^{-2}}$. Thus the same force acting on a smaller area exerts a larger pressure. That is why it is easier to cut with a sharp knife than with a blunt one. Since the area of the edge of a sharp blade is extremely small, even a small force will exert a very high pressure.



(a) (b)

Fig. 10.13

Let us now calculate the pressure exerted by a liquid at the bottom of a container. Take equal mass, say 5 kg, of water in two rectangular containers as shown in Fig. 10.13. The force exerted by a mass of 5 kg of water is 5 kg × $9.8\ \mathrm{ms^{-2}} = 49$ N. In Fig. 10.13(a) water exerts this force over an area $= 20\ \mathrm{cm} \times 5\ \mathrm{cm} = 100\ \mathrm{cm^2} = 0.01\ \mathrm{m^2}$. Hence the pressure exerted by water on the base of the container $= \frac{49\ \mathrm{N}}{0.01\ \mathrm{m^2}} = 4900\ \mathrm{Nm^{-2}}$. In Fig. 10.13(b) the same force now acts over an area $= 10\ \mathrm{cm} \times 5\ \mathrm{cm} = 50\ \mathrm{cm^2} = 0.005\ \mathrm{m^2}$. Hence the pressure exerted in this case is $\frac{49\ \mathrm{N}}{0.005\ \mathrm{m^2}} = 9800\ \mathrm{Nm^{-2}}$. Thus water exerts the same force but not the same pressure at the bottom of the container in these two cases. We have seen that, like solids liquids also exert pressure. But unlike solids, liquids exert pressure not only at the bottom but also on the sides of the vessel in which they are contained. If you make a hole in the vessel, the liquid begins to come out of it. If you place your finger on the hole, you can feel the pressure.

10.7 PASCAL's LAW

Blaise Pascal (1623—1662), a French scientist, discovered a principle which tells us how force (or pressure) can be transmitted in a fluid. Pascal's law states that *pressure in a fluid in equilibrium is the same everywhere* (if the effect of gravity can be neglected).

Experimental Demonstration of Pascal's Law

In a liquid at rest, the force exerted by the liquid on any surface is prependicular to the surface. Any force acting parallel to the surface will make the liquid flow and the liquid will not stay at rest. Whenever such a liquid is subjected to a pressure, it transmits equally in all directions. To illustrate this take a glass flask having small tubes J jetting out from the side as well as the bottom of the flask. The flask is filled with water and is fitted with a piston, which can be pushed down by applying a force at the handle H (Fig. 10.14). Apply a force at the handle and push the piston down. The jets of water will be seen to rise to the same height. This shows that the pressure in a liquid acts equally in all directions.

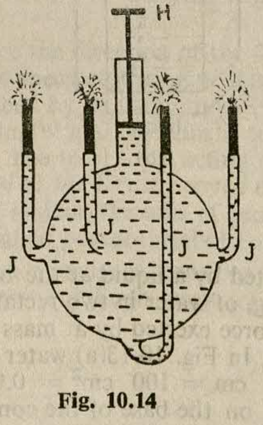


Fig. 10.14

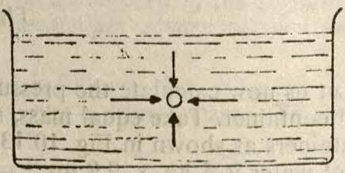


Fig. 10.15

Figure 10.15 explains why the pressure should be the same in all directions. Consider a tiny sphere of a liquid that is so small that we can ignore the gravitational force on it. Then the pressure on one side of it must be equal to that on the side opposite to it, as shown in the diagram. If the pressures were not equal, the tiny piece of liquid would move. In other words, the liquid will flow. Hence when the liquid is at rest, the pressure must have the same magnitude in all directions.

Proof of Pascal's Law

We will show that the pressure at any two points P_1 and P_2 in a fluid is the same (see Fig. 10.16). Imagine that we have a right circular cylinder in the fluid with the line $P_1 P_2$ as its axis. Let A be the area of the circular end-faces of the cylinder.

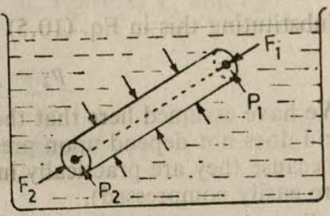


Fig. 10.16 A cylinder in a fluid; the arrows indicate the direction of pressure

Let us suppose that the cylinder is filled with a fluid which is in equilibrium under the influence of the fluid outside. The forces from the outside fluid act normally on the surfaces of the cylinder as shown. It is clear that the forces F_1 and F_2 on the circular faces at P_1 and P_2 are perpendicular to the forces on the cylindrical surfaces. Hence forces F_1 and F_2 must add to zero because the cylinder is at rest. Since the areas of the circular end-faces are equal, it follows that the pressures at P_1 and P_2 are also equal. Similar arguments show that the pressure at a point does not depend on the direction of the surface on which it acts.

Effect of Gravity

We will now find out how Pascal's law is modified if gravity is taken into account. Consider again a vessel containing a fluid. Imagine a cylindrical column of the fluid of height h as shown in Fig. 10.17. Then the pressure at a point P_1 on the top face of the cylinder will be the same as that on a point P_2 on the bottom face, if the forces due to gravity are neglected. Will these pressures remain equal if the cylinder effect of gravity is included? The answer is no.

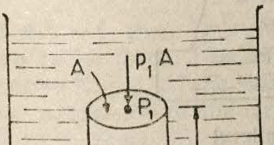


Fig. 10.17 The pressure-depth formula

The cylindrical column of the fluid is in equilibrium which means that the forces acting on it are balanced. The horizontal forces acting on the sides of the cylinder are mutually balanced. The vertical forces acting on the cylinder are:

(i) the force p_1A acting on the top face vertically downwards, where p_1 is the fluid pressure at P_1 and A is the cross-sectional area of the top face, (ii) the weight mg of the fluid of mass m in the cylinder acting vertically downwards and (iii) the force p_2A at the bottom face acting vertically upwards. Since the cylinder is in equilibrium, we have

$$p_1A + mg = p_2A$$

or

$$(p_2 - p_1)\,A = mg \tag{10.5}$$

If ρ is the mass density (mass per unit volume) of the fluid, then the mass (m) of the fluid in the cylinder of height h and base area A is

$$m = \text{density} \times \text{volume} = \rho h A$$

Substituting this in Eq. (10.5) we have

$$p_2 - p_1 = h\rho g \tag{10.6}$$

We have assumed here that the mass density of the fluid is uniform and does not depend upon pressure. This assumption is true for liquids (because they are practically incompressible) but not for gases (which are easily compressed).

Equation (10.6) describes how Pascal's law $(p_2 - p_1) = 0$ is modified due to gravity. This equation tells us that the pressure difference between two points in a fluid depends upon h, the vertical height between the two points, ρ the density of the fluid and g, the acceleration due to gravity. Since the base area A does not appear in Eq. (10.6) it is evident that the pressure difference $(p_2 - p_1)$ does not depend on the amount (i.e. mass) of the fluid column but only on its vertical height.

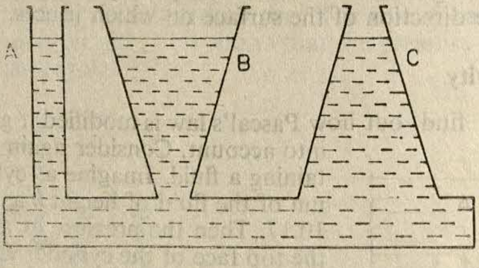


Fig. 10.18 Liquid seeks its own level

It is for this reason that a *liquid seeks its own level.* Pour some water in the communicating tubes A, B and C as shown in Fig. 10.18. When equilibrium is established, the water level will be the same in each tube, although the amount of water in tubes B and C is much greater than that in tube A.

Applications of Pascal's Law

We have seen that the pressure difference between two points in a fluid is either zero (if they are at the same horizontal level) or is a definite quantity depending on their height difference. So if the pressure at some point in a fluid is changed, there will be an equal change in pressure at any other point. Thus fluids, especially liquids (because they are incompressible) are ideal for transmitting pressure. This fact is used in hydraulic machines, such as hydraulic press, hydraulic brakes, hydraulic jacks, etc.

Principle of a Hydraulic Machine

Consider two cylindrical tubes of areas of cross-section A_1 and A_2 connected by a horizontal tube T as shown in Fig. 10.19. The apparatus is filled with a liquid. There are water tight pistons P_1 and P_2.

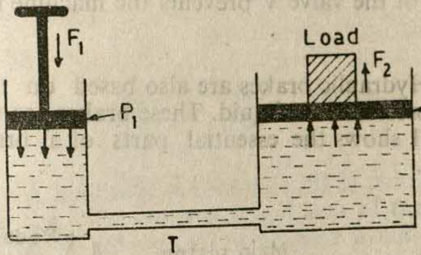


Fig. 10.19 Principle of a hydraulic machine

If a force F_1 is applied at piston P_1 the pressure exerted at any point on it is F_1/A_1. This pressure is transmitted unchanged to piston P_2. Therefore the upward force F_2 exerted on piston P_2 is

$$F_2 = \frac{F_1 \times A_2}{A_1}$$

or

$$\frac{F_2}{F_1} = \frac{A_2}{A_1}$$

If $A_2 > A_1$ then $F_2 > F_1$. Thus a small force can be used to exert a much larger force. This relation is valid only if the two pistons are at the same height. If the smaller piston is at a height h above the larger one, the pressure at the larger piston will be the sum of the pressure due to a force F_1 on the smaller piston and the pressure due to a column of liquid of height h.

Hydraulic Lift The hydraulic lift is used for lifting heavy loads such as cars or other heavy vehicles. The principle of its working is the same as that of a hydraulic press. Figure 10.20 shows a simple hydraulic lift.

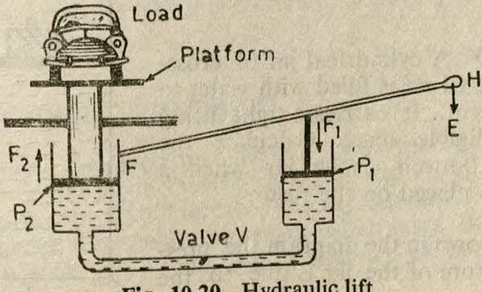


Fig. 10.20 Hydraulic lift

The effort is applied on the smaller piston P_1. The car is placed on the platform of the larger piston P_2. The whole apparatus is filled with oil. When an effort is applied on the handle H, valve V opens and oil is forced into the larger cylinder. The piston P_2 is raised, thus raising the car. When the car has reached the desired height, the handle is released; the closing of the valve V prevents the machine from running backward.

Hydraulic Brakes Hydraulic brakes are also based on the principle of transmission of pressure in a liquid. These brakes are used in automobiles. Figure 10.21 shows the essential parts of a braking device used in automobiles.

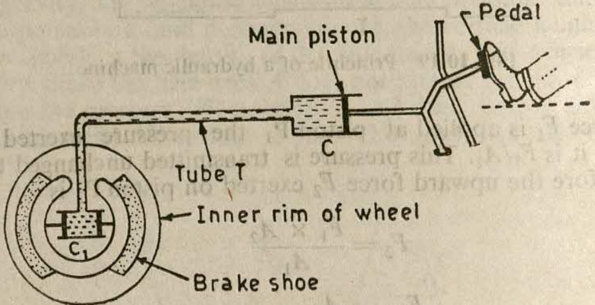


Fig. 10.21 Hydraulic brake

It consists of a tube T containing oil. One end of this tube is connected to the cylinder C fitted with a piston which is connected to the pedal as shown. The other end of the tube is connected to a cylinder C_1 having two pistons which are connected to the brake shoes. The cylinder C_1 has a larger diameter than that of C. When the pedal is pressed, increase in the pressure is transmitted through the oil to the pistons in cylinder C_1 which are pushed towards the rim of the wheel. When the shoe presses against the rim, the brake is applied, retarding the motion. In this way a small force applied to the pedal causes a much larger retarding force.

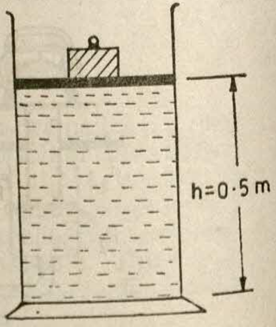


Fig. 10.22

EXAMPLE 10.10 A cylindrical jar of cross-sectional area 0.01 m^2 is filled with water to a height of 50 cm. It carries a tight fitting piston of negligible mass. Calculate the pressure at the bottom of the jar when a mass of 1 kg is placed on the piston.

Solution As shown in the diagram the pressure at the bottom of the jar is due to the weight of a column of water 50 cm high and the weight of the load (Fig. 10.22).

Total force acting on the base

$$= h\rho g A + mg$$
$$= 0.5 \times 1000 \times 9.8 \times 0.01 + 9.8$$
$$= 5 \times 9.8 + 9.8$$
$$= 6 \times 9.8$$
$$= 58.8 \text{ N}$$

$$\therefore \quad \text{Pressure} = \frac{\text{force}}{\text{area}} = \frac{58.8}{0.01} = 5880 \text{ Nm}^{-2} \text{ or Pa}$$

EXAMPLE 10.11 A hydraulic automobile lift is used to lift a car with a mass of 3000 kg. The cross-sectional area of the piston on which the car is supported is 425 cm^2. What pressure would the smaller piston have to bear if (a) both the pistons are at the same horizontal level and (b) the bigger piston is 3 m above the smaller piston. The density of oil filling the hydraulic machine is 800 kg m^{-3}.

Solution

$$\text{Force on bigger piston} = 3000 \times 9.8 = 2.94 \times 10^4 \text{ N}$$

$$\text{Area of bigger piston} = 425 \text{ cm}^2 = 4.25 \times 10^{-2} \text{ m}^2$$

(a) $$\text{Pressure at any point on bigger piston} = \frac{2.94 \times 10^4}{4.25 \times 10^{-2}}$$
$$= 6.92 \times 10^5 \text{Nm}^{-2} \text{ or Pa}$$

Since pressure is transmitted unchanged in the fluid, the smaller piston would have to bear a pressure of 6.92×10^5 pa.

(b) The additional pressure due to a column of oil of height 3m and density 800 kg m^{-3} = $h\rho g$

$$= 3 \times 800 \times 9.8 = 2.35 \times 10^4 \text{ Pa} \simeq 0.23 \times 10^5 \text{ Pa}$$

Therefore, the total pressure on the smaller piston, in this case, is

$$6.92 \times 10^5 + 0.23 \times 10^5 = 7.15 \times 10^5 \text{ Pa}$$

The atmospheric pressure which is common to both pistons has been ignored.

10.8 ATMOSPHERIC PRESSURE

Like liquids, gases also exert pressure. The atmosphere of the earth is a mixture of gases. Near the surface of the earth, air is made up of about 78% nitrogen, 21% oxygen, slightly less than 1% argon, tiny amounts of carbon dioxide and traces of other gases. Because of the pull of gravity, these gases tend to accumulate near the surface of the earth. Thus air is the densest at sea-level and becomes thinner and thinner as we go up.

Like all gases, air also has weight and hence exerts pressure. Just as water pressure is caused by the weight of water, the weight of all the air above the earth causes an atmospheric pressue of about $10^5\ Nm^{-2}$. The atmosphere exerts this pressure not only on the earth's surface, but also on the surface of all objects on the earth including living beings. The surface area of an average human being is about $2\ m^3$. Therefore the total force exerted over his body by the atmosphere = atmospheric pressure × area = 2×10^5 N. This is an enormous force. But we do not feel such an enormous force on our body. Why are we not crushed by this huge force? The answer lies in the fact that the blood in the cells of our body and bodies of all other organisms exerts an equal pressure. Thus the pressure inside a cell equals the atmospheric pressure outside. When placed in a vacuum, the cells of any organism actually explode because of this internal pressure.

Measurement of Atmospheric Pressure: The Mercury Barometer

A barometer is a device for measuring the atmospheric pressure. It was invented by the Italian physicist, E. Torricelli in 1643. It is called the simple barometer. A simple barometer can be made by taking a glass tube about 1 m long and closed at one end. It is filled with mercury up to the brim and a finger is placed over the open end as shown in Fig. 10.23(a). The tube is then inverted and the open end, with the finger securely placed over it, is dipped in a dish containing mercury, as shown in Fig. 10.23(b). The finger is then removed. The moment this is done, the mercury level drops to a height has shown. At sea-level the height *h* at which the mercury column stands is about 76 cm. This height is the same regardless of the diameter of the tube or the length of the vacuum space at the top.

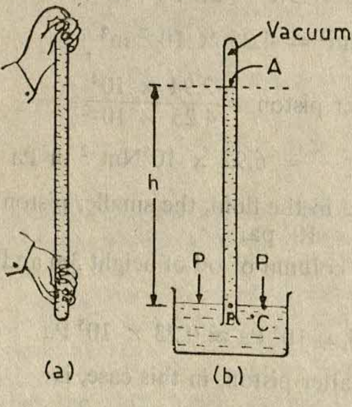


Fig. 10.23 The simple barometer

The pressure at point C is due to the atmosphere above. Since points B and C are at the same horizontal level, the pressure at B must be the same as that at C, i.e. the presssure at B is atmospheric. The pressure at B is due to a column of mercury of height *h* plus pressure if any, at point A due to some traces of air above the point A. But, the space above A is vacuum. Hence the pressure at A is zero. Thus the pressure at B is

$$P = h\rho g$$

where ρ is the density of mercury. At sea-level the value of h is 0.76 m. The atmospheric pressure at sea level is

$$\begin{aligned} P &= 0.76 \text{ m} \times 13600 \text{ kg m}^{-3} \times 9.8 \text{ m s}^{-2} \\ &= 1.01 \times 10^5 \text{ Nm}^{-2} \text{ or Pa} \end{aligned}$$

The SI unit of pressure is Nm^{-2} called *pascal* (Pa). Pressure is also expressed in terms of the height of mercury column as *centimetres* or *millimetres of Hg*. The atmospheric pressure at sea-level is 76 cm of Hg. The pressure of blood flowing in major arteries is about 120 mm of Hg in a normal person. A pressure of 1 mm of Hg is called a *Torr* (in honour of Torricelli). Pressure is also measured in units of atmospheric pressure. The pressure at a depth of 10 m in water is about 2 *atmospheres*. The unit of pressure used for meteorological purposes is called the *bar*. One bar is about 10^5 Nm^{-2}. A smaller unit is the *millibar* which is 10^2 Nm^{-2}.

The reason for using mercury in a liquid barometer is that it is the densest of all liquids. Consequently only a 76-cm height of mercury column is sufficient to balance atmospheric pressure. With any other liquid, the height would be much larger. For example, with water it is 10.34 m which is inconveniently large. Another reason is that the vapour pressure of mercury is negligible. If any other liquid is used, more vapour will be present in the vacuum above the liquid column. Due to the pressure exerted by the vapour, the barometer will give less pressure than the true value.

Aneroid Barometer

The mercury barometer, on account of its large size and the risk of spilling the mercury, cannot be easily taken from one place to another. For this purpose, a light and portable barometer called the aneroid barometer is used. In this barometer no liquid is used. The word 'aneroid' means 'without liquid'.

Figure 10.24 shows the essential parts of an aneroid barometer. It consists of an air-tight metal box B which is fitted with a lid D of thin

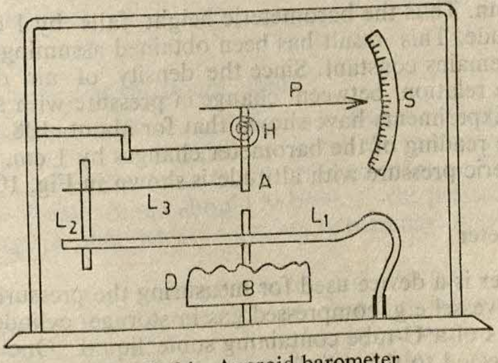


Fig. 10.24 Aneroid barometer

corrugated metal. The inside of the box is partially evacuated with a vacuum pump. The box is then sealed. The lid or diaphragm D is elastic and yields readily to changes in the atmospheric pressure.

The lid rises and falls with changes in the atmospheric pressure. The lid rises as the pressure decreases and falls when the pressure increases. The motion of the lid is very small for small changes in pressure. This motion is magnified by a system of levers L_1, L_2 and L_3 which communicate this motion to an axle A to which is attached a hair spring H. The spring H winds or unwinds as the pressure changes. Consequently, a pointer P attached to the axle moves in a clockwise or anti-clockwise direction. So a circular scale S over which the pointer moves is calibrated by comparison with a mercury barometer to read the pressure directly in centimetres of mercury.

10.9 VARIATION OF ATMOSPHERIC PRESSURE WITH ALTITUDE

Since the atmospheric pressure at a place is due to the weight of the air above it, it is obvious that the pressure will decrease as we go higher up above the sea-level. Thus the atmospheric pressure is less on mountains than on plains. If we take a barometer up through the atmosphere, the height of the mercury column will decrease. On the other hand, if we take it into a mine, the height of the mercury column will be greater than 0.76 m. Since the barometric height varies with the altitude, it is clear that the barometric reading at a place can be used in finding the height of the place of observation above the sea-level.

If the density of air is taken to be 1.29 kg m^{-3}, then 1 cm (0.01 m) of mercury column is equivalent to

$$\frac{0.01\ \text{m} \times 13600\ \text{kg m}^{-3}}{1.29\ \text{kg m}^{-3}}$$
$$= 105\ \text{m}$$

of air column. Thus the barometeric height falls by 1 cm for about 105 m altitude. This result has been obtained assuming that the density of air remains constant. Since the density of air decreases with altitude, the relation between change in pressure with altitude is not so simple. Experiments have shown that for about 108 m change in altitude, the reading of the barometer changes by 1 cm. The variation of atmospheric pressure with altitude is shown in Fig. 10.25.

The Manometer

A manometer is a device used for measuring the pressure of a gas enclosed in a vessel e.g. compressed gas in storage cylinders. It essentially consists of a U-tube containing some liquid. One end of the U-tube is connected to the cylinder containing gas and the other is open to the atmosphere (Fig. 10.26). If the pressure of the gas in the cylinder is equal to the atmospheric pressure the liquid will stand at the same level in the two arms of the manometer.

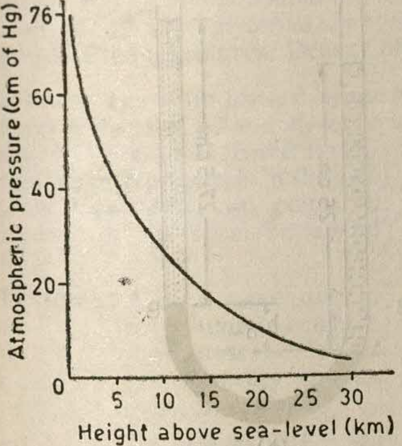


Fig. 10.25 Variation of atmospheric pressure with altitude

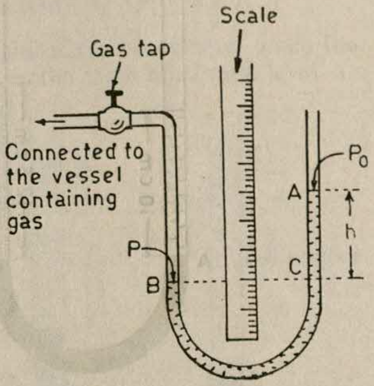


Fig. 10.26 A manometer

Let us assume that the pressure P of the gas enclosed in the vessel is more than P_0, the atmospheric pressure. When the gas tap is opened the pressure P forces the liquid to a level B in the arm connected to the vessel. The liquid in the other arm rises to level A. Let h be the difference between the two levels.

Now since the points B and C are at the same horizontal plane, the pressure at B = pressure at C. The level at A is open to the atmosphere. Hence the pressure at A = P_0, the atmospheric pressure.

P = Pressure at C

= Pressure P_0 + Pressure due to a height h of a column of liquid.

$= P_0 + h\rho g$

where ρ is the density of the liquid used.

The excess pressure is $P - P_0 = h\rho g$ is called the gauge pressure. Hence by measuring the level difference h, we can measure the pressure P or pressure difference $P - P_0$. In order to measure low pressures $(P - P_0)$ a low density liquid should be used. If the pressure difference $(P - P_0)$ is large, a high-density liquid (e.g. mercury) may be used.

EXAMPLE 10.12 A U-tube contains water and alcohol separated by mercury. The mercury columns in the two arms are level with 10.0 cm of water in one arm and 12.5 cm of alcohol in the other. What is the relative density of alcohol? If 15.0 cm more of water and alcohol each are poured into the respective arms of the U-tube, what will be the difference in the levels of mercury in the two arms? Relative density of mercury = 13.6.

Solution

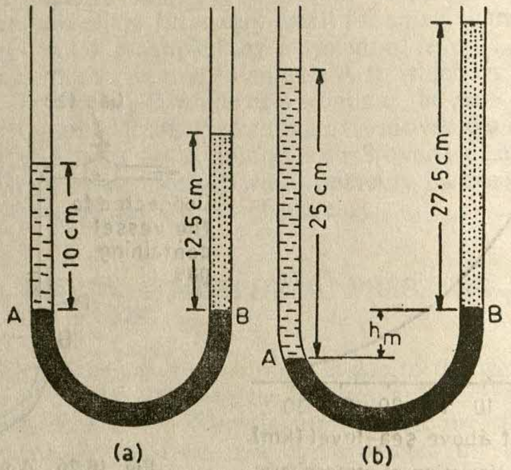


Fig. 10.27

Referring to Fig. 10.27(a), since the levels A and B of mercury are at the same height, the pressure due to 10.0 cm of water exerted at A equals that due to 12.5 cm of alcohol exerted at B. Thus

$$h_w \rho_w g = h_a \rho_a g$$

where subscripts *w* and *a* refer to water and alcohol respectively. Therefore the relative density of alcohol is

$$\frac{\rho_a}{\rho_w} = \frac{h_w}{h_a} = \frac{10.0}{12.5} = 0.800$$

If 15.0 cm more of each liquid is poured into the respective arms, the level of mercury in arm A falls while that in arm B rises. Let h_m be the difference in the mercury levels (Fig. 10.27(b)). It is clear that

Pressure of water column at A = pressure of alcohol column at B + pressure of mercury column of height h_m

i.e. $$25 \rho_w g = 27.5 \rho_a g + h_m \rho_m g$$

or $$25 = 27.5 \frac{\rho_a}{\rho_w} + h_m \frac{\rho_m}{\rho_w}$$

Now $\frac{\rho_a}{\rho_w} = 0.800$ and $\frac{\rho_m}{\rho_w} = 13.6$. Therefore, we have

$$25 = 27.5 \times 0.800 + 13.6\, h_m$$

or $$h_m = 0.221 \text{ cm}$$

EXAMPLE 10.13 A vertical U-tube of uniform inner cross-section contains mercury on both sides of its arms. A glycerine (density 1300 kg m^{-3}) column of length 10 cm is introduced in one of its arms. Oil of density 800 kg m^{-3} is poured into the other arm until the upper surfaces of oil and glycerine are at the same horizontal level. Find the length of the oil column. Density of mercury = 13600 kg m^{-3}.

Solution Let h_0 (in metres) be the height of the oil column when the upper surfaces of oil and glycerine are at the same horizontal level, as shown in Fig. 10.28. E and B indicate the mercury levels in the two arms. A and B are two points in mercury at the same horizontal level.

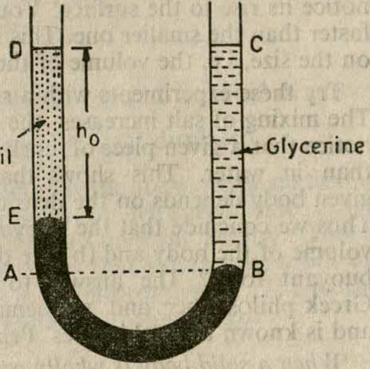


Fig. 10.28

Pressure at A = pressure due to oil column of height h_0 + pressure due to mercury column of height EA = $(0.1 - h_0)$

$$= h_0 \rho_0 g + (0.1 - h_0) \rho_m g \quad \text{(i)}$$

Pressure at B = pressure due to glycerine column of height h_g

$$= h_g \rho_g g \quad \text{(ii)}$$

Since A and B are at the same horizontal level,

Pressure at A = pressure at B

From (i) & (ii) we get

i.e. $$h_0\rho_0 + (0.1 - h_0) \rho_m = h_g\rho_g \quad \text{(iii)}$$

Now $\rho_0 = 800$ kg m^{-3}, $\rho_m = 13600$ kg m^{-3}, $\rho_g = 1300$ kg m^{-3},

and $h_g = 10$ cm = 0.1 m. Substituting these numerical values in (iii) we get

$$h_0 = 0.096 \text{ m} = 9.6 \text{ cm}$$

10.10 ARCHIMEDES' PRINCIPLE

Bodies seem to weigh less when immersed in a liquid. Fill a bathtub with water. Take a brick or a stone which you find difficult to lift. Immerse it in the bath water. Now slowly lift it out of the water. You will notice that as long as the brick remains under water, it is readily lifted; but the moment it comes out of water, the same brick seems too heavy to be easily lifted. We conclude that a liquid exerts an upward force on a body immersed in it. The tendency of a fluid to exert an upward force on a body immersed in it is called *buoyancy* and the upward force is called *buoyant force*. It is obvious that the buoyant force is equal to the apparent loss in the weight.

Let us try another home experiment. Take a small piece of cork and take it to the bottom of a jug full of water and leave it. It rises to the surface, although the force of gravity is pulling it downwards. It rises due to the fact that, in this case, the upward force exerted by the liquid on the cork is greater than its own weight.

Now try this experiment with a bigger piece of cork and carefully notice its rise to the surface. You will see that the bigger cork rises faster than the smaller one. This shows that the buoyant force depends on the size, i.e. the volume of the body.

Try these experiments with a saturated solution of salt and water. The mixing of salt increases the density of the solution. You will notice that a given piece of cork rises much faster in the salt solution than in water. This shows that the buoyant force experienced by a given body depends on the density of the fluid in which it is immersed. Thus we conclude that the buoyant force depends on (a) the size or volume of the body and (b) the density of the fluid. How large is this buoyant force? The answer to this question was discovered by the Greek philosopher and mathematician, Archimedes (287-212 B.C.) and is known as Archimedes' Principle which states that:

'When a solid body is wholly or partly immersed in a fluid, it experiences an upthrust or buoyant force equal to the weight of the fluid displaced by it.' The word 'fluid' includes both liquids and gases. The principle is a general one and holds for solids of any shape and for all fluids.

Theoretical Proof of Archimedes' Principle

Consider a solid cylinder having a cross-sectional area a and height h completely immersed in a fluid of density ρ (Fig. 10.29). The fluid exerts pressure on the top and bottom faces as well as on the sides of the cylinder. The thrust due to the pressure of the fluid at a point on the side of the cylinder is cancelled by an equal and opposite thrust acting at a point diametrically opposite to it in the same horizontal line, as shown by the arrows pointing towards the sides. But the thrusts on the two faces of the cylinder do not cancel out.

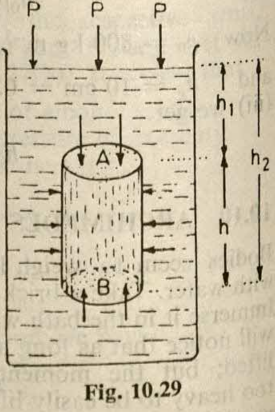


Fig. 10.29

Let the top face A of the cylinder be at a depth h_1 below the free surface of fluid and the bottom face B at the depth h_2 below the free surface of fluid. Let P be the atmospheric pressure. Then,

$$\text{Total downward thrust on face A} = (P + h_1 \rho g)\, a$$

Total upward thrust on face B $= (P + h_2 \rho g) a$

Since $h_2 > h_1$, the upward thrust > downward thrust.
Resultant upthrust on the cylinder.

$$= (P + h_2\rho g) a - (P + h_1\rho g) a$$
$$= (h_2 - h_1) \rho g a$$
$$= h\rho g a \qquad (\because \; h_2 - h_1 = h)$$

But $h \times a =$ Volume (V) of the cylinder which is evidently the volume of the fluid displaced by it.

$\therefore$ Upthrust $= V \rho g = Mg$

$(\because \; M = V \times \rho$ where M is the mass of the fluid displaced)

i.e. Upthrust = weight of the fluid displaced.

It may be worthwhile to mention that the above mathematical proof of Archimedes' Principle has been given for the special case of a solid of a regular shape immersed in a fluid of uniform density. The most remarkable feature of this principle is that it holds for solids of any shape immersed in a fluid of even non-uniform density. The mathematical proof of the general case is extremely complicated.

Floatation

When a body is placed in a fluid, two forces act upon it: (a) the weight W of the body acting downward and (b) the buoyant force W' acting upward. W may be greater than equal to or less than W'.

Case I $W > W'$: The downward pull of the weight of the body is higher than the upthrust. The body, therefore, sinks in the fluid (Fig. 10.30 (a)).

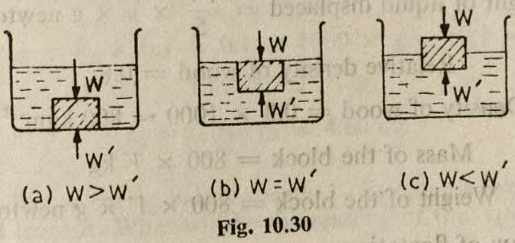


Fig. 10.30

Case 2 $W = W'$: This case is shown in Fig. 10.30(b). The weight of the body is just balanced by the upthrust and, therefore, the body just floats or is at rest in the body of the fluid.

Case 3 $W > W'$: In this case the downward gravitational force W is less than the upward force W'. Therefore, the body, when held below the surface of the fluid, experiences a net upward force to the surface.

It comes to rest when the upthrust or buoyant force becomes equal to the weight of the body (see Fig. 10.30 (c)). Similarly when a block of wood is placed on the surface of water, it begins to sink. The sinking stops when the buoyant force W' equals the weight W of the block. When this happens the block is partly submerged.

From the above discussion it is clear that the necessary condition for a body to float in a fluid is that the weight of the fluid displaced by it must be equal to that of the body. This is the law of floatation. This law does not express a new principle. It is nothing but Archimedes' principle applied to the case of a floating body.

EXAMPLE 10.14 A block of wood floats in a liquid with four-fifths of its volume submerged. If the relative density of wood is 0.800, find the density of the liquid.

Solution

Let the volume of the block be V m^3

$$\text{Volume of block under liquid} = \frac{4V}{5}\ \text{m}^3$$

$$\therefore \quad \text{Volume of liquid displaced} = \frac{4V}{5}\ \text{m}^3$$

Now let the density of the liquid be ρ kg m^{-3}

Mass of liquid displaced = (volume of liquid displaced) × (density of liquid)

$$= \frac{4V}{5}\rho\ \text{kg}$$

$$\text{Weight of liquid displaced} = \frac{4V}{5} \times \rho \times g\ \text{newton}$$

Relative density of wood = 0.8

$$\therefore \quad \text{Density of wood} = 0.8 \times 1000 = 800\ \text{kgm}^{-3}$$

$$\therefore \quad \text{Mass of the block} = 800 \times V\ \text{kg}$$

$$\text{Weight of the block} = 800 \times V \times g\ \text{newton}$$

From the law of floatation.

Weight of block = weight of liquid displaced

$$\text{or} \quad 800 \times V \times g = \frac{4V}{5} \times \rho \times g$$

$$\text{or} \quad \rho = 800 \times \frac{5}{4} = 1000\ \text{kgm}^{-3}$$

Notice that although the buoyant force on the block depends on the value of g (the acceleration due to gravity), the fraction of the volume of the block submerged under a liquid is independent of g and depends on the density of the block relative that of the liquid.

EXAMPLE 10.15 A cubical block of steel 10 cm on each side is floating on mercury in a vessel. The density of steel is 7.8×10^3 kgm^{-3} and that of mercury is 13.6×10^3 kgm^{-3}. (a) What is the height of the block above mercury level? (b) Water is poured into the vessel until it just cavers the steel block. What is the height of the water column?

Solution

Volume of steel block $= 10 \times 10 \times 10 = 10^3 \text{ cm}^3 = 10^{-3} \text{ m}^3$

(a) Let h_1 be the height of the block above the mercury surface.

$$\text{Volume of mercury displaced} = (0.1 - h_1) \times 0.1 \times 0.1 \text{ m}^3$$

$$\therefore \quad \text{Weight of mercury displaced} = (0.1 - h_1) \times 0.1 \times 0.1 \times 13.6 \times 10^3\, g \text{ newton}$$

This must be equal to the weight of block which is

$$7.8 \times 10^3 \times 10^{-3} \times g \text{ newton} = 7.8 \times g \text{ newton}$$

$$\therefore \quad (0.1 - h_1) \times 0.1 \times 0.1 \times 13.6 \times 10^3 \times g = 7.8 \times g$$

which gives $\quad h_1 = 0.0426 \text{ m} = 4.26 \text{ cm}$

(b) Let h_2 be the height of the water column required to just submerge the steel block. Thus

weight of the block = weight of water displaced + weight of mercury displaced

$$\text{i.e} \quad 7.8 \times g = h_2 \times 0.1 \times 0.1 \times 1000 \times g + (0.1 - h_2) \times 0.1 \times 0.1 \times 13.6 \times 10^3 \times g$$

which gives $\quad h_2 = 0.046 \text{ m} = 4.60 \text{ cm}$

EXAMPLE 10.16 A spring balance reads 10 kg when a bucket of water is suspended from it. What will be the reading of the spring balance when

(a) an ice cube of mass 1.5 kg is put into the bucket, and

(b) an iron piece of mass.7.2 kg suspended by another string is immersed with half its volume inside the water in the bucket? Relative density of iron is 7.2.

Solution (a) When an ice cube of mass 1.5 kg is dropped into the bucket, the total mass of the system suspended from the spring balance will be 10 kg + 1.5 kg = 11.5 kg. Hence the balance will read 11.5 kg.

(b) Since the relative density of iron is 7.2, its density is 7.2×10^3 kg m^{-3}. Therefore, the volume of the iron piece is

$$V = \frac{\text{mass}}{\text{density}} = \frac{7.2 \text{ kg}}{7.2 \times 10^3 \text{ kg m}^{-3}} = 10^{-3} \text{ m}^3$$

Now, volume of iron piece under water is $\frac{V}{2}$, which is the volume of water displaced by it. Hence weight of water displaced = volume displaced × density of water × g

$$= \frac{V}{2} \times 1000 \times g \text{ newton}$$

$$= \tfrac{1}{2}\, g \text{ newton} = \text{weight of } 0.5 \text{ kg}$$

($\because$ $V = 10^{-3}$ m^3)

This is the buoyant force exerted by water on the iron piece. Hence the iron piece will exert an equal force on water in the downward direction, which will increase the reading of the balance by 0 5 kg. Hence, the spring balance will read 10 kg + 0.5 kg = 10.5 kg.

10.11 VISCOSITY

We have so far dealt with fluids at rest. We will now consider fluids in motion. The branch of science that deals with the motion of liquids is known as *hydrodynamics*. The study of motion of liquids finds important applications in various fields, such as meteorology, marine engineering, civil engineering, etc. Liquids while flowing obey the same fundamental laws of motion as obeyed by particles. But the description of motion of a liquid is often more complex than that of a solid, for the simple reason that the intermolecular forces in liquids are not as strong as those in solids. To avoid complexities, we often make certain simplifying assumptions regarding the nature of the liquid and its flow. We shall assume that the liquid is incompressible and that the flow is steady or uniform. We will first discuss a very important property of a liquid which manifests itself when the liquid flows and has a profound influence on its motion. This property is called viscosity. Later we shall discuss the flow of liquids through pipes, a problem that finds numerous industrial applications.

Steady (or Streamlined) Flow

The flow of a liquid is said to be steady or streamlined if the velocity at every point in the fluid remains constant in magnitude as well as direction. In other words, in a steady flow, each infinitesimally small volume element of the liquid (which we may call a particle of the liquid) follows exactly the same path and has the same velocity as its preceding particle. Consider the flow of a liquid along the path ABC (Fig. 10.31(a)), where A, B and C are points inside the liquid. Each new particle of the liquid arriving at A will always have the same velocity V_A in a direction tangential to the curve ABC at point A. Similarly, a particle arriving at point B will always have the same velocity V_B which may or may not be the same as V_A. The same is true for point C. The line ABC along which the particles of the liquid move, one after another, with their velocities constant (but not necessarily equal) at the various points on it and directed along the tangents to these points is called a *streamline*.

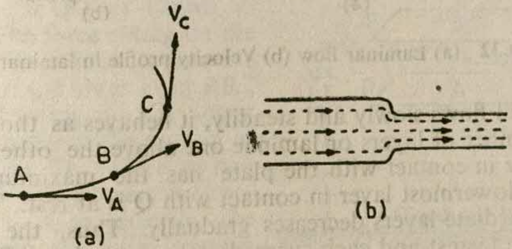


Fig. 10.31 Steady or streamlined flow

A streamline may be defined as a curve, the tangent to which at any point gives the direction of the liquid flow at that point. A streamline may be straight or curved. Figure 10.31(b) shows the streamlines of the steady flow of a liquid through a pipe of variable cross-section. Since the volume of liquid crossing any section of the pipe per unit time is the same, the velocity of water flowing through the narrower part must be greater than that flowing through the broader part. But every particle of the liquid passes a particular section of the pipe with the same velocity and in the same direction. The streamlines are more crowded in the narrower part where the flow of the liquid is faster. They never cross each other.

The liquid flow remains steady or streamlined if its velocity does not exceed a limiting value, called its *critical velocity* (see below), beyond which the streamlines bend in an irregular manner causing the liquid to move in disorderly fashion, often breaking up into eddies and whirls. Such a disorderly, churning motion of a liquid is known as *turbulence* and the flow of the liquid is said to be turbulent. We shall, however, deal only with a steady or streamlined flow of liquids.

Laminar Flow

Imagine that a liquid is in contact with a fixed horizontal surface Q at its bottom (Fig. 10.32(a)). Place a glass plate P on its surface and move it forward as shown. The liquid begins to flow because a liquid, by definition, cannot withstand the shearing strain caused by the moving plate. The particles of the liquid in contact with plate P move with the velocity, say v, of the plate, but the particles in contact with the stationary horizontal surface Q are practically at rest. The particles of the liquid lying between Q and P move with velocities lying in the range zero to *v*.

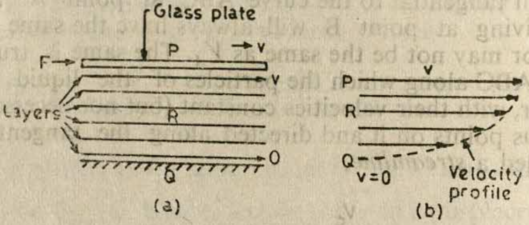


Fig. 10.32 (a) Laminar flow (b) Velocity profile in laminar flow

If the liquid flows slowly and steadily, it behaves as though it consists of a number of layers or laminae one above the other. The uppermost layer in contact with the plate has the maximum velocity *v* whereas the lowermost layer in contact with Q is at rest. The velocity of the intermediate layers decreases gradually. Thus, the uppermost layer P is the fastest and each succeeding layer, such as *R*, is slower than the one above it, with the lowermost layer Q in contact with the fixed horizontal surface Q remaining at rest. In Fig. 10.32(b), the magnitude of the velocities of the various layers is represented by the length of the arrowed lines. The greater the distance of a layer from the fixed surface Q, the greater is the velocity of the layer. The dotted curve in Fig. 10.32(b) represents what is called the *velocity profile* or *velocity shape* of the flow.

Similarly, when a liquid flows slowly and steadily through a pipe, the layer A of the liquid in contact with the pipe is practically stationary, but the central layer C moves with a maximum velocity (Fig. 10.33). At other layers between A and C, such as B, the velocity is less than that at C. The lengths of the arrows represent the magnitude of the velocities of various layers. The layers are cylindrical in form, one inside the other. The velocity of the layers increases from zero at the walls of the pipe to a maximum along its axis. The velocity profile is parabolic as shown in the diagram.

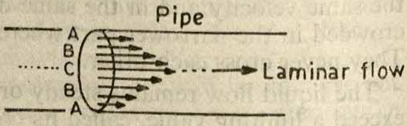


Fig. 10.33 Laminar flow in a pipe

Such a liquid-flow in which the different layers or laminae glide over one another at a slow and steady velocity, without intermixing, is called a *laminar* or *viscous flow*.

Viscosity

If we swim in a pool of water, we experience a resistance to our motion. If a liquid in a vessel is stirred and left to itself, the motion will disappear after sometime. This shows that there is some kind of a frictional force in liquids. This force is called viscous force in the case of fluids.

To understand the origin of the viscous force in a liquid, let us again consider a steady laminar flow of a liquid over a fixed horizontal surface Q (Fig. 10.32(a)). The experiment shows that (as stated above) the uppermost layer has the maximum velocity, which decreases progressively as we go to other layer below the uppermost layer until we reach the lowermost layer which is at rest.

Let us consider two adjacent layers PP′, and RR′ of the liquid at certain distances from the lowermost fixed layer QQ′ as shown in Fig. 10.34. The force acting on the upper layer PP′ tends to accelerate the motion of the lower layer RR′, whereas that acting on the lower layer tends to retard the motion of the upper layer. Thus there are two forces which act on layers in opposite directions. Consequently, the liquid between the two layers is subjected to a shearing stress. But a shearing stress cannot produce shearing strain in a liquid. it will instead cause the liquid to flow so that the stress disappears. We conclude that the force acting on the adjacent layers tends to destroy the relative motion between them. This force is called the *viscous force or viscous drag*. The property of a liquid which gives rise to such a viscous force which tends to oppose the relative motion between its layers is called *viscosity*.

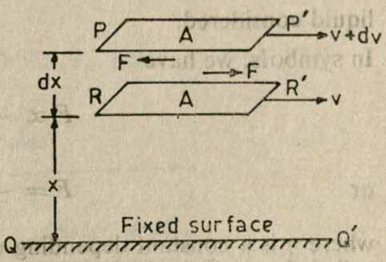


Fig. 10.34 Newton's law of viscous laminar flow

Viscosity is sometimes called the *internal friction* because it acts between two layers of the same substance. Like friction between two solid surfaces, the viscous force acts tangentially in a direction opposite to the relative motion between the different layers of the liquid. It follows, therefore, that if a relative velocity between the layers of a liquid is to be maintained, an external force must be applied to it. In the absence of the external force, the relative motion between the layers is destroyed and the flow of the liquid ceases. It is for this reason that the motion of a liquid disappears after it is stirred and left to itself. Once the liquid stops flowing, the force due to viscosity stops acting.

The viscosity of liquids such as water, alcohol, etc, which readily flow over a solid surface, is said to be low. Liquids such as glue, glycerine, coal tar etc. have a high viscosity. The viscosity of a lubricating oil is one of the factors which decides whether it is suitable for use in the engine of a machine. Thus the subject of viscosity has considerable practical importance.

Coefficient of Viscosity

Newton was the first person to study the factors that govern the viscous force in a liquid. Consider a layer RR′ of a liquid moving with a velocity v and situated at a distance x from the fixed horizontal surface QQ′ (Fig. 10.34). PP′ is another layer at a very small distance dx above RR′ moving with a velocity $(v + dv)$, where dv represents a very small increase in velocity. Then the ratio dv/dx is the rate of change of velocity with distance in a direction perpendicular to the direction of the flow of the liquid. This ratio is called the ***velocity gradient*** at the part of the liquid in consideration.

According to ***Newton's law of viscous flow***, the magnitude of the viscous force F on a certain layer of liquid is proportional to (i) the area A of the layer, and (ii) the velocity gradient $\left(\frac{dv}{dx}\right)$ at the layer of the liquid considered.

In symbols, we have

$$F \propto -A\frac{dv}{dx}$$

or

$$F = -\eta A \frac{dv}{dx} \tag{10.7}$$

where η is a constant depending upon the nature of the liquid and is called the *coefficient of viscosity* of the liquid. The negative sign indicates that the viscous force acts in a direction opposite to the direction of liquid flow. Equation (10.7) is applicable only when the flow is laminar.

It is worthwhile to recall that in the case of a solid sliding over another solid, the frictional force is independent of the area of contact and of the relative velocity between the solid surfaces concerned. This is not true of the frictional force which is called into play when a liquid layer slides over another.

From Eq. (10.7) it follows that if $A = 1\ \text{m}^2$ and $\frac{dv}{dx} = 1\ \text{m s}^{-1}\ \text{m}^{-1}$ the numerical value of η equals F newton. Thus, *the coefficient of viscosity of a liquid may be defined as the tangential viscous force per unit area which is called into play between layers of a liquid in which a unit velocity gradient exists in a direction normal to the layers.* Alternatively, *the coefficient of viscosity may be defined as the tangential force per unit area required to maintain a unit velocity gradient normal to the direction of liquid flow.*

Units of Coefficient of Viscosity It follows from Eq. (10.7) that the numerical value of η is given by

$$\eta = \frac{F}{A\,(dv/dx)}$$

Its dimensions in terms of dimensions of force, length and time are

$$\frac{F}{L^2\left(\frac{L/T}{L}\right)} = FL^{-2}\,T$$

Thus in the SI system, η is experessed in $\mathrm{Nsm^{-2}}$ (newton second per square metre) which is also called *Poiseiulles* (symbol Pl).

Dimensions of Coefficient of Viscosity In terms of the fundamental dimensions of mass, length and time, the dimensions of η are:

$$\frac{\text{Dimensions of force}}{\text{Dimensions of area} \times \text{Dimensions of velocity gradient}}$$

$$= \frac{M\,LT^{-2}}{L^2\left(\frac{L/T}{L}\right)} = \mathrm{ML^{-1}}\,T^{-1}$$

Viscosities of a few fluids are given in Table 10.2 below at the indicated temperatures.

Table 10.4 Coefficients of viscosity of some fluids

Fluid	*Temperature* (°C)	η ($\mathrm{Nsm^{-2}}$ *or* Pl)
Blood	37	1.5×10^{-3}
Light machine oil	16	0.113
Water	20	1×10^{-3}
Air	0	1.7×10^{-5}
Water vapour	100	1.25×10^{-5}
Honey	20	2×10^{-1}
Til oil	20	4×10^{-2}

Critical Velocity and Reynold's Number As mentioned earlier, Eq. (10.7) holds only for laminar flow. The flow remains streamlined or laminar only so long as its velocity does not exceed a certain limiting value called the *critical velocity*. Beyond the critical velocity, the flow becomes unsteady and disorderly with the velocity changing abruptly and haphazardly. Such a flow is called *turbulent* flow and most of the energy needed to drive the liquid is dissipated in setting up whirlpools, vortices and eddies in it.

It was Osborne Reynold who, in 1883, showed experimentally that the value of the critical velocity v_c of a liquid is given by

$$v_c = \frac{k\eta}{\rho r} \tag{10.8}$$

where η and ρ stand for the coefficient of viscosity and density of the liquid respectively, r is the radius of the tube in which the liquid flows and k is a number called *Reynold's number*.

The value of Reynold's number is usually very high. For pure water k is about 1100. The flow of a viscous liquid is said to be laminar if k of the liquid lies between 0 and about 2000. For values of k above 3000, the flow is turbulent.

It follows from Eq. (10.8) that the value of the critical velocity v_c will be high if η is large, and ρ and r are small. If follows, therefore, that the flow of liquids of higher viscosity and lower density through narrow pipes tends to be streamlined whereas the flow of liquids of lower viscosity and higher density through broad pipes tends to become turbulent, since in this case, the critical velocity becomes very small.

A Sphere Falling through a Fluid: Stokes' Law It is a common experience that a body falling through a viscous medium (liquid or gas) experiences a retarding force resulting in absorption of energy by the medium in the form of heat. As it falls, the body drags the layer of the fluid with it. The fluid at a large distance from the body is unaffected by it and is, therefore, at rest. Thus the motion of the body produces a relative motion between the different layers of the fluid. Consequently, it experiences a viscous force which tends to retard its motion. When a small object, such as a small steel ball, is released in a viscous liquid like glycerine it is found to accelerate first, but soon begins to experience a retarding force. If the weight of the ball is sufficiently small, it is found that after attaining a certain velocity the retarding force just becomes equal to the effective (apparent) weight of the ball in the liquid. The ball then does not experience any force (and hence any acceleration) and, therefore, falls with a constant velocity known as the *terminal velocity*.

Stokes' law Sir George Stokes made extensive experimental studies on the motion of small spherical bodies in various fluids and found that if a small spherical body of radius r moves with a small uniform velocity v through an infinitely extensive, homogeneous fluid of coefficient of viscosity η, the retarding force experienced by it is given by

$$F = 6\pi\eta\, rv \tag{10.9}$$

This relation is known as *Stokes' law*.

Expression for terminal velocity If ρ is the density of the falling spherical body, then the mass m of the body is given by

$$m = \text{volume} \times \text{density}$$

$$= \frac{4}{3}\pi r^3 \rho$$

The weight W of the body is given by

$$W = mg = \frac{4}{3}\pi r^3 \rho g$$

The volume of the fluid displaced by it is $\frac{4}{3}\pi r^3$ and if σ is the density of the fluid, the weight of the fluid displaced by it is the upthrust U it experiences.

Thus

$$U = \frac{4}{3}\pi r^3 \sigma g$$

Thus the effective (or apparent) weight of the body inside the fluid is given by

$$W - U = \frac{4}{3}\pi r^3 (\rho - \sigma) g$$

When the body attains a terminal velocity v_t, this downward force on it must just be balanced by the viscous force F acting upwards against motion. We therefore, have (see Fig. 10.35)

$$F = W - U$$

Using Stokes' law, we get

$$6\pi\eta r v_t = \frac{4}{3}\pi r^3 (\rho - \sigma) g$$

or

$$v_t = \frac{2}{9}\frac{(\rho - \sigma) r^2 g}{\eta} \qquad (10.10)$$

This relation tells us that the terminal velocity v_t of a small spherical body falling under gravity in a fluid depends on the radius and density of the body and on the density and viscosity of the fluid.

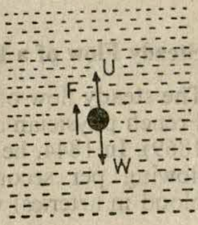


Fig. 10.35 Motion of a small spherical body through a homogeneous fluid

The graph in Fig. 10.36 illustrates how the velocity of a small sphere dropped from rest into a viscous fluid varies with time. Notice that initially the body is accelerated and after some time it acquires its terminal velocity v_t.

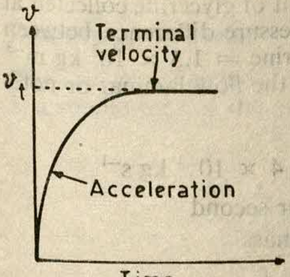


Fig. 10.36

EXAMPLE 10.17 An oil drop falls through air with a terminal velocity of 5×10^{-4} m s^{-1}. (a) Calculate the radius of the drop. (b) What will be the terminal velocity of the drop half this radius? Viscosity of air $= 1.8 \times 10^{-5}$ N s m^{-2}, density of oil $=$ 900 kg m^{-3}. Neglect the density of air compared to that of oil.

Solution

(a) Neglecting σ compared to ρ in Eq. (10.10) we have

$$v_t = \frac{2}{9}\frac{\rho r^2 g}{\eta}$$

or

$$r = \left(\frac{9\, v_t \eta}{2\rho g}\right)^{1/2}$$

$$= \left(\frac{9 \times 5 \times 10^{-4} \times 1.8 \times 10^{-5}}{2 \times 900 \times 9.8}\right)^{1/2}$$

$$= 2.14 \times 10^{-6}\ \text{m}$$

(b) The terminal velocity v_t for r^2, from the above expression. Hence when the radius is decreased by half, the terminal velocity becomes one-fourth.

$$\text{New terminal velocity} = \tfrac{1}{4} \times 5 \times 10^{-4}$$
$$= 1.25 \times 10^{-4}\ \text{m s}^{-1}$$

Steady Flow of a Fluid through a Pipe: Poiseiulle's Formula

The steady flow of a fluid through a pipe was first investigated by the French physician and physiologist Jean Louis-Marie Poiseiulle (1799-1869). In 1844, he discovered a formula for the volume (Q) of a fluid flowing per second through a pipe (while working on the flow of blood in arteries) which is

$$Q = \frac{\pi p r^4}{8\,\eta l} \tag{10.11}$$

where p = pressure difference at the ends of a pipe of radius r and length l where a fluid of viscosity η flows through it without turbulence. This formula is known as Poiseiulle's formula. A derivation of this formula, based on first principles, is given in Supplement S2 at the end of this chapter.

EXAMPLE 10.18 Glycerine flows steadily through a horizontal pipe of length 1.5 m and radius 1.0 cm. If the amount of glycerine collected at one end is 4.0×10^{-3} kg s^{-1}, what is the pressure difference between the two ends of the pipe? Density of glycerine = 1.3×10^3 kg m^{-3} and viscosity of glycerine = 0.83 N sm^{-2}. Is the flow laminar or not?

Solution

Rate of flow of mass of glycerine = 4×10^{-3} kg s^{-1}

$\therefore$ Volume of glycerine flowing per second

$$(Q) = \frac{\text{rate of flow of mass}}{\text{density}}$$
$$= \frac{4 \times 10^{-3}\ \text{kg s}^{-1}}{1.3 \times 10^{3}\ \text{kg m}^{-3}}$$
$$= 3.077 \times 10^{-6}\ \text{m}^3\ \text{s}^{-1}$$

Radius of pipe $(r) = 1.0 \text{ cm} = 1.0 \times 10^{-2}$ m

Length of the pipe $(l) = 1.5$ m

Viscosity of glycerine $(\eta) = 0.83 \text{ N sm}^{-2}$

From Eq. (10.11) the pressure difference between the two ends of the pipe is given by

$$p = \frac{8\eta l Q}{\pi r^4}$$

Substituting the values, we get

$$p = 9.75 \times 10^2 \text{ Pa}$$

To find out whether the flow is laminar or not, we find the Reynold's number

$$k = \frac{\rho r v}{\eta}$$

The velocity (v) of the flow $= \dfrac{Q}{\pi r^2} = 0.98 \times 10^{-2} \text{ m s}^{-1}$, which gives

$$k = 0.15$$

Since k is small, the flow is laminar.

10.12 FLUID FLOW

In the preceding section, we have discussed a very important property of a fluid in motion. We have found that every fluid (liquid or gas) has viscosity. In this section we shall be concerned with the equations of motion of a fluid through pipes. These equations, for a viscous fluid, are very complicated. But it turns out that many physical phenomena associated with fluid flow are satisfactorily explained even when the effect of viscosity is neglected.

Equation of Continuity

The *equation of continuity* is a fundamental equation of fluid flow and is a special case of the general law of conservation of mass.

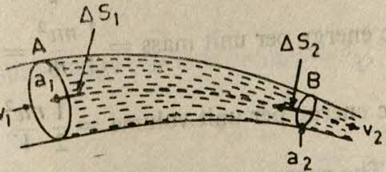


Fig. 10.37 Continuity of flow

Consider an incompressible and a non-viscous fluid flowing slowly and steadily through a pipe of a non-uniform cross-section. Let A and B be two different sections of a pipe having areas a_1 and a_2 respectively. (Fig. 10.37). Let v_1 and v_2 be the respective velocities of the liquid flow through these sections. Let Δs_1 and Δs_2 be the small distances covered by the fluid at A and B respectively in time Δt. Then, if ρ_1 and ρ_2 are the densities of the fluid at A and B respectively, we have

$$\text{Mass of fluid entering per unit time at A} = \frac{\text{Volume} \times \text{Density}}{\text{Time}}$$

$$= \frac{a_1\,\Delta s_1 \times \rho_1}{\Delta t} = a_1 v_1 \rho_1 \qquad \left(\because \quad v_1 = \frac{\Delta s_1}{\Delta t}\right)$$

Similarly the mass of fluid leaving per unit time at B $= a_2 v_2 \rho_2$. Now from the principle of conservation of mass, there can be no accumulation of mass anywhere, i.e.

$$a_1 v_1 \rho_1 = a_2 v_2 \rho_2$$

Since the fluid is incompressible, $\rho_1 = \rho_2$. Hence we have

$$a_1 v_1 = a_2 v_2$$

or

$$av = \text{Constant} \qquad (10.12)$$

This equation is called the *equation of continuity* for streamline flow. It shows that the quantity av remains constant throughout the flow provided it is steady or streamlined. Since av is constant, it follows that $v \propto \frac{1}{a}$, i.e. the velocity of fluid flow at any section of the pipe is inversely proportional to the area of cross-section of the pipe at that section.

Energy of a Fluid in a Steady Flow

A fluid in a steady or streamlined flow can have three kinds of energy; (i) kinetic energy, (ii) potential energy, and (iii) pressure energy.

Kinetic energy The energy possessed by a body by virtue of its velocity is called its kinetic energy. If m is the mass of a fluid of volume V and density ρ flowing with a velocity v, its kinetic energy (KE) is given by

$$\text{Kinetic energy of mass } m = \tfrac{1}{2} mv^2$$

$$\therefore \quad \text{Kinetic energy per unit mass} = \frac{1}{2}\frac{mv^2}{m} = \frac{1}{2}v^2$$

and

$$\text{Kinetic energy per unit volume} = \frac{1}{2}\frac{mv^2}{V} = \frac{1}{2}\rho v^2$$

Potential energy The energy possessed by a body by virtue of its position above the earth's surface is called its potential energy. If a mass m of a fluid is raised to a height h above the earth's surface, its potential energy is given by

$$\text{Potential energy of mass } m = mgh$$

$$\therefore \quad \text{Potential energy per unit mass} = gh$$

and $$\text{Potential energy per unit volume} = \rho\, gh$$

Pressure energy The energy possessed by a fluid by virtue of its pressure is called its pressure energy. A fluid under pressure can do work and thus possesses potential energy. For example, a fluid in a cylinder can derive a piston as shown in Fig. 10.38. Let p be the pressure

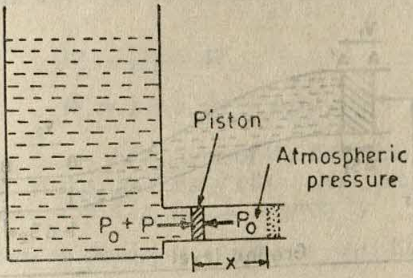


Fig. 10.38 Pressure energy in fluid

exerted by the fluid on a frictionless piston of area A. Let us say that the piston moves through a distance x due to the excess pressure p above the atmospheric pressure. Then the force exerted on the piston = Excess pressure × Area = $p \times A$. Therefore the work dW done by the fluid in giving a small displacement dx is $p \times A \times dx$. Hence the total work done in giving a displacement x to the piston is given by

$$W = \int dW = \int_0^x p \times A \times dx = pA \int_0^x dx = pAx = pV$$

where $V = Ax$ is the volume swept by the piston. This work is equivalent to the energy contained in a volume V of the fluid on account of its pressure. Thus

$$\text{Pressure energy in volume } V = pV$$

Therefore,

$$\text{Pressure energy per unit volume} = \frac{pV}{V} = p$$

$$= \text{its excess pressure}$$

and $$\text{Pressure energy per unit mass} = \frac{pV}{m} = \frac{p}{\rho}.$$

Thus the total energy per unit mass of a flowing fluid

$$= \frac{p}{\rho} + gh + \frac{v^2}{2}$$

Total energy per unit volume of a flowing fluid

$$= p + \rho gh + \tfrac{1}{2}\rho v^2$$

Bernoulli's Theorem

In the year 1738, Daniel Bernoulli proposed a theorem which states that *the total energy of an incompressible, non-viscous fluid in a streamline flow remains constant through the flow.* The theorem may be deduced as follows.

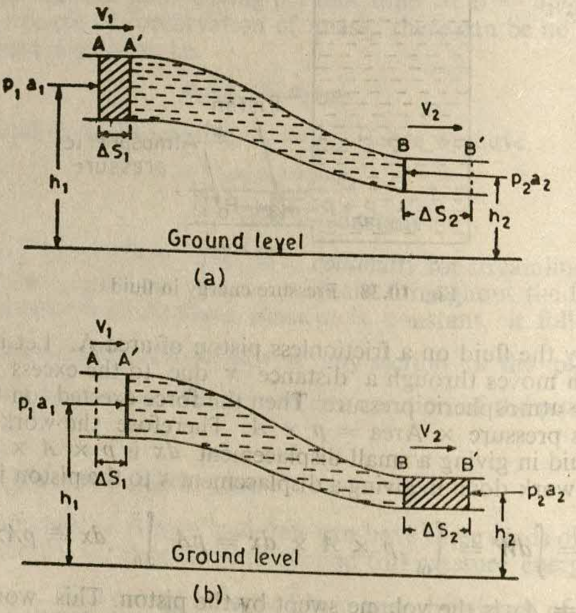


Fig. 10.39 Bernoulli's theorem. A portion of the fluid moves through a section of a pipe from the position shown in (a) to that shown in (b)

Consider a non-viscous and incompressible fluid flowing steadily through a pipe of a non-uniform cross-section (Fig. 10.39). Let us consider the flow of the fluid between two sections A and B of the tube. Let a_1 be the area of cross-section at A, v_1 the fluid velocity, p_1 the fluid pressure and h_1 the mean height above the ground level. Let a_2, v_2, p_2 and h_2 be the values of the corressponding quantities at B.

In a small time interval Δt the fluid layer A advances to A′ by a small distance $\Delta s_1 = v_1 \Delta t$ under the influence of a force $p_1 a_1$. Hence the work done by the force p_1a_1 on the fluid element enclosed in AA′ is given by, say W_1, where

$$W_1 = \text{force} \times \text{distance} = p_1a_1v_1\Delta t$$

In the same time interval Δt, the fluid layer at B advances to B′ by a small distance $\Delta s_2 = v_2\,\Delta t$, against the force p_2a_2 exerted by the fluid pressure p_2 on the layer of area a_2. Hence the work done against the force p_2a_2 by the fluid element enclosed in BB′ is given by, say W_2, where

$$W_2 = p_2a_2v_2\,\Delta t$$

Thus the net work ΔW done on the fluid by the fluid pressure is given by

$$\Delta W = W_1 - W_2$$

or

$$W = p_1a_1v_1\,\Delta t - p_2a_2v_2\,\Delta t \tag{10.13}$$

Now $a_1v_1\,\Delta t = a_1\,\Delta s_1$ is the volume of the fluid that crosses the section A in time Δt. If ρ is the density of the fluid, the mass Δm_1 of the fluid crossing the section A in time Δt is given by

$$\Delta m_1 = \text{density} \times \text{volume} = \rho a_1v_1\,\Delta t$$

Since the fluid is incompressible, the density of the fluid crossing the section B must also be ρ. Therefore, the mass Δm_2 of the fluid crossing the section B in time Δt is given by

$$\Delta m_2 = \rho a_2v_2\,\Delta t$$

But from the principle of conservation of mass, there can be no accumulation of mass anywhere in the pipe. Hence

$$\Delta m_1 = \Delta m_2 = \Delta m \text{ (say)}$$

or

$$\rho a_1v_1\,\Delta t = \rho a_2v_2\,\Delta t = \Delta m$$

or

$$a_1v_1\,\Delta t = a_2v_2\,\Delta t = \frac{\Delta m}{\rho}$$

or

$$a_1v_1 = a_1v_2 \tag{10.14}$$

This is the equation of continuity already obtained earlier. Substituting in Eq. (10.13), we have

$$\Delta W = \frac{\Delta m}{\rho}(p_1 - p_2) \tag{10.15}$$

As the fluid has moved from a position AB to a new position A′B′, there is a decrease in its potential energy which is equal to the difference between the potential energy (Δmgh_1) of mass Δm when it was enclosed between AA$'$ and the potential energy ($\Delta m g h_2$) of mass Δm when it is now enclosed between BB$'$, i.e.

$$\begin{aligned}\text{Decrease in potential energy} &= \Delta m g h_1 - \Delta m g h_2 \\ &= \Delta m g\,(h_1 - h_2) \qquad (10.16)\end{aligned}$$

The work ΔW done by the fluid and the decrease in its potential energy together are responsible for an increase in the kinetic energy of the fluid. Now,

$$\begin{aligned}\text{Increase in kinetic energy} &= \tfrac{1}{2}\,\Delta m v_2^2 - \tfrac{1}{2}\,\Delta m v_1^2 \\ &= \tfrac{1}{2}\,\Delta m\,(v_2^2 - v_1^2) \qquad (10.17)\end{aligned}$$

Now from the principle of conservation of energy, we have

Work done by the fluid + decrease in PE = increase in KE

Using Eqs (10.15), (10.16) and (10.17), we have

$$\frac{\Delta m}{\rho}(p_1 - p_2) + \Delta m g\,(h_1 - h_2) = \frac{1}{2}\,\Delta m\,(v_2^2 - v_1^2)$$

or

$$\frac{p_1}{\rho_1} + g h_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho_2} + g h_2 + \frac{1}{2} v_2^2$$

i.e.

$$\frac{p}{\rho} + g h + \frac{1}{2} v^2 = \text{constant} \qquad (10.18)$$

This is the familiar form of Bernoulli's theorem. If we consider a unit mass of the fluid, this equation implies that the sum of pressure energy (p/ρ), potential energy (gh) and kinetic energy ($\frac{1}{2}v^2$) of a unit mass of a fluid remains constant. Thus the total energy of a unit mass of a fluid remains constant. Now, what is true of a unit mass is also true of any mass of the fluid. Therefore we conclude that *the total energy of any incompressible and non-viscous fluid in a streamline flow remains constant throughout the flow*. This is the statement of *Bernoulli's theorem.*

Since density ρ of an incompressible fluid is constant, we can multiply Eq. (10.18) by ρ and write

$$p + \rho g h + \tfrac{1}{2}\rho v^2 = \text{constant} \qquad (10.19)$$

This is another form of Bernoulli's theorem in terms of pressure. It can easily be seen that every term in Eq. (10.19) has the dimensions of pressure. The term ($p + \rho g h$) is called the *static* pressure, because it is the pressure of the fluid even if it is at rest, and the term $\frac{1}{2}\rho v^2$ is called the *dynamic* pressure of the fluid which is the pressure by virtue of its velocity v. Thus we have

Static pressure + dynamic pressure = constant

We can write Bernoulli's theorem in yet another form. Since g is also a constant for a given place, Eq. (10.19) can be rewritten as

$$\frac{p}{\rho g} + h + \frac{1}{2}\frac{v^2}{g} = \text{constant}$$

This is Bernoulli's theorem in terms of lengths of *heads*. Every term here has dimensions of a length and is called a *head*. Thus $\frac{1}{2}\frac{v^2}{g}$ is the *velocity head*, $\frac{p}{\rho g}$ is the *pressure head* and h is the *gravitational head*, so that pressure head + gravitational head + velocity head = constant.

We will now consider a few special cases of Bernoulli's theorem.

Case 1: *Pressure at a Certain Depth in a Stationary Liquid* Consider a fluid at rest in a vessel (Fig. 10.40). Its velocity is zero everywhere. At point A on the surface of the fluid, the pressure is atmospheric, let us call it P_0. Let p be the pressure; at a point B at the depth h below the surface. Thus we may write: For point A.

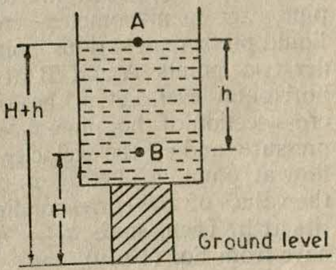


Fig. 10.40 Hydrostatic pressure

$$p_1 = P_0, \quad h_1 = H + h, \quad v_1 = 0$$

and for point B.

$$p_2 = P_0 + p \quad h_2 = H, \quad v_2 = 0$$

Substituting in Eq. (10.19), we have

$$P_0 + \rho g\,(H + h) + 0 = (P_0 + p) + \rho g H + 0$$

or

$$p = \rho g h$$

which is Eq. (10.6) obtained in Sec. 10.7.

Case 2: *Velocity of Efflux* (*Torricelli's Theorem*) Let us consider a vessel that has a small opening (or orifice) through which the liquid gradually flows out (Fig. 10.41). We assume that the flow is so slow that the speed of the fall of the surface of the liquid can be neglected. Now the pressure at a point, such as A, of the surface is atmospheric. The pressure of the liquid at the end of the opening B is also atmospheric. say, P_0.

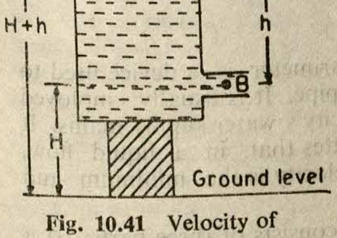


Fig. 10.41 Velocity of efflux

Now referring to Fig. 10.41, for point A.

$$p_1 = P_0, \quad h_1 = H + h \quad v_1 = 0$$

and for point B,

$$p_2 = P_0, \quad h_2 = H, \quad v_2 = v \quad \text{(say)}$$

where v is the velocity of efflux of the liquid. It is the velocity with which the liquid flows out of the opening. Substituting in Eq. (10.19), we have

$$P_0 + \rho g\,(H + h) + 0 = P_0 + \rho g H + \tfrac{1}{2}\,\rho\, v^2$$

or
$$\tfrac{1}{2}\rho v^2 = \rho g h$$

or
$$v = \sqrt{2\,gh}$$

This shows that the velocity of the efflux through an orifice at a certain depth below the free surface of a liquid is the same as that attained by another body falling through the same height. This result is known as Torricelli's theorem.

Case 3 *Liquid Flowing through a Horizontal Non-uniform Pipe* Consider a liquid flowing steadily through a horizontal non-uniform pipe as shown in Fig. 10.42. Narrow vertical tubes P and Q, which are fused to the pipe, act as manometers to measure liquid pressure at various points. Consider two points A and B at the same horizontal level. Let a_1 be the area of cross-section of the pipe, p_1 the liquid pressure and v_1 the velocity of liquid flow at point A. Let a_2, p_2, and v_2 be the values of the corresponding quantities at B. Then, since $h_1 = h_2 = h$, we have from Eq. (10.19)

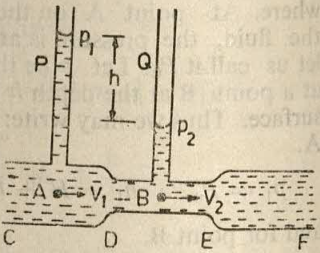


Fig. 10.42 A venturimeter

$$p_1 + \rho g h + \tfrac{1}{2}\,\rho v_1^2 = p_2 + \rho g h + \tfrac{1}{2}\,\rho v_2^2$$

or
$$(p_1 - p_2) = \tfrac{1}{2}\,\rho\,(v_2^2 - v_1^2) \qquad (10.20)$$

Now since the area of cross-section at B is smaller than that at A, it follows that $v_2 > v_1$. Equation (10.20) than tells us that $p_1 > p_2$. This result is called Bernoulli's principle which may be stated as 'when an incompressible and non-viscous liquid flows steadily through a non-uniform pipe, the pressure of the liquid is lower where the liquid velocity is higher, (i.e. where the cross-section is smaller) and vice versa'. This principle is used in a venturimeter as described below.

Applications of Bernoulli's Theorem

Venturimeter (*or Flowmeter*) The venturimeter is a device used to measure the rate of flow of a liquid in a pipe. It is usually employed to measure the rate of flow of water in a city's water supply mains. It is based on Bernoulli's principle which states that, in a liquid flow, the pressure is the maximum where the velocity is the minimum and vice versa.

As shown in Fig. 10.42 a venturimeter consists of three parts: (i) a pipe CD of area a_1 at its outer end C and tapering down to a much smaller cross-sectional area a_2 at the other end, (ii) a short horizontal tube DE of uniform cross-sectional area a_2 and (iii) a pipe EF with cross-sectional area a_1 at its outer end F.

Let p_1 and p_2 be the liquid pressures at points A and B respectively where the velocities of the liquid flow are v_1 and v_2. Then from Eq. (10.20) above, we have

$$p_1 - p_2 = \tfrac{1}{2}\rho\,(v_2^2 - v_1^2)$$

$$= \frac{1}{2}\rho v_1^2\left(\frac{v_2^2}{v_1^2} - 1\right)$$

Now from the equation of continuity, we have

$$v_1 a_1 = v_2 a_2 \qquad \text{or} \qquad \frac{v_2}{v_1} = \frac{a_1}{a_2}$$

$$\therefore \qquad p_1 - p_2 = \frac{1}{2}\rho v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right)$$

From Fig 10.42 it is clear that $p_1 - p_2 = h\rho g$. Therefore

$$h\rho g = \frac{1}{2}\rho v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right)$$

or

$$v_1 = \sqrt{\frac{2\,gh}{\left(\dfrac{a_1^2}{a_2^2} - 1\right)}}$$

knowing g, h, a_1 and a_2, velocity v_1 at A is determined. The rate of liquid flow is then obtained from the equation

$$\text{Volume crossing section A per second} = v_1 a_1$$

The Pitot Tube The Pitot tube is a simple device, based on Bernoulli's principle, for measuring the velocity of flow at any depth in a flowing liquid. Henry Pitot, in the year 1730, accidentally dipped an open-ended L-shaped tube (Fig. 10.43) in a river and observed that water rose in the tube. Such a tube came to be known as a Pitot tube.

The arrangement illustrated in Fig. 10.43 consists of a vertical tube Q and an L-shaped Pitot tube P. The tubes have a small aperture at ends A and B. The plane of the aperture at A and the plane of the aperture at B are respectively parallel and perpendicular to the direction of the flow. The ends A and B are placed very near each other and at the same height, say, H from the ground level. Let v be the velocity of the liquid at this plane and let its pressure at A be p_1. Total energy per unit volume at point A is $(p_1 + \frac{1}{2}\rho v^2 + \rho g H)$. Now, since the plane of the aperture at B is

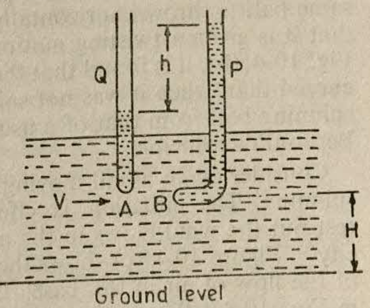


Fig. 10.43 A Pitot tube

normal to the liquid flow, the aperture is directly facing the liquid flow. The flow will, therefore, be stopped at the nose B of tube P. This point B is called the *stagnation point*. The whole of the kinetic energy of the liquid is converted into potential energy which will cause the liquid to rise in the tube P to a certain height. Thus the total pressure at B will be different from that at A. Let this pressure be p_2. Since $v = 0$ at B, the total energy per unit volume at point B is $(p_2 + \rho gH)$. According to Bernoulli's theorem.

Total energy at A = Total energy at B

or $$p_1 + \frac{1}{2}\rho v^2 + \rho gH = p_2 + \rho gH$$

or $$\frac{1}{2}\rho v^2 = p_2 - p_1$$

or $$\frac{1}{2}\rho v^2 = \rho gh \quad (\because\ p_2 - p_1 = \rho gh)$$

as shown in Fig. 10.43.

or $$v = \sqrt{2\,gh}$$

Knowing g and h, the velocity of the liquid at point A is determined.

Sprayer or Atomizer The working of an atomizer that is used to spray liquids is based on Bernoulli's principle. Fig. 10.44 shows the essential parts of an atomizer. When the rubber balloon is pressed the air rushes out of the horizontal tube B decreasing the pressure in this tube to p_2 which is less than the atmospheric pressure p_1 in the container. Consequently the liquid rushes up the vertical tube A. When it collides with the high speed air in tube B, the liquid breaks up into a fine spray.

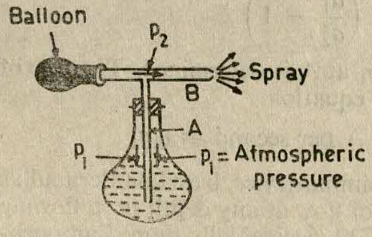


Fig. 10.44 The atomizer

Curved Path of a Spinning Ball Take a cricket ball and throw it horizontally with a certain initial velocity. The ball follows a curved path as shown in Fig. 10.45(a) (see projectile motion, Chapter 4). When the same ball is thrown horizontally with the same velocity in such a way that it is given a twisting motion so as to cause it to spin as shown in Fig. 10.45(b), it is found that the path of the ball is much more sharply curved than when it was not spinning. The deviation of the path of the spinning ball from that of a usual projectile can be easily explained by Bernoulli's principle.

Consider again a ball moving horizontally with a velocity v without spinning. The situation is effectively the same as if the ball were at rest but the wind blows in the opposite direction with the same velocity v. Figure 10.45(c) shows the shape and direction of the streamlines of the flow of air in this case. But when the ball is spinning as shown in Fig. 10.45(d), on the upper side A of the ball, the air moves in a direction opposite to that of the spin. Since the spinning ball is not smooth, it drags some air with it while spinning. This causes a decrease in the resultant velocity of air above A. But on the lower side B, the air velocity is helped by the spinning motion; both being in the same direction. Consequently the resultant velocity of air below B increases. According to Bernoulli's principle, a reduction in velocity must be accompanied by a corresponding increase in pressure and vice versa. Thus the air pressure above A increases while that below B decreases. As a result the ball curves downwards in the direction of the spin as shown in the figure.

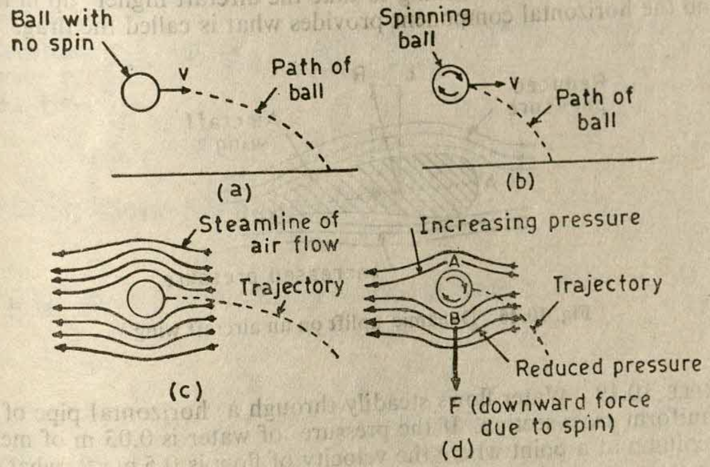


Fig. 10.45 Curving of a spinning ball

Dynamic Uplift If the ball moving to the right were also spinning in an anticlockwise sense about an axis perpendicular to the plane of the paper (rather than in clockwise sense as shown in Fig. 10.45 d), then on the upper side A of the ball the air would move in the same direction as that of the spin, resulting in an increase in the velocity of air above A. On the lower side B, the air would move in a direction opposite to that of the spin. The resultant air velocity, therefore, decreases. From Bernoulli's principle, the pressure above A is now lower than that below B. Consequently the ball would experience an uplift and its trajectory would be less curved than when it had no spin. This uplift is called the *dynamic uplift* which must be distinguished from the *static uplift* which is the buoyant force that acts on an object in a fluid in accordance with Archimedes' principle. Similarly, an aircraft experiences an uplift by virtue of its motion through air.

Aerofoil: Lift on an Aircraft Wing *Aerofoil* is the name given to an object shaped in such a way as to cause dynamic uplift by virtue of its horizontal motion through air. This upward force makes aeroplanes fly.

Figure 10.46 shows a section of a wing of an aircraft in the plane of the paper. The wing is so designed that its upper surface is slightly more curved (and hence longer) than the lower surface and the front edge A is broader than the rear edge B. As the aircraft moves in the indicated direction, the streamlines of the air flow curve around the wing at A and meet at B at the same time. Thus the velocity of air moving along the upper surface is higher than that along the lower surface. Now, according to Bernoulli's principle, the air pressure on the upper surface is reduced below atmospheric pressure whereas that on the lower surface is increased above atmospheric pressure. Hence, the wing experiences an upthrust R. The vertical component of the thrust provides the lift L tending to take the aircraft higher up in the air and the horizontal component provides what is called the drag.

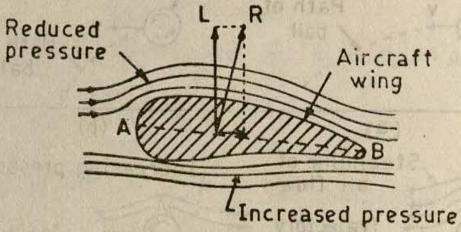


Fig. 10.46 Dynamic uplift on an aircraft wing

EXAMPLE 10.19 Water flows steadily through a horizontal pipe of a non-uniform cross-section. If the pressure of water is 0.05 m of mercury column at a point where the velocity of flow is 0.5 m s^{-1}, what is the pressure at another point where the velocity of flow is 0.75 m s^{-1}?

Solution According to Bernoulli's theorem, for a horizontal flow at a height h from ground level,

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g h = p_2 + \tfrac{1}{2}\rho v_2 + \rho g h$$

$\therefore$ $$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

or $$p_2 = p_1 + \tfrac{1}{2}\rho\,(v_1^2 - v_2^2)$$

Here $p_1 = 0.05$ m of Hg $= 0.05 \times 9.8 \times 13600$ N m^{-2}

$$v_1 = 0.5 \text{ m s}^{-1}$$

$$v_2 = 0.75 \text{ m s}^{-1}$$

$$\rho = 1000 \text{ kg m}^{-3}$$

$$\therefore\ p_2 = 0.05 \times 9.8 \times 13600 + \tfrac{1}{2} \times 1000\,\{(0.5)^2 - (0.75)^2\}$$

$$= 6539.56 \text{ N m}^{-2}$$

$$= \frac{6539.56}{9.8 \times 13600} = 0.049 \text{ m of Hg column}$$

EXAMPLE 10.20 Water stands at a depth H in a tank whose side walls are vertical. A hole is made in one of the walls at a depth h below the water surface. The emerging stream of water strikes the floor at a distance R from the tank. (i) Obtain an expression for R in terms of H and h, (ii) Show that R is maximum if $h = H/2$.

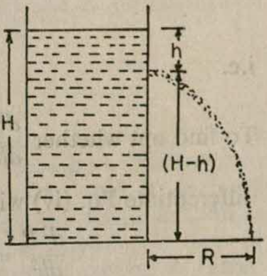


Fig. 10.47

Solution (i) Referring to Fig. 10.47, let h be the depth of the hole below the free surface of water.

According to Torricelli's theorem, the velocity of efflux v of water through the hole is given by

$$v = \sqrt{2gh} \tag{i}$$

The height through which water falls is

$$S = H - h$$

If t is the time taken by water to strike the floor, then

$$S = \tfrac{1}{2} gt^2$$

or

$$H - h = \tfrac{1}{2} gt^2$$

giving

$$t = \sqrt{\frac{2(H-h)}{g}} \tag{ii}$$

The distance R where the emerging stream strikes the floor is given by

$$R = vt$$

Substituting for v and t from Eqs (i) and (ii), we get

$$R = \sqrt{2gh} \times \sqrt{\frac{2(H-h)}{g}} = 2 \times \sqrt{h(H-h)}$$

which is the required expression.

(ii) R will be maximum at value of h for which $\frac{dR}{dh} = 0$

and

$$\frac{d^2R}{dh^2} < 0$$

Now

$$R = 2(hH - h^2)^{1/2} \tag{iii}$$

Differentiating with respect to h, we have

$$\frac{dR}{dh} = 2 \times \frac{1}{2}(hH - h^2)^{-1/2}(H - 2h) = \frac{(H-2h)}{(hH - h^2)^{1/2}} \tag{iv}$$

It is clear that $\frac{dR}{dh}$ will be zero at a value of h given by

$$H - 2h = 0$$

i.e.

$$h = \frac{H}{2}$$

To find out whether $\frac{d^2R}{dh^2}$ is negative at this value of h, we differentiate Eq. (iv) with respect to h to get

$$\frac{d^2R}{dh^2} = -\frac{2h}{(H-h)^{1/2}}\left\{1 + \frac{1}{4}\frac{(H-2h)}{h(H-h)}\right\}$$

Putting $h = H/2$, we have

$$\left(\frac{d^2R}{dh^2}\right)_{h=\frac{H}{2}} = -\sqrt{2H}$$

which is negative. Hence R will be maximum at $h = H/2$. Thus for range R to be maximum the hole must be exactly in the middle of the tank.

SECTION C: LIQUIDS

10.13 SURFACE TENSION

We will now consider a liquid surface phenomenon called surface tension. We will show that this phenomenon is due to intermolecular attraction.

It is a well-known fact that some insects are able to walk on the surface of water. We have often seen a drop of water which remains suspended for some time from a tap before it falls, as if some force were holding it there. When a big drop of mercury splits, it gathers into small droplets. We know that when we gently place a sewing needle on the surface of water in a beaker, it may float on water although its density is eight times that of water (see Fig. 10.48). If you watch closely, you will see that immediately below the needle, there is a slight depression on the water surface as if there were some kind of a membrane stretched so as to support the needle. If the needle is heavy enough it can pierce this 'membrane' and sink to the bottom of the beaker. These observations sug-

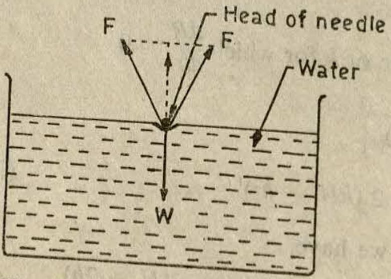


Fig. 10.48 A needle floating on water

gest that the surface of a liquid behaves like a very fine elastic skin or stretched membrane. This property by virtue of which the free surface of a liquid at rest behaves like a stretched membrane is called surface tension.

Molecular Theory of Surface Tension

The fact that a liquid surface is in a state of strain or tension can be explained in terms of the intermolecular forces (see Ch 9). The horizontal line PQ in Fig. 10.49 represents the surface of a liquid in a large container. Let us first consider the forces acting on a molecule such as A which is well below the surface, i.e. which is situated in the bulk of the liquid. It is clear that there is a certain range around A within which it can feel the influence of its neighbours. The molecules far away from A exert negligible forces on A. The range of influence has been shown by drawing a sphere around A. The molecules present in this sphere exert forces on A. The molecule A is surrounded by an equal number of molecules in all directions within the sphere around it. The closer molecules exert repulsive forces on A while the farther ones exert attractive forces. Since, by symmetry, the molecules are uniformly distributed around A, the attractive forces balance the repulsive forces; the net effect is that the average intermolecular force between A and the surrounding molecules is zero.

Consider next a molecule such as B which lies near the liquid surface and has a part of the upper half of its sphere of influence lying outside the surface as shown in the figure.

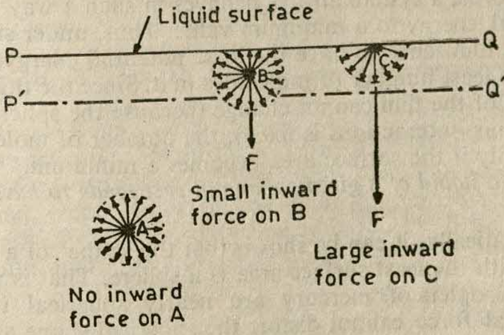


Fig. 10.49 Molecular forces in a liquid

Since the number of molecules of the liquid lying in the lower half of the sphere around B is much larger than that in the upper half, the molecule B experiences some resultant force tending to bring in inside the liquid.

Consider now a molecule such as C which lies on the surface PQ and has the upper half of its sphere of influence lying completely outside the liquid surface. There are very few particles on the vapour side above C compared to those in the liquid below. Consequently this molecule experiences the maximum force directed inward towards the interior of the liquid. This applies to all the other molecules on the surface of the liquid. Thus the surface itself experiences a net force or tension.

Energy of a Liquid Surface

Let us now construct an imaginary plane P′Q′ parallel to the free surface PQ of the liquid so that the distance PP′ (or QQ′) is equal to the radius of the sphere of influence. The liquid enclosed between the planes PQ and P′Q′ is called the surface film. It is evident that the molecules below the surface film do not experience any net force, but those in this film experience a net inward force. Therefore, if you try to pull a molecule in this film near the free surface, work will have to be done to overcome the inward force. The molecules at the surface PQ experience a maximum inward force and hence require maximum amount of work to be pulled out of the surface. In other words, the molecules in the surface film have potential energy by virtue of their position there. It is clear that the molecules on the surface of a liquid have more potential energy than those below it. This energy is called the *surface energy.*

Shape of Drops

The potential energy of any system in stable equilibrium is minimum. In other words, a system always behaves in such a way as to reduce its potential energy to a minimum value. Thus, under surface tension, the surface film tends to have the least potential energy, i.e. it tends to have the least number of molecules in it. Since the thickness PP′ (see Fig. 10.49) of the film cannot change (because the sphere of influence of molecular interactions is fixed), the number of molecules in it can decrease only if the surface area becomes a minimum. Thus *the free surface of a liquid of a given volume at rest tends to have the least surface area.*

Mathematically, it can be shown that the shape of a given volume of liquid with the least surface area is a sphere. That is why raindrops and small droplets of mercury are nearly spherical in shape. The gravitational force cannot distort the spherical shape appreciably due to the very small mass of tiny droplets.

A soap bubble is almost exactly spherical because the gravitational force on it is extremely small and the shape of the liquid is then mainly due to surface tension.

Let us consider a small drop of mercury. Although the density of mercury is high, a very small drop of it is spherical. Let us say that its radius is r. Then its surface area is $4\pi r^2$ and its volume is $\frac{4\pi r^3}{3}$. The ratio of surface area and volume is clearly proportional to $\frac{1}{r}$. Thus the smaller the radius, the greater is the influence of surface tension compared to the weight (or gravitation). Larger drops of mercury are flattened on top due to the relatively large weight of mercury. In this case, the shape is such that the sum of the gravitational potential energy and the surface potential energy must be a minimum. Hence, the centre of gravity must move down as low as possible, which explains the flattening of larger drops.

Definition of Surface Tension

We have seen that the free surface of a liquid behaves like an elastic membrane under tension, tending to contract so as to have minimnm surface area. This behaviour of a liquid surface suggests that there exists a force on the surface of a liquid that creates a surface of the smallest area. The property by virtue of which the free surface of a liquid at rest behaves like an elastic membrane under tension which tends to contract the surface so as to have minimum surface area is called surface tension.

A blown-up balloon has a surface in a state of tension. Draw a line on its surface. Since any point on this line is acted upon by two equal and opposite forces (because it is in equilibrium) we can say that the two equal and opposite forces act on the line itself. If the balloon is cut along the line with a razor blade, the rubber is drawn away from the line by the two forces present and the balloon bursts open.

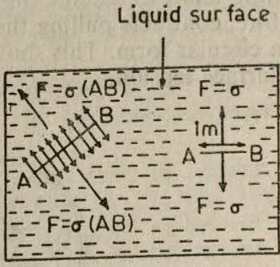


Fig. 10.50 Definition of surface tension

Similarly any imaginary line (such as AB in Fig. 10.50) drawn on the flat surface of a liquid is acted upon by equal and opposite forces, perpendicular to the line on either side. Since every molecule of the liquid in the line AB experiences equal and opposite forces when in equilibrium, it is clear that the force acting on this line will be proportional to the length of the line. If l is the length of the imaginary line and F the total force acting on either side of it, we have

$$F \propto l$$

$$F = \sigma l$$

where σ, the constant of proportionality, is called the surface tension of the liquid. If $l = 1$, $\sigma = F$.

Thus, *surface tension is measured by the force (or tension) acting per unit length of an imaginary line on the liquid surface, the direction of the force being perpendicular to this line and tangential to the liquid surface*. A much more meaningful definition of surface tension in terms of surface energy will be given in Sec. 10.14.

It may be mentioned that the magnitude of surface tension is quite independent of the area of the liquid surface. The above example of an inflated balloon is, therefore, not quite appropriate because here the tension on the surface of the balloon increases if its surface area is increased (by inflating).

Unit and Dimensions of Surface Tension

In the SI system, surface tension σ is measured in newton per metre (N m^{-1}). The dimensions of σ are

$$\frac{\text{Dimensions of force}}{\text{Dimensions of length}} = \frac{\text{ML T}^{-2}}{\text{L}} = \text{MT}^{-2}$$

Experiments that Illustrate Surface Tension

1. Take a circular wire frame and support a loop of cotton thread by means of two small threads tied to its frame as shown in Fig. 10.51(a). Dip the ring in soap solution. Remove it from the solution. A soap film in the ring is formed on which the loop of thread forms an irregular outline. Now prick the film within the loop by a needle. The film in the loop suddenly vanishes and the loop of thread assumes a circular form as shown in Fig. 10.51(b). Since for a given perimeter, a circle encloses the maximum area, it follows that the film surrounding the loop tends to assume the least area. The film outside the loop, therefore, contracts pulling the loop equally at all points so that it assumes a circular form. This shows that a liquid film tends to contract due to surface tension.

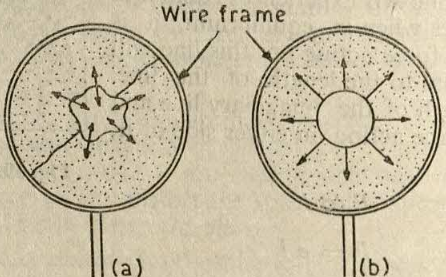


Fig. 10.51 A soap film in a circular frame

2. We have mentioned above that, due to surface tension, the free surface of a liquid tries to contract to have a minimum area. For a given volume, a sphere has the least area. That is why tiny raindrops are spherical. A tiny mercury globule on a plane glass plate is also spherical because of surface tension. We have explained above why a

bigger globule of mercury will be flattened on top due to a gravitational force which can no longer be neglected.

If the gravitational force were absent, the drop would have been perfectly spherical in shape. This has been beautifully demonstrated by what is called Plateau's spherule experiment illustrated in Fig. 10.52. He prepared a mixture of water and alcohol of density equal to that of olive oil. He put the mixture in a beaker and carefully introduced a drop of olive oil in it with the help of a pipette as shown. Since the density of the olive oil drop is the same as that of the surrounding liquid, the weight of the drop is counterbalanced by the upthrust of the surrounding liquid. Thus the effect of the gravitational force is eliminated. Therefore, the drop is perfectly spherical.

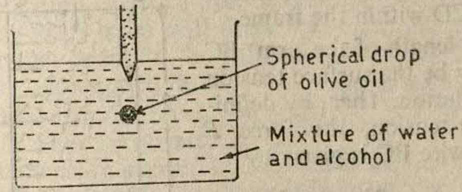


Fig. 10.52 Plateau's spherule experiment

3. The fact that a liquid drop becomes spherical due to surface tension in the absence of external forces is used in the manufacture of lead shot. A fine stream of molten lead is poured down from a tower into a tank of cold water. The stream breaks into tiny droplets of lead which assume a spherical shape and solidify as they drop. The tiny lead drops have very little mass and hence the gravitational force is too small to distort their shape.

4. A tiny sewing needle floats on the surface of water as shown in Fig. 10.48. The floating needle causes a little depression in the surface of water below it. The forces F due to surface tension of the curved surface are inclined as shown in the figure. The vertical components of these two forces support the weight of the needle.

10.14 SURFACE ENERGY

We have seen above that the molecules on the free surface of a liquid experience an inward force due to molecular interactions. These molecules possess potential energy since work will have to be done to pull them from the surface to overcome the inward force due to surface tension. As explained earlier, a molecule on the surface has a higher potential energy than one deep in the interior of the liquid. In creating a surface the liquid has to pull these molecules up to the surface.

Thus work has to be done in increasing the number of molecules on the surface. *The energy required to create a surface is called surface energy and is defined as the amount of work done to produce a fresh surface of liquid film of a unit area.* Alternatively surface energy may be defined as *the potential energy per unit area of the liquid surface.*

Calculation of Surface Energy

Take a rectangular frame ABCD of a wire. The arm BC of the frame is not fixed to the frame but can slide parallel to side AB resting on the arms AB and CD as shown in Fig. 10.53. The frame gently dipped in a soap solution to form a soap film ABCD within the frame.

Let l be the length of the arm BC (or AD) and σ be the surface tension of the soap solution. Then, by definition of surface tension, the force F acting on the wire BC is given by

$$F = \sigma \times 2l = 2\sigma\, l$$

The factor 2 appears in this expression because the film has two surfaces, each of length l.

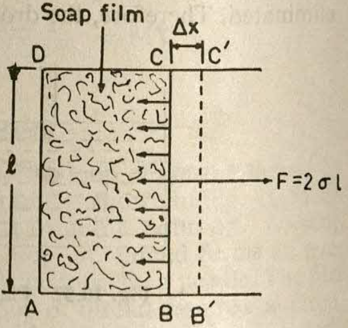


Fig. 10.53 Stretching of a film

Now let us slide the arm BC to a new position B′ C′ by a small distance Δx. The film is now stretched. The increase in the surface area of the film is given by

$$\Delta S = 2l\, \Delta x$$

It is clear that, in stretching the film, work has to be done on the film against the inward force F due to surface tension. Let the work done be ΔW, which is given by

$$\Delta W = F\, \Delta x = (2\sigma\, l)\, \Delta x$$

$$= \sigma\, \Delta S$$

or work done = surface tension × increase in the area of the film. If $\Delta S = 1$, $\Delta W = \sigma$. Thus, the work done to increase the surface area of a film by unity is equal to the surface tension of the liquid. This gives us a much more meaningful definition of surface tension. *The surface tension of a liquid is the amount of work done in increasing the surface area of a film of the liquid by a unit amount.* This work is stored as the surface energy in the soap film. Thus the surface tension is numerically equal to the free surface energy per unit area. Defined in this way, surface tension is measured in the units of work (or energy) per unit area, i.e. J m^{-2}.

Measurement of Surface Tension

In practice, the force acting on wire BC (Fig. 10.53) can be measured by a very simple experiment. Take a rectangular frame of a clean wire and suspend it from one arm of a sensitive beam balance (Fig. 10.54). A beaker containing soap solution is placed under the frame and its height adjusted till the frame is completely dipped in the solution. The beam is balanced by adding mass on the other pan. The beaker is now gently lowered a little so that the portion of the frame which is now outside the solution carries a film of the soap solution. More mass will have to be added for the beam to be balanced again. Let the mass added be m.

The film has two surfaces, each of length, say, l. The downward force N due to surface tension exerted by the film is $2\sigma\, l$. We have neglected the mass of the film itself. This force is balanced by the weight mg.

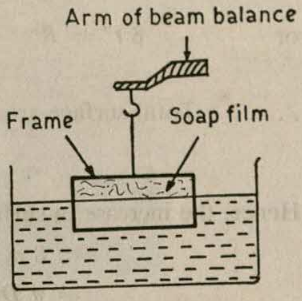


Fig. 10.54 Measuring force due to surface tension

Thus,

$$2\sigma\, l = mg$$

or

$$\sigma = \frac{mg}{2l}$$

This experiment demonstrates the action of surface tension and its measurement in a very simple manner.

EXAMPLE 10.21 Calculate the work done to break up a drop of mercury of diameter 1.0×10^{-2} m into eight drops, all of the same size. Surface tension of mercury is 0.035 N m^{-1}.

Solution To calculate the work done, we have to determine the increase in the surface area. Let R be the radius of the big drop.

$$\text{Surface area of big drop} = 4\pi R^2$$

$$\text{Volume of the big drop} = \frac{4}{3}\pi R^3$$

Let r be the radius of each tiny drop. Then the total surface of eight drops $= 8 \times 4\pi r^2 = 32\pi r^2$

$$\text{Total volume of eight drops} = 8 \times \frac{4\pi}{3} r^3 = \frac{32\pi}{3} r^3$$

As the total volume remains unchanged, we have

$$\frac{32\,\pi}{3}\,r^3 = \frac{4\pi}{3}\,R^3$$

or $\quad 8\,r^3 = R^3 \quad$ or $\quad 2r = R \quad$ or $\quad r = \frac{R}{2}$

$\therefore \quad$ Total surface area of eight drops $= 32\,\pi\,r^2 = 32\,\frac{\pi\,R^2}{4}$

$$= 8\,\pi\,R^2$$

Hence, the increase in surface area

$$= 8\,\pi\,R^2 - 4\,\pi\,R^2 = 4\,\pi\,R^2$$
$$= \pi\,D^2$$

where $D = 2R$, is the diameter of the big drop.

Thus,

$$\text{Work done} = \text{surface tension} \times \text{increase in area}$$
$$= \sigma\,\pi\,D^2$$
$$= 0.035 \times 3.142 \times (1.0 \times 10^{-2})^2$$
$$= 1.1 \times 10^{-5}\ \text{J}$$

EXAMPLE 10.22 Calculate the work done in blowing a soap bubble of radius 5 cm. Surface tension of soap solution $= 3.0 \times 10^{-2}$ N m^{-1}.

Solution

$$\text{Radius of bubble } (r) = 5\ \text{cm} = 5 \times 10^{-2}\ \text{m}$$

Since a bubble has two surfaces in contact with air,

Total surface area of the bubble

$$= 2 \times (4\,\pi\,r^2) = 8\,\pi\,r^2$$
$$= 8 \times 3.142 \times (5 \times 10^{-2})^2\ \text{m}^2$$

$\therefore \quad$ Work done $=$ surface tension $\times$ increase in area

$$= 3.0 \times 10^{-2} \times 8 \times 3.142 \times (5 \times 10^{-2})^2$$
$$= 1.88 \times 10^{-3}\ \text{J}$$

10.15 ANGLE OF CONTACT

So far we have considered only a free liquid surface. A liquid has to be kept in a container. So the liquid surface near the walls of the container is not free, it is in contact with a solid (i.e. walls of the container). It is also in contact with the liquid vapour (i.e. gas) just above its surface. Thus, at the boundary of a liquid in a container, three surfaces meet, namely, solid, liquid and gas. As explained below, they exert forces on the liquid surface, as a result of which the liquid surface near the boundary is generally curved.

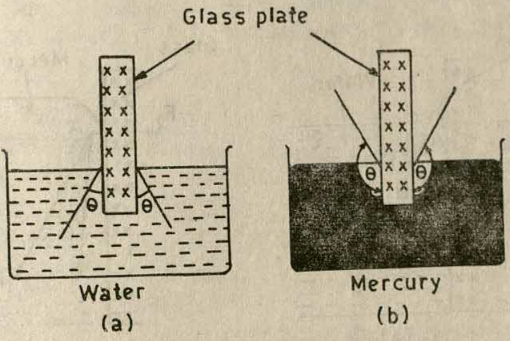


Fig. 10.55 Angle of contact

Take a glass plate and dip it in water with its sides vertical. Water is drawn up near the point of contact and assumes a curved surface as shown in Fig. 10.55(a). The shape of the surface is *concave upward.* But if the same plate is dipped in mercury, the surface of mercury near the point of contact is again curved but is depressed below the horizontal surface as shown in Fig. 10.55(b). The shape of the surface is *convex upward.*

The angle θ between the tangent plane to the liquid surface at the point of contact and the solid surface inside the liquid is called the angle of contact.

The value of angle of contact depends upon the nature of surfaces in contact. For a clean glass plate in contact with pure water with vapour above water, the angle of contact (θ) is zero. For ordinary tap water θ is acute and about 8°. The angle of contact between mercury and glass when air is above mercury is obtuse, being about 130°. Generally, the angle of contact is acute for liquids which wet glass and obtuse for those which do not.

Dyers often use wetting agents. They are added to decrease the angle of contact between the fabric and the dye so that the dye may penetrate well. On the other hand, water-proofing agents, such as waxes, are used to increase the angle of contact between the fabric and water to prevent water from penetrating the cloth.

Angle of Contact and Molecular Forces

When a glass rod is dipped in water and then taken out, a little water still sticks to the glass. This implies that the adhesive forces between water and glass molecules are stronger than the cohesive forces between water molecules. But when a glass rod is dipped in mercury and taken out, no mercury sticks to glass, showing that the cohesive forces between mercury molecules are stronger than the adhesive forces between mercury and glass molecules. Using these ideas, we can explain why the angle of contact between water and glass is acute, and between mercury and glass obtuse.

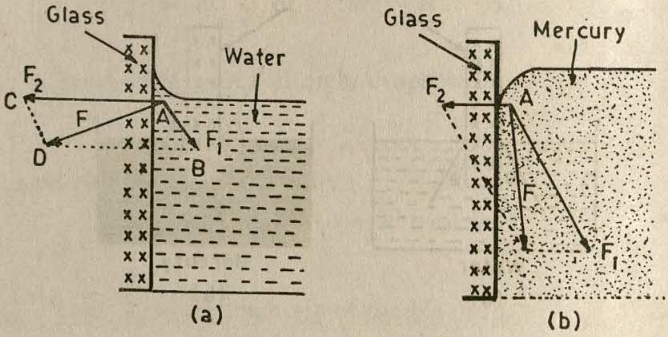


Fig. 10.56 (a) The meniscus at water-glass interface
(b) The meniscus at mercury-glass interface

Let us consider a liquid molecule A near the glass wall as shown in Fig. 10.56(a). It experiences a net cohesive force $\vec{F_1}$ acting inwards along AB within the liquid. At the same time, it experiences a net adhesive force $\vec{F_2}$ acting perpendicular to the glass face pointing along AC as shown. As explained above, $\vec{F_2} > \vec{F_1}$. The resultant of these two forces can be obtained by the law of parallelogram of forces which tells us that the resultant force $\vec{F}$ acts along the diagonal AD. Since the liquid is at rest, the force $\vec{F}$ must have no component parallel to the liquid surface (because this would cause mass motion). Hence $\vec{F}$ must act normal to the liquid surface, i.e. the surface near A must adjust itself to make the force $\vec{F}$ act normal to it. This can happen if the surface assumes the shape (concave upward) as shown in Fig. 10.56(a).

Let us now consider a molecule A of mercury near the glass wall as shown in Fig. 10.56(b). The adhesive force $\vec{F_2}$ is much smaller than the cohesive force $\vec{F_1}$. The resultant force $\vec{F}$ now acts within mercury. The mercury surface can become normal to $\vec{F}$ by curving itself in the shape (convex upward) as shown.

Shape of Meniscus and Angle of Contact

We have seen above how we can have a general idea about the shape of a liquid surface in contact with a solid by knowing the relative strengths of cohesive and adhesive forces. To obtain the exact shape of the meniscus, we must consider the force due to surface tension acting on various surfaces involved.

At the boundary of a liquid in a container, three surfaces are in contact with each other as shown in Fig. 10.57. These three surfaces are those of the liquid, solid and vapour. Therefore, there are three forces acting at the boundary of the three surfaces. These forces are:

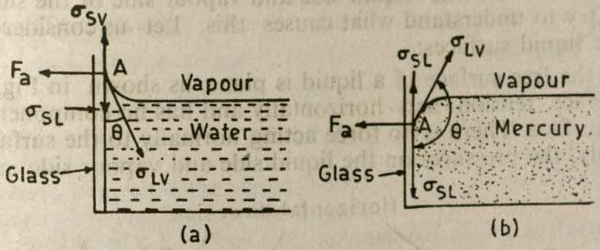


Fig. 10.57 Surface tension forces at the boundary of three surfaces (solid, liquid and vapour)

1. Surface tension σ_{LV} of the liquid-vapour film acting tangentially to the liquid surface.
2. Surface tension σ_{SV} due to solid-vapour film acting parallel to the wall of the container.
3. Surface tension σ_{SL} due to solid-liquid surface acting parallel to the wall of the container directed into the liquid.

In addition, there is an adhesive force F_a between the molecules of the container (solid) and the liquid. This force is always normal to the wall of the container. The point A where the forces meet is in equilibrium as there is no flow of the liquid.

Let us first consider water in a glass vessel. Figure 10.57(a) shows the various forces at A. For equilibrium there should be no net force on A in any direction. Let θ be the angle of contact for water-glass interface. Then the components of σ_{LV} parallel and perpendicular to water surface are $\sigma_{LV} \sin\theta$ and $\sigma_{LV} \cos\theta$ respectively. For equilibrium, we must have

$$F_a = \sigma_{LV} \sin\theta$$

and
$$\sigma_{SV} = \sigma_{SL} + \sigma_{LV} \cos\theta$$

or
$$\cos\theta = \frac{\sigma_{SV} - \sigma_{SL}}{\sigma_{LV}} \tag{10.21}$$

Now, in the case of water in a glass container, $\sigma_{SV} > \sigma_{SL}$, so that $\cos\theta$ is positive. Hence $\theta < 90°$, i.e. the angle of contact is acute and the meniscus is concave upward as shown in Fig. 10.57(a).

Let us now consider a liquid, such as mercury, in a glass container. In this case, $\sigma_{SV} < \sigma_{SL}$ and $\cos\theta$ is negative. Hence θ lies between

90° and 180°, i.e. the angle of contact is obtuse and the meniscus is convex upward as shown in Fig. 10.57(b).

10.16 DROPS AND BUBBLES: PRESSURE DIFFERENCE ACROSS A FILM

When the free surface of a liquid is curved, there is a difference of pressure between the liquid side and vapour side of the surface. Let us first try to understand what causes this. Let us consider the three possible liquid surfaces:

1. If the free surface of a liquid is plane, as shown in Fig. 11.58(a) the surface tension acts horizontally and has no component normal to the surface. There is no force acting normally to the surface. Consequently, the pressures on the liquid side and vapour side are equal.

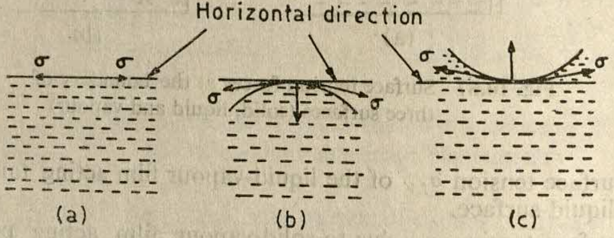


Fig. 10.58 Surface tension and excess pressure across a surface film

2. If the liquid surface is convex (Fig. 10.58(b)) the forces of surface tension have components acting downwards. Consequently, the pressure inside the liquid increases so that the surface remains in equilibrium.

3. If the liquid surface is concave (Fig. 10.58(c)), there will be a resultant upward force due to surface tension acting on the surface. For the surface to remain in equilibrium, the pressure on the vapour side must be greater than that on the liquid side.

Thus, it follows that, if the surface of a liquid is curved, any element of it can be in equilibrium only if the pressure on the concave side is greater than that on the convex size, otherwise, the surface will not stay in equilibrium and will contract on account of surface tension.

Expression for Excess Pressure

We shall now obtain an expression for the excess pressure across a curved surface and show that it depends on the surface tension of the liquid and the radius of curvature of the surface.

Consider a spherical drop of radius R of a liquid of surface tension σ (Fig. 10.59). Due to its spherical shape the pressure inside the drop will be more than that outside. Let the pressure outside be P and that inside be $P + p$, where p is the excess pressure. Let us now enlarge the size of the drop slightly so that its radius increases from R to $R + \Delta R$ where the increase ΔR is so small that the pressure $(P + p)$ inside remains unchanged. Let the surface area of the drop increase from A to $A + \Delta A$.

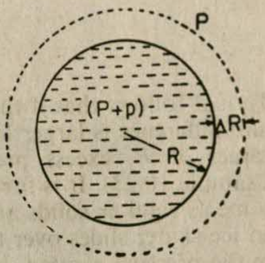


Fig. 10.59 Excess pressure across the surface of a liquid drop

Initial surface area $A = 4\pi R^2$

Final surface area $A + \Delta A$

$$= 4\pi (R + \Delta R)^2$$

$$= 4\pi (R^2 + 2R \Delta R)$$

where we have neglected the term $(\Delta R)^2$ which is negligibly small, being the square of a very small term ΔR. Thus the increase ΔA in the surface area is given by

$$\Delta A = (A + \Delta A) - A = 4\pi (R^2 + 2R \Delta R) - 4\pi R^2$$

or $\quad \Delta A = 8\pi R \Delta R$

Now, by definition of surface tension σ, the work done dW in enlarging the drops is given by

$$dW = \text{Surface tension} \times \text{increase in surface area}$$

$$= \sigma \times \Delta A$$

or $$dW = 8\pi\sigma R\Delta R \tag{10.22}$$

The work dW is also given by the excess force times the distance moved. The excess force dF is given by

$$dF = \text{Excess pressure} \times \text{Area}$$

$$= p \times 4\pi R^2$$

Since the distance moved is ΔR, the work done must be given by

$$dW = dF \times \Delta R$$

or $$dW = 4\pi R^2 p \Delta R \tag{10.23}$$

Equating the two expressions [Eqs (10.22) and (10.23)] for the work done, we have

$$4\pi R^2 p \Delta R = 8\pi\sigma R \Delta R$$

or $$p = \frac{2\sigma}{R} \tag{10.24}$$

In the case of a bubble, such as a soap bubble, there are two surfaces in contact with vapour. Hence, for a bubble, the excess pressure will be twice that for a drop which has only one surface in contact with vapour. Thus for a bubble, we have

$$p = \frac{4\sigma}{R} \qquad (10.25)$$

Thus it follows that if the surface tension remains constant, the pressure difference is larger for smaller values of R. For extremely small values of R, excess pressure p can assume an enormous value (see Example 10.23). It is for this reason that very tiny liquid drops become as rigid as solids and resist fairly large deforming forces. When an ice-skater slides over the surface of smooth ice, some ice melts due to the pressure exerted by the sharp metal edges of the skates. The skater runs along the tiny water droplets as if he were on ball bearings. For the same reason it is easir to walk on wet sand than on dry sand.

Example 10.23 Calculate the excess pressure across a soap bubble of diameter 3 mm. Surface tension of soap solution is 1.5 N m^{-1}.

Solution

Here $R = \frac{3}{2}\text{ mm} = 1.5\text{ mm} = 1.5 \times 10^{-3}\text{ m}$

$$\sigma = 1.5\text{ N m}^{-1}$$

$$\text{Excess pressure } p = \frac{4\sigma}{R}$$

$$= \frac{4 \times 1.5}{1.5 \times 10^{-3}}$$

$$= 4000\text{ N m}^{-2}$$

10.17 CAPILLARITY

The word capillary means ‘hair-like’. A tube having a very fine bore is called a capillary tube. If a glass capillary tube open at both ends is dipped in a liquid such as water or alcohol, which wets glass (i.e. a liquid whose angle of contact with glass is acute), the liquid rises up in the tube to a certain height above the liquid level outside (Fig. 10.60(a)). But if the tube is dipped in a liquid, such as mercury, which does not wet glass (i.e. whose angle of contact with glass is obtuse), it gets depressed below the liquid level outside (Fig. 10.60(b)). The rise or fall of liquids in a capillary tube is known as the phenomenon of *capillarity*. We shall show that the rise or fall of a given liquid is greater in a tube of smaller diameter.

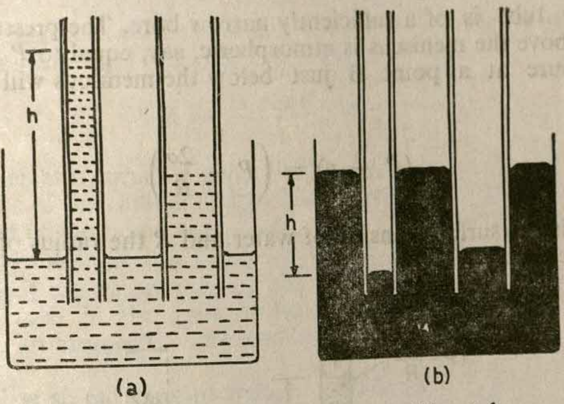


Fig. 10.60 (a) The rise of water in a capillary tube
(b) The fall of mercury in a capillary tube

Illustrations of Capillarity

We come across various illustrations of capillarity in everyday life. A few examples are:

1. The fibres of wood in the stem of a plant form very fine capillaries, and this helps the leaves of the plant receive moisture from the earth.

2. Kerosene oil rises in the wick of an oil lamp through the narrow spaces between the strands (threads) of the fibre in the wick, which serve as capillary tubes.

3. A piece of blotting paper soaks ink by capillary action. The pores of the blotting paper serve as capillary tubes.

4. Melted wax in a candle rises up the wick by capillary action.

5. Capillary action plays an important role in farming. In porous soil, water rises to the surface by capillary action. The spaces between the grains of a compact soil serve as capillary tubes. To prevent this loss of water, the soil is loosened and split into pieces so that the spaces between pores become large and the rise to the surface by capillarity is prevented. Thie is sometimes called dry farming.

Height of a Liquid in a Capillary Tube

We shall now obtain an expression for the height to which a liquid rises in a capillary tube. It is instructive to learn the various ways by which the expression can be obtained.

Capillary Rise on the Basis of Excess Pressure Considerations

Let us consider water in a capillary tube (Fig. 10.61(a)). The shape of the meniscus of water in the capillary tube is nearly spherical if the capillary tube is of a sufficiently narrow bore. The pressure at point A just above the meniscus is atmospheric, say, equal to P. Therefore, the pressure at a point B just below the meniscus will be (see Eq. 10.24).

$$(P - p) = \left(P - \frac{2\sigma}{R}\right)$$

where σ is the surface tension of water and R the radius of curvature of the meniscus.

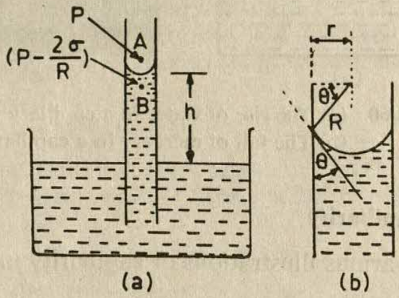


Fig. 10.61 The height of a liquid in a capillary tube by excess pressure considerations]

Since the pressure at A is greater than that at B, the water will rise in the tube and reach a height h above the horizontal surface outside the tube until the excess pressure $p\left(=\frac{2\sigma}{R}\right)$ becomes equal to the hydrostatic pressure due to a height h of the liquid column. The hydrostatic pressure due to a height h is $h\rho g$, where ρ is the density of the liquid and g the acceleration due to gravity. Thus, at equilibrium, we have

$$h\rho g = \frac{2\sigma}{R}$$

or

$$h = \frac{2\sigma}{R\rho g} \tag{10.26}$$

Now referring to Fig. 10.61(b) (which shows a magnified view of the meniscus), we have

$$R \cos \theta = r$$

or

$$R = \frac{r}{\cos \theta} \tag{10.27}$$

where r is the radius of the capillary tube and θ is the angle of contact. Substituting Eq. (10.27) in Eq. (10.26), we get

$$h = \frac{2\sigma \cos \theta}{r\rho g} \tag{10.28}$$

Thus, the height to which a given liquid rises is inversely proportional to the radius of the capillary tube. The narrower the tube, the greater is the height of the liquid that is supported by the pressure difference across the meniscus, which is due to surface tension.

Similarly, it can be proved that the fall h of a liquid, such as mercury, in a tube is also given by Eq. (10.28). The reason for this has been left as an exercise for the student.

Capillary Rise on the Basis of Balance of Forces

To find the height of a column of a liquid that can be supported by surface tension, consider a liquid, such as water, in a capillary tube made of glass. The angle of contact θ is acute, as shown in Fig. 10.62. The forces due to surface tension act along the tangent at the point of contact to the spherical liquid meniscus.

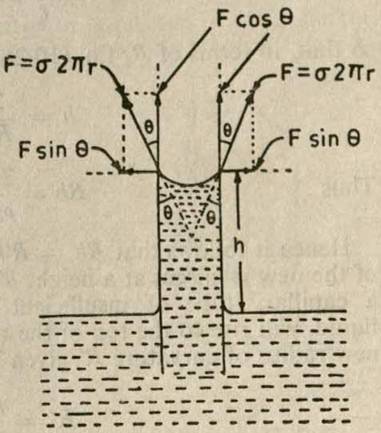


Fig. 10.62 The height of a liquid in a capillary tube by balance of forces

Resolving these forces along the horizontal and vertical directions, we find that the horizontal components $\sigma \sin \theta$ balance each other and the vertical components $\sigma \cos \theta$ support the column of liquid of height h. If r is the radius of the capillary tube, then the forces F on the circular surface, which has a length $2\pi r$, are given by $2\pi r\sigma$, by the definition of surface tension. Thus the resultant upward force on the surface of the meniscus is $F \cos \theta = 2\pi r\sigma \cos \theta$. This force balances the force due to hydrostatic pressure due to a height h of the liquid.

Pressure due to height $h = h\sigma g$

$$\begin{aligned}\text{Downwarde force} &= \text{Pressure} \\ &\quad \times \text{Area of surface} \\ &= h\rho g \times \pi r^2\end{aligned}$$

Thus, at equilibrium, we have

$$h\rho g \times \pi r^2 = 2\pi r\sigma \cos \theta$$

or

$$h = \frac{2\sigma \cos \theta}{r\rho g}$$

which is Eq. (10.28) obtained above.

Capillary of Non-Uniform Bore

It is worthwhile to note that Eq. (10.28) holds even when the capillary tube has non-uniform cross-section. In such cases, the quantity r is the radius of the capillary tube at the meniscus of the liquid.

Capillary Rise in a Tube of Insufficient Length

It is interesting to consider what will happen if the length of the capillary tube is less than the height h given by Eq. (10.28) up to which a liquid will rise. The liquid cannot flow out in the form of a fountain because then the liquid rushing out of the tube can be used to produce perpetual work by making it fall on, say, the blades of a toy fan which would begin to rotate. This would violate the principle of conservation of energy. No perpetual work is possible without the supply of energy.

In fact, what happens in that the meniscus changes its shape till an equilibrium at a smaller height h' is reached, where h' is the length of the capillary tube. From Eq. (10.27) we have seen that the radius of curvature R of the meniscus and the radius r of the capillary are related as

$$r = R\cos\theta$$

so that, in terms of R, Eq. (10.28) becomes

$$h = \frac{2\sigma}{R\rho g}$$

Thus $$Rh = \frac{2\sigma}{\rho g} = \text{constant}$$

Hence it follows that $Rh = R'h'$, where R' the radius of curvature of the new meniscus at a height h'. Since $h' < h\ R' > R$. Thus, when a capillary tube of insufficient length h' is inserted in a liquid, the liquid, will rise to the top of the tube and will spread out to acquire a new radius of curvature R' given by

$$R' = \frac{hR}{h'}$$

EXAMPLE 10.24 A glass tube of 1 mm bore is dipped vertically into a container of mercury with the lower end 2 cm below the mercury surface. What must be the gauge pressure of air in the tube to blow a hemispherical bubble at its lower end? Surface tension of mercury is $4.0 \times 10^{-1}\ \text{N m}^{-1}$.

Solution The gauge pressure must be given by where R = radius of hemispherical bubble = 0.5 mm

$= 0.5 \times 10^{-3}$ m

σ = surface tension of mercury = 4.0×10^{-1} N m^{-1}

ρ = density of mercury = 13,600 kg m^{-3}

g = acceleration due to gravity = 9.8 m s^{-2}

h = depression of mercury = 2 cm = 2×10^{-2} m

$$\therefore \quad P = \frac{2 \times 4.0 \times 10^{-1}}{0.5 \times 10^{-3}} + 13{,}600 \times 9.8 \times 2 \times 10^{-2}$$

$$= 4265.6 \text{ N m}^{-2}$$

EXAMPLE 10.25. A glass U-tube is such that the diameter of one limb is 3.0 mm and that of the other 6.0 mm. The tube is inverted vertically which the open ends below the surface of water in a beaker. What is the difference between the heights to which water rises in the two limbs? Surface tension of water is 0.07 N m^{-1}. Assume that the angle of contact between water and glass is zero.

Solution Since the angle of contact (θ) is zero, it follows from Eq. (10.26) that the radius of curvature (R) of the concave water surface in a limb is equal to the radius (r) of the tube in that limb.

Referring to Fig. 10.63 and remembering that the pressure on the concave side of the liquid is greater than that on the other side by $\frac{2\sigma}{R}$, we have

$$P_A = P_B + \frac{2\sigma}{r_2}$$

and

$$P_C = P_D + \frac{2\sigma}{r_1}$$

Since the pressures at A and C are equal, say, P_0, we have

$$P_A = P_C = P_0$$

$$\therefore \quad P_0 = P_B + \frac{2\sigma}{r_1}$$

and

$$P_0 = P_D + \frac{2\sigma}{r_2}$$

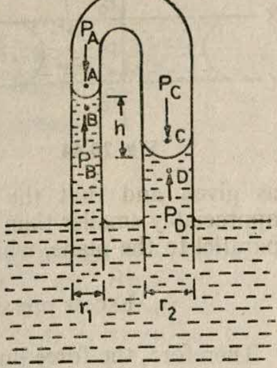


Fig. 10.63

Equating the two equations, we get

$$P_B - P_D = 2\sigma\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

or $h\rho g = 2\sigma\left(\frac{1}{\ } - \frac{1}{\ }\right)$

where h = difference in water levels in the two limbs

$\rho = 1000 \text{ kg m}^{-3}$

$g = 9.8 \text{ m s}^{-2}$

$\sigma = 7.0 \times 10^{-2} \text{ N m}^{-1}$

$r_1 = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$

$r_2 = 3.0 \text{ mm} = 3.0 \times 10^{-3} \text{ m}$

Thus

$$h = \frac{2\sigma}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{2 \times 7.0 \times 10^{-2}}{1000 \times 9.8}\left(\frac{1}{1.5 \times 10^{-2}} - \frac{1}{3.0 \times 10^{-3}}\right)$$

$$= 0.476 \times 10^{-2} \text{ m}$$

$$= 4.76 \text{ mm}$$

EXAMPLE 10.26 A small drop of water is squeezed between two clean glass plates so that a thin layer of thickness 10^{-6} m and area 10^{-2} m^2 is formed between them. Assuming that the surface tension of water is 0.07 Nm^{-1} and angle of contact is zero, calculate the force required to pull the plates apart.

Solution Referring to Fig. 10.64, X and Y are the two plates, P_0 is the external atmospheric pressure and P is the pressure inside the very thin layer of water of thinkness d.

Now it is easy to show that the excess pressure $(P_0 - P)$ is given by $\frac{\sigma}{r}$ in place of $\frac{2\sigma}{r}$ obtained in the case of a spherical drop, where $r = d/2$. We have assumed that $\cos\theta = 1$, i.e. $\theta = 0°$ as given and that the radius of curvature (r) of the meniscus is appreciably smaller than the radius of curvature of the water film parallel to the plane, which indeed is true. Thus

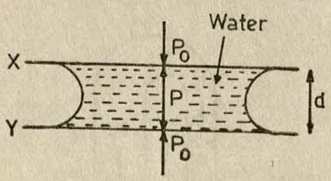


Fig. 10.64

$$\text{Excess pressure} = \frac{\sigma}{r} = \frac{\sigma}{d/2} = \frac{2\sigma}{d}$$

Therefore, the force due to surface tension pushing the two plates together is given by

$$F = \text{excess pressure} \times \text{area of layer}$$

$$= \frac{2\sigma}{d} \times \text{area}$$

$$= \frac{2 \times 0.07 \times 10^{-2}}{10^{-6}}$$

$$= 1400 \text{ N}$$

Thus a force as large as 1400 N is required to pull the plates apart. Notice that if the thickness d is bigger, F becomes smaller so that it is now easier to pull the plates apart.

EXAMPLE 10.27 Water rises to a height of 10 cm in a capillary tube dipped in water. When the same tube is dipped in mercury it is depressed by 2.60 mm. Compare the surface tensions of water and mercury. Density of water = 1000 kg m^{-3}, that of mercury = 13600 kg m^{-3}. Angle of contact for water is 5° and that for mercury is 135°.

Solution We know that

$$\sigma_1 = \frac{h_2\rho_1 gr}{2\cos\theta_1}, \qquad \sigma_2 = \frac{h_2\rho_2 gr}{2\cos\theta_2}$$

where subscripts 1 and 2 refer to water and mercury respectively. Therefore,

$$\frac{\sigma_1}{\sigma_2} = \frac{h_1}{h_2}\cdot\frac{\rho_1}{\rho_2}\,\frac{\cos\theta_2}{\cos\theta_1}$$

Here $h_1 = 10 \text{ cm} = 0.1 \text{ m}$

$h_2 = -\ 2.6 \text{ mm} = -\ 2.6 \times 10^{-3}$ m (negative sign represents depression)

$\rho_1 = 1000 \text{ kg m}^{-3}$

$\rho_2 = 13600 \text{ kg m}^{-3}$

$\theta_1 = 5°$

$\theta_2 = 135°$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{0.1}{-(2.6\times 10^{-3})} \times \frac{1000}{13600} \times \frac{(-1/2)}{0.996}$$

$\because$ $(\cos 135° = -\ 1/2, \text{ and } \cos 5° = 0.996)$

Thus $\dfrac{\sigma_1}{\sigma_2} = 2.01$

EXAMPLE 10.28 The internal diameter of the glass tube of a mercury barometer is 2 mm. The barometer reads 74.45 cm Hg. Find the correct reading of the barometer after allowing for the error due to surface tension. Density of mercury is 13600 kg m^{-3}, its surface tension = 3.5×10^{-1} N m^{-1} and its angle of contact = 135°.

Solution The depression due to surface tension is

$$h = \frac{2\sigma\cos\theta}{r\rho g}$$

$$= \frac{2\times(3.5\times 10^{-1})(\cos 135°)}{(1.0\times 10^{-3})(13600)(9.8)}$$

$$= -\ 0.0037 \text{ m} = -\ 0.37 \text{ cm}$$

The negative sign indicates that the barometer reading is lower by 0.37 cm. Therefore,

The correct reading $= 74.45 + 0.37 = 74.82$ cm Hg

10.18 EFFECT OF TEMPERATURE ON SURFACE TENSION

Surface Tension of a Liquid Decreases with Rise in Temperature

Sprinkle some light dust or lycopodium power on the surface of water in a flat metal dish. Heat one side of the dish with a candle. You will notice the particles of the powder being swept away from the heated portion, indicating that the force due to surface tension can no longer hold the particles in their previous positions.

10.19 DETERGENTS AND SURFACE TENSION

The addition of detergents or soap increases the cleansing ability of a solution. Dirty clothes contain grease and oil stains which stick to the fabric. When washed with water alone, the grease stains are not removed because water does not wet grease, and so there is very little area of contact between them. If, somehow, the surface tension of water could be decreased, the area of contact can be increased. This is exactly what detergents do when mixed with water. The molecules of a detergent have the shape of a hairpin, one end of which remains attached to water while the other is attracted to molecules of grease, thus forming a water-grease interface. In other words, the surface tension is considerably reduced. If the mixture of water and detergent is heated, a further decrease in surface tension results, thereby increasing the cleansing ability of the solution.

This process of using surface-active detergents (called *surfactants*) is used not only for cleaning but also for recovering oil, mineral ores, etc.

SECTION D: GASES

10.20 INTRODUCTION

In Chapter 9 we have learnt that every substance (solid, liquid or gas) consists of extremely tiny particles called molecules. In a gas, the molecules move rapidly because the forces between them are weak. Various phenomena, such as diffusion, evaporation, elasticity and Brownian movement, provide conclusive evidence of molecular motion.

Based on these observations and the fact that heat can be identified with molecular motion (see Chapter 11), a theory has been developed which satisfactorily explains physical concepts like temperature, pressure, elasticity, etc. This theory is called the kinetic theory of matter.

The kinetic theory of solids and liquids is too complicated to be discussed. We shall consider only the kinetic theory of gases because of its simplicity.

10.21 KINETIC THEORY OF GASES

The development of a theory in physics is always based on certain simplifying assumptions about the system whose behaviour the theory intends to explain. The founders of kinetic theory R. Clausius (1822-1888) of Germany and J.C. Maxwell (1831-1875) of scotland conceived a model of a gas. The model, which is an idealization, is called a *perfect* or *ideal gas*. The kinetic theory is based on the following fundamental assumptions regarding the model of a gas.

1. *A gas consists of particles called molecules.* We consider all molecules of a gas to be identical.

2. *The molecules are in random motion and obey Newton's laws of motion.* The molecules move about in all directions with various speeds. We assume that Newton's laws of motion are applicable at the microscopic level.

3. *The total number of molecules is large.* Any particular molecule will follow a zig-zag path due to collisions with other molecules or with the walls of the container. But, because there are so many particles, we assume that the resulting large number of collisions maintains an overall distribution of molecular velocities (see Supplement S4 at the end of this chapter for distribution of molecular velocities).

4. *The volume of the molecules is a negligibly small fraction of the volume occupied by the gas.* Even though there are many molecules, they are extremely small and occupy a very small volume.

5. *No appreciable force acts on the molecules during collision.* In other words, a molecule travels with a uniform velocity in a straight line between two collisions. The average distance through which a molecule moves freely between two collisions is called the *mean free path* (see Supplement S5 at the end of this chapter).

6. *Collisions are elastic and of negligible duration.* The law of conservation of momentum should certainly hold when a molecule collides with another molecule or with the walls of the container. We further assume that kinetic energy is also conserved. Because the collision time is negligible compared to the time spent by a molecule between collisions, the kinetic energy which is converted into potential energy during a collision is again (almost instantaneously) available as kinetic energy without any decrease. Collisions are said to be *elastic* if there is no dissipation of energy in a collision, i.e. the kinetic energy after collision is the same as that before it. These simplifying assumptions help avoid mathematical complications. They define an ideal gas. Strictly speaking, none of these assumptions is really true. However, this model, though approximate, satisfactorily explains a large number of experimental facts about gases. We will now use this model to see how far this can explain the experimental facts.

Kinetic Theory and Gas Pressure

The molecules of a gas move in all directions with various speeds. During its random motion, a fast molecule often strikes against the walls of the container. The collision is assumed to be perfectly elastic, i.e. the molecule bounces back with the same speed with which it strikes. Since the number of molecules is assumed to be very large, it is evident that billions of molecules strike against the walls of the container every second. These molecules collectively exert a sizeable force on the wall. The force experienced per unit area is the pressure exerted by the gas on the walls. Thus, *the pressure of a gas is the result of the continuous bombardment of the gas molecules against the walls of the container.*

Expression for Gas Pressure on the basis of Kinetic Theory

After every collision with the wall, the velocity of a molecule is reversed. Hence, the molecule undergoes a change in momentum. We will calculate the total change in momentum of all particles in a certain time and then calculate the rate of change of momentum which, from Newton's second law, is the force experienced by the molecules colliding with the walls. From Newton's third law, this is also the force the molecules exert on the wall. The total force exerted on the walls per unit area then gives the gas pressure.

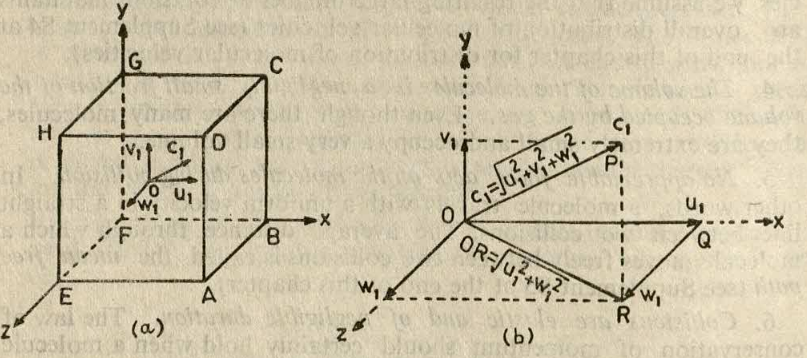


Fig. 10.65 (a) Velocity components of a molecule of a gas in a vessel (b) Resolution of velocity vector into three rectangular components

Consider a gas enclosed in a vessel in the form of a cube of edge l as shown in Fig. 10.65(a). Let n be the number of molecules of the gas contained in the vessel and m the mass of each molecule. The volume of the vessel and hence of the gas is l^3. Consider a molecule moving with a velocity $\vec{c}_1$. We label it as molecule number one. The molecule moves in a space of three dimensions x, y and z. The directions x, y and z are taken to be parallel to the edges of the cube itself as shown. Let u, v and w be the three rectangular components of vector $\vec{c}_1$ along

x, y and z axes respectively. The vector $\vec{c_1}$ does not lie in the plane of the figure. It is a vector in space. It is clear from Fig. 10.65, that

$$c_1^2 = u_1^2 + v_1^2 + w_1^2$$

Consider a molecule striking a face, say, ABCD of the cube. It is clear that only the x-component (i.e. u_1) of the velocity is effective because it is perpendicular to the face. There will be no effect on v_1 and w_1 as they are parallel to this face. The molecule strikes the face with an initial momentum mu_1. Since the collision is assumed to be perfectly elastic, the kinetic energy of the molecule after the collision is the same as that before collision, but the direction of the velocity is reversed (see Fig. 10.66). Thus the molecule rebounds with a velocity $-u_1$, so that its momentum after collision is $-mu_1$. The change in the momentum of the molecule due to the collision is equal to

$$\text{final momentum} - \text{initial momentum}$$

$$= - mu_1 - (mu_1) = - 2\, mu_1$$

From the law of conservation of momentum, it follows that the momentum imparted to the face ABCD by the molecule is $+ 2\, mu_1$.

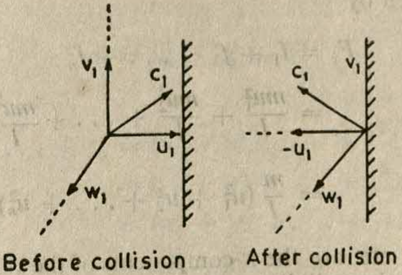


Fig. 10.66 Velocity components before and after collision with a face of the cube

Now suppose that after rebound, the molecule reaches the opposite face EFGH without colliding with any other particle on the way. The time taken by the molecule in going from face ABCD to face EFGH is, obviously, $\frac{l}{u_1}$. At the face EFGH, the molecule will again rebound with the same velocity u_1 with its direction again reversed. The molecule returns to face ABCD and strikes it the second time. It is clear that the time interval between two successive collisions of a molecule with the same face is given by

$$t = \frac{2l}{u_1}$$

Therefore, the number of collisions with the same face per second is

$$\frac{1}{t} = 1 \div \frac{2l}{u_1} = \frac{u_1}{2l}$$

Hence the total momentum imparted by a molecule to the face ABCD in one second is the momentum imparted in one collision times the number of collisions per second

$$= (2\,mu_1) \times \frac{u_1}{2l} = \frac{mu_1^2}{l}$$

According to Newton's second law, the force exerted by the molecule on this face is the rate at which momentum is imparted to the wall by the molecule. Thus the x-component of the force exerted by the molecule is

$$f_1 = \frac{mu_1}{l}$$

The total force exerted on face ABCD by all the n molecules of the gas is clearly given by

$$\begin{aligned} F_x &= f_1 + f_2 + \ldots + f_n \\ &= \frac{mu_1^2}{l} + \frac{mu_2^2}{l} + \ldots + \frac{mu_n^2}{l} \\ &= \frac{m}{l}(u_1^2 + u_2^2 + \ldots + u_n^2) \end{aligned}$$

where $u_2, \ldots, u_n$, etc. are the x-components of the velocities $c_2, c_3, \ldots, c_n$ of other molecules colliding with the face ABCD. Since the area of the face is l^2, the pressure exerted by all the molecules on face ABCD is

$$P_x = \frac{F_x}{l^2} = \frac{m}{l^3}(u_1^2 + u_2^2 + \ldots + u_n^2)$$

Similarly, the pressure exerted on the other faces of the cube, which are prependicular to the y- and z-axes, are respectively given by

$$P_y = \frac{m}{l^3}(v_1^2 + v_2^2 + \ldots + v_n^2)$$

$$P_z = \frac{m}{l^3}(w_1^2 + w_2^2 + \ldots + w_n^2)$$

These expressions can be written in a simpler form as

$$P_x = \frac{mn}{l^3}\,\overline{u^2}$$

$$P_y = \frac{mn}{l^3}\,\overline{V^2}$$

$$P_z = \frac{mn}{l^3}\,\overline{w^2} \tag{10.29}$$

where $\overline{u^2} = \dfrac{u_1^2 + u_2^2 + \ldots + u_n^2}{n}$

is the mean of the squares of the x-components of the velocities of all molecules. Similarly $\overline{v^2}$ and $\overline{w^2}$ are respectively the mean of the squares of the y- and z-components of the velocities of all particles.

Now, since

$$\begin{aligned} c_1^2 &= u_1^2 + v_1^2 + w_1^2 \\ c_2^2 &= u_2^2 + v_2^2 + w_2^2 \\ \ldots &\quad\ \ldots \quad\ \ldots \\ c_n &= u_n^2 + v_n^2 + w_n^2 \end{aligned}$$

it follows that

$$\begin{aligned} c_1^2 + c_2^2 + \ldots + c_n^2 &= (u_1^2 + u_2^2 + \ldots + u_n^2) \\ &= (v_1^2 + v_2^2 + \ldots + v_n^2) \\ &= (w_1^2 + w_2^2 + \ldots + w_n) \\ &= n\overline{u^2} + n\overline{v^2} + n\overline{w^2} \\ &= n\,(\overline{u^2} + \overline{v^2} + \overline{w^2}) \end{aligned}$$

$$\therefore \qquad \overline{u^2} + \overline{v^2} + \overline{w^2} = \frac{c_1^2 + c_2^2 + \ldots + c_n}{n}$$

The quantity $\dfrac{c_1^2 + c_2^2 + \ldots + c_n^2}{n}$ is clearly the mean of the squares of the velocities of individual molecules and is called the mean square speed. Let us denote it by the symbol $\overline{c^2}$.

Thus

$$\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2} \tag{10.30}$$

Now, since the gas is in equilibrium and has no bulk motion (the gas is not moving as a whole), there is no preferred direction for molecular motion. In other words, the motion of molecules is completely random which means that mean square speed of the molecules in all directions is the same, i.e.

$$\overline{u^2} = \overline{v^2} = \overline{w^2} \tag{10.31}$$

From Eqs (10.30) and (10.31), it follows that

$$\overline{c^2} = 3\,\overline{u^2} \quad \text{or} \quad \overline{u^2} = \overline{v^2} = \overline{w^2} = \tfrac{1}{3}\,\overline{c^2} \tag{10.32}$$

From Eqs (10.29) and (10.32) we find that

$$P_x = P_y = P_z = \frac{1}{3}\,\frac{mn}{l^3}\,\overline{c^2}$$

This is the Pascal's law which states that the pressure exerted by a gas is the same in every direction. Hence the pressure P $(= P_x = P_y = P_z)$ exerted by an ideal gas is given by

$$\mathbf{P} = \frac{1}{3}\,\frac{mn}{V}\,\overline{c^2} = \frac{1}{3}\,\rho\,\overline{c^2} \tag{10.33}$$

where $V = l^3$ is the volume of the gas and $\rho = \frac{mn}{V}$ is the density of gas, since $mn = M$ is the total mass of the gas.

We shall now make a few interesting remarks regarding the validity of Eq. (10.33).

1. Equation (10.33) relates a macroscopic quantity, namely, pressure P, to a mean value of a microscopic quantity, namely, the mean square velocity $\overline{c^2}$ of the molecules.

2. Equation (10.33) is independent of the shape of the container. We would obtain the same equation for any container. We chose a cube merely because it simplified the calculation.

3. Although we have derived Eq. (10.33) by neglecting collisions between molecules themselves, the equation is true even when collisions are included. The reason is that since the molecules are identical and collisions between them are assumed elastic, the molecules merely exchange velocities (masses being the same) in collisions. Hence, there will always be one molecule that will collide with face EFGH with momentum mu_1 corresponding to the one that leaves the face ABCD with the same momentum. What one molecule loses, another one gains. Thus, our neglect of collisions between molecules themselves is merely a convenient simplification.

4. We have neglected the weight of the gas. The molecules moving vertically downward experience a negligible gravitational force because of their extremely small mass. In the case of a gas of a very large extent (such as the atmosphere) we are not justified in neglecting the weight of the gas.

The Root-Mean Square Speed The root-mean square speed (abbreviated as the rms speed) of molecules of a gas is the square-root of the mean of the squares of the velocities of the molecules. It is denoted by the symbol c_{rms}.

Thus $$c_{rms} = \sqrt{\overline{c^2}}$$

From Eq. (10.33) the rms speed of the molecules of an ideal gas of density ρ and at pressure P is given by

$$c_{rms} = \sqrt{\overline{c^2}} = \sqrt{\frac{3P}{\rho}} \tag{10.34}$$

The rms speed is a sort of average speed of the molecules of the gas. It can be determined by the measured values of the pressure and density of the gas (see Example 10.29). Table 10.4 gives the rms speeds of the molecules of some common gases at 0°C.

Table 10.5 Root-mean square speeds for some common gases

Gas	*Root-mean square speed at 0 °C* ($m\ s^{-1}$)
Oxygen	460
Nitrogen	490
Air	485
Hydrogen	1840
Helium	1310
Carbon dioxide	393
Neon	584

EXAMPLE 10.29 Assuming hydrogen to be an ideal gas, calculate the root-mean square speed of the gas molecules at 0.0 °C and a pressure of 1.0 atmosphere. The density of hydrogen at 0.0 °C = 9.0×10^{-2} kg m^{-3}.

Solution

$$P = 1.0 \text{ atmosphere} = 0.76 \text{ m of Hg}$$
$$= 0.76 \times 13600 \times 9.8 = 1.01 \times 10^5 \text{ N m}^{-2}$$
$$\rho = 9.0 \times 10^{-2} \text{ kg m}^{-3}$$

Using Eq. (10.34) we have,

$$c_{rms} = \sqrt{\frac{3P}{\rho}} = \left(\frac{3 \times 1.01 \times 10^5}{9.0 \times 10^{-2}}\right)^{\frac{1}{2}} = 1840 \text{ m s}^{-1}$$

Mean Translational Kinetic Energy

As mentioned earlier, the root-mean square speed c_{rms} may be regarded as some kind of an average speed of the molecules of a gas. Hence, the mean translational kinetic energy (denoted by $\bar{E}$) of a molecule of a gas is given by

$$\bar{E} = \tfrac{1}{2}\, m\, c_{\text{rms}}^2 = \tfrac{1}{2}\, m\, \overline{c^2}$$

In terms of $\bar{E}$, Eq. (10.33) becomes

$$P = \frac{1}{3}\, \rho\, \overline{c^2} = \frac{1}{3}\, \frac{m \times n}{V}\, \overline{c^2} = \frac{2}{3}\, \frac{n}{V}\left(\frac{1}{2}\, m\, \overline{c^2}\right) \quad (10.35)$$

$$= \frac{2}{3}\, n\, \frac{\bar{E}}{V}$$

Now, $n\bar{E}$ is the total translational mean kinetic energy of all the n molecules of the gas. Let us denote it by U.

Thus

$$P = \frac{2\,U}{3\,V}$$

In other words, the pressure exerted by a gas is equal to two-thirds of the total mean translational kinetic energy of the molecules per unit volume of the gas.

$U = n\bar{E}$ is also called the *internal energy* of the gas. We will learn more about internal energy in Chapter 11.

10.22 GAS LAWS

Boyle's Law

Among the earliest experiments on gases were those of an English physicist Robert Boyle (1627-1691). He discovered a law, now known as Boyle's law, which states that

"*At any given temperature, the volume of a given mass of a gas is inversely proportional to the pressure applied to it*", i.e. $V \propto P$ or PV = constant (temperature remaining constant). (10.35)

This law can be derived from Eq. (10.33) obtained above. According to the kinetic theory of gases, the pressure exerted by an ideal gas is given by

$$P = \frac{1}{3}\, \frac{mn}{V}\, \overline{c^2}$$

where V is the volume of the enclosed gas, n the number of molecules in volume V, m the mass of each molecule and $\overline{c^2}$ is the mean square speed. Thus,

$$PV = \tfrac{1}{3}\, mn\, \overline{c^2}$$

or

$$PV = \tfrac{1}{3}\, n\, (m\overline{c^2})$$

Since n is a constant (matter can neither be created nor destroyed), it is clear that the product PV will remain constant if $m\overline{c^2}$ is constant (since m is constant, this also means that $\overline{c^2}$ remains constant). But $\frac{1}{2}\,m\overline{c^2}$ is the average kinetic energy of any molecule of the gas. Hence, if the average kinetic energy of any molecule of the gas remains constant, the product of pressure and volume will be constant.

We know from experience that if temperature remains constant, there is no flow of heat energy in the system. Kinetic theory assumes this energy to be totally kinetic. Hence, if temperature remains constant, there will be no change in the average kinetic energy which means that $\overline{c^2}$ (or $\frac{1}{2}\,m\overline{c^2}$) remains unchanged. In Sec. 10.24, we will give a kinetic interpretation of temperature and justify it. This argument immediately leads to Boyle's law which states that *the product of the pressure and volume of* a *given mass of a gas is constant, provided the temperature remains constant.*

Experiments with real gases have shown that Boyle's law holds for them if their density is low and temperature is high. Thus at low density and high temperature, a real gas behaves like a perfect or ideal gas (see Supplement S3 on van der Waals' equation for real gases, at the end of the chapter).

Charles' Law

It was Jacques Charles (1747-1823) who found experimentally that, when the pressure of a gas is kept constant, its volume increases with temperature at a constant rate. If V_0 is the volume of a gas a 0 °C, then its volume V at t °C is given by (provided pressure is kept constant)

$$V = V_0(1 + \alpha_v t)$$

where α_v is the coefficient of volume expansion at constant pressure. The value of α_v has been determined experimentally in the case of various gases, such as air, hydrogen, oxygen, nitrogen, etc. The most striking observation is that α_v is the same for all gases and has been found to be $\frac{1}{273}$ per °C. Hence for all gases,

$$V = V_0\left(1 + \frac{t}{273}\right)$$

or

$$\frac{V}{V_0} = \frac{t + 273}{273} = \frac{T}{T_0}$$

where T and T_0 are absolute temperatures corresponding to t °C and 0 °C. This relation tells us that

$$\frac{V}{T} = \text{constant} \tag{10.36}$$

This is Charles' law which states that '*if the pressure of a gas is kept constant, the volume af the gas is directly proportional to its absolute temperature.*

Instead of keeping the pressure constant, experiments can be done in which the volume of the gas is kept constant as the temperature is changed. Such experiments of heating a gas in a closed rigid container were performed by Joseph Gay-Lussac (1778—1850) who found that '*if the volume of a gas is kept constant, the pressure of the gas is directly proportional to its absolute temperature*; i.e.

$$\frac{P}{T} = \text{constant} \tag{10.37}$$

Experiments have shown that laws (10.36) and (10.37) do not hold at low temperatures and low volumes. The departures of real gases from an ideal behaviour become significant at low temperatures and low volumes (or high densities).

Avogadro's Law

Consider two gases A and B at the same temperature and pressure. Let the pressure be P. Then from Eq. 10.33) we have

$$P = \frac{1}{3}\frac{m_A n_A}{V_A}\overline{c_A^2}$$

$$P = \frac{1}{3}\frac{m_B n_B}{V_B}\overline{c_B^2}$$

which give

$$\frac{n_A}{V_A}(m_A \overline{c_A^2}) = \frac{n_B}{V_B}(m_B \overline{c_B^2})$$

Since the temperature of the two gases is the same, the average kinetic energy of any molecule of gas A will be equal to that of any molecule of gas B, (see Sec. 10.23). Thus

$$\tfrac{1}{2} m_A \overline{c_A^2} = \tfrac{1}{2} m_B \overline{c_B^2}$$

Hence

$$\frac{n_A}{V_A} = \frac{n_B}{V_B}$$

If the volumes are also equal, i.e. $V_A = V_B$, then

$$n_A = n_B$$

This means that *equal volumes of all gases at the same temperature and pressure contain an equal number of molecules.* This is Avogadro's law. The number of molecules in one mole of any gas is 6.0255×10^{23}

which is called the *Avogadro number*. The amount of a substance which contains as many elementary units as there are in 0.012 kg (or 12 g) of carbon-12 is called a *mole*. In other words, a mole of a substance is the number of grams equal to its molecular mass in grams. Thus, the mass of 1 mole of carbon is 12 g or 0.012 kg and that of 1 mole of oxygen is 32 g (0.032 kg).

10.23 EQUATION OF STATE OF AN IDEAL GAS

The relationship between the pressure P, volume V and and absolute temperature T of a gas is called its *equation of state*. Combining Eqs (10.35), (10.36) and (10.37) we have

$$\frac{PV}{T} = \text{constant} \tag{10.38}$$

The value of the constant depends upon the amount or mass of the gas.

Molar Gas Constant

For one mole of gas, the corresponding constant in Eq. (10.38) is called the *molar gas constant* and is denoted by the symbol R. So if one of mole of a gas occupies a volume $V = V_m$ at temperature T and pressure P, we write

$$PV_m = RT$$

Since n moles of the gas will occupy a volume $V = n\,V_m$ at the same temperature and pressúre, the equation of state for n moles is thus

$$PV = nRT \tag{10.39}$$

From Avogadro's law, it follows that one mole of all gases, at the same temperature and pressure, occupies equal volume. Experiments confirm that one mole of any gas occupies 22.4 litres at STP. Consequently, for one mole the ratio $\frac{PV}{T}$ is constant for all gases. This constant is the molar gas constant and can be evaluated as follows:

At STP $\quad V = 22.4 \text{ litre} = 22.4 \times 10^{-3}\ \text{m}^3$

$$P = 0.76 \text{ m Hg} = 1.013 \times 10^5\ \text{N m}^{-2}$$

$$T = 273\ \text{K}$$

$$\therefore \quad R = \frac{PV}{T} = \frac{1.013 \times 10^5 \times 22.4 \times 10^{-3}}{273}$$

$$= 8.315\ \text{J mol}^{-1}\ \text{K}^{-1}$$

10.24 KINETIC INTERPRETATION OF TEMPERATURE

Let us, for simplicity, consider one gram molecule (or simply one mole) of a gas having a volume V at absolute temperature T. Let N

be the number of molecules in one mole of the gas and let m be the mass of each molecule. From Eq. (10.33), the pressure exerted by a gas is given by

$$P = \tfrac{1}{3}\frac{mN}{V}\overline{c^2}$$

or

$$PV = \tfrac{1}{3}\, mN\overline{c^2} \tag{10.40}$$

where $\bar{c}$ is the root-mean speed of any molecule of the gas. Now, the equation of state (or the gas equation) for one mole of an ideal gas is

$$PV = RT \tag{10.41}$$

where R is the molar gas constant. Equating Eqs (10.40) and (10.41) we have

$$\tfrac{1}{3}\, mN\overline{c^2} = RT$$

or

$$\frac{2}{3}\; N\left(\tfrac{1}{2}\, m\overline{c^2}\right) = RT$$

or

$$\tfrac{1}{2}\, m\overline{c^2} = \frac{3}{2}\frac{R}{N}T \tag{10.42}$$

where, as stated before, N is the number of molecules in one mole of a gas. N is called Avogadro's number and its value is

$$N = 6.023 \times 10^{23}$$

It is the same for all gases. The ratio $\frac{R}{N}$ is called *Boltzmann's constant* and is usually denoted by k. Thus

$$k = \frac{R}{N} = \frac{8.315\ \text{J K}^{-1}\ \text{mol}^{-1}}{6.023 \times 10^{23}\ \text{molecules mol}^{-1}}$$
$$= 1.380 \times 10^{-23}\ \text{J K}^{-1}/\text{molecule}$$

Now $\frac{1}{2}\, m\overline{c^2}$ is the average kinetic energy $\bar{E}$ of any molecule of the gas, then from Eq. (10.42), we have

$$\bar{E} = \frac{3}{2}\, kT \tag{10.43}$$

Remember that the average kinetic energy of all the N molecules is

$$U = N\bar{E}$$

which from Eq. (10.39) is given by

$$U = N\bar{E} = \tfrac{1}{2}\, mN\overline{c^2} = \frac{3}{2}\, RT \tag{10.44}$$

Equation (10.43) shows that temperature is related to the average kinetic energy of any molecule of the gas. It tells us that *the average kinetic energy of any molecule of an ideal gas is proportional to its absolute temperature.* Thus, the temperature of a gas is a measure of the average kinetic energy of the molecules of the gas. Whenever a gas gains energy, its temperature rises. If the temperature is constant, $\bar{E}$ $(= \frac{1}{2} m\overline{c^2})$ remains constant. Since N is constant, it also implies that U $(= N\bar{E})$, the total average kinetic energy of all molecules in a mole of an ideal gas, is proportional to its absolute temperature.

According to this interpretation of temperature, the average kinetic energy U [see Eq. (10.44)] is zero at $T = 0$. Hence, according to the kinetic theory, the motion of molecules ceases altogether at absolute zero. This result is meaningless because the gas ceases to be a gas and solidifies before $T = 0$ is reached and the kinetic theory of gases does not apply at low temperatures.

Example 10.30 At what temperature would the root-mean square speed of a gas molecule have twice its value at 100 °C?

Solution From Eq. (10.42) we have

$$\overline{c^2} = 3\,\frac{RT}{Nm} = 3\,\frac{k}{m}\,T$$

Thus $$\overline{c_1^2} = \frac{3\,kT_1}{m}$$

$$\overline{c_2^2} = \frac{3\,kT_2}{m}$$

giving $$\frac{\overline{c_1^2}}{\overline{c_2^2}} = \frac{T_1}{T_2}$$

Here $$\bar{c}_2 = 2\bar{c}_1$$

$$T_1 = 273 + 100 = 373 \text{ K}$$

$$\therefore \quad T_2 = T_1\left(\frac{\overline{c_2^2}}{\overline{c_1^2}}\right) = 373 \times 4$$

$$= 1492 \text{ K}$$

$$= 1492 - 273 = 1219 \text{ °C}$$

Example 10.31 Calculate the root mean square speed of oxygen molecules at 1092 K. Given density of oxygen at STP $= 1.424$ kg m^{-3}.

Solution First calculate the root-mean square speed of oxygen at STP.

$$P_0 = 0.76 \text{ m of Hg} = 1.01 \times 10^5 \text{ N m}^{-2}.$$

$$\rho_0 = 1.424 \text{ kg m}^{-3}$$

$\therefore$ $\bar{c}_0$, the root-mean square speed at 0 °C is

$$\bar{c}_0 = \sqrt{\frac{3\,P_0}{\rho_0}}$$

$$= \left(\frac{3 \times 1.01 \times 10^5}{1.424}\right)^{1/2}$$

$$= 4.61 \times 10^2 \text{ m s}^{-1}$$

Now c_{rms} is also given by

$$c_{\text{rms}} = \sqrt{3\,\frac{kT}{m}}$$

$$\therefore \quad \frac{c_{\text{rms}}}{\bar{c}_0} = \sqrt{\frac{T}{T_0}}$$

Here $T_0 = 273$ K and $T = 1092$ K

$$\therefore \quad c_{\text{rms}} = \bar{c}_0\sqrt{\frac{T}{T_0}} = 4.61 \times 10^2 \times \sqrt{\frac{1092}{273}}$$

$$= 9.22 \times 10^2 \text{ m s}^{-1}$$

EXAMPLE 10.32 An electric bulb of volume 250 cm³ was sealed off during manufacture at a pressure of 10^{-3} mm of Hg at 27 °C. Find the number of air molecules in the bulb.

Solution Let N be the number of air molecules in the bulb. It is given that

$$P = 10^{-3} \text{ mm of Hg} = 10^{-4} \text{ cm of Hg}$$

$$V = 250 \text{ cm}^3$$

$$T = 273 + 27 = 300 \text{ K}$$

Now $\quad PV = NkT \quad$ (i)

where k is a constant.

We know that at STP, one mole of a gas occupies a volume $V_0 = 22400$ cm³ and contains $N_0 = 6.02 \times 10^{23}$ molecules (N_0 is the Avogadro's number) and the pressure $P_0 = 76$ cm of Hg and $T_0 = 273$ K. Also

$$P_0 V_0 = N_0 k T_0 \quad \text{(ii)}$$

Dividing Eqs (i) and (ii), we get

$$N = N_0 \times \frac{T_0}{T} \times \frac{P}{P_0} \times \frac{V}{V_0}$$

$$= (6.02 \times 10^{23}) \times \left(\frac{273}{300}\right) \times \left(\frac{10^{-4}}{76}\right) \times \left(\frac{250}{22400}\right)$$

$$= 8.045 \times 10^{15} \text{ molecules}$$

EXAMPLE 10.33 A cylinder of volume 30 litres contains oxygen at a gauge pressure of 15 atm and a temperature of 27 °C. When some oxygen is ejected from the cylinder, the gauge pressure falls to 11 atm and its temperature to 17 °C. Determine the mass of oxygen ejected. $R = 8.3$ J mol^{-1} K^{-1} and molecular mass of oxygen = 32.

Solution Let P_1 and T_1 be the initial pressure and initial absolute temperature of oxygen and P_2 and T_2 their values after some oxygen is ejected. Let N_1 be the number of molecules initially present in the cylinder and N_2 the number of molecules left over. Then (since the volume V remains unchanged) we have

$$P_1V = N_1kT_1 \tag{i}$$

$$P_2V = N_2kT_2 \tag{ii}$$

where $k = \frac{R}{N_0}$, where $N_0 = 6.023 \times 10^{23}$ is the Avogadro number

i.e. $k = \frac{8.3 \text{ J mol}^{-1} K^{-1}}{6.023 \times 10^{23} \text{ molecule mol}^{-1}} = 1.38 \times 10^{-23} \text{ JK}^{-1}$ per molecule

Also

$$P_1 = 15 \text{ atm} = 15 \times 1.01 \times 10^5 \text{ Pa}$$

$$P_2 = 11 \text{ atm} = 11 \times 1.01 \times 10^5 \text{ Pa}$$

$$V = 30 \text{ litres} = 30 \times 10^{-3} \text{ m}^3$$

$$T_1 = 273 + 27 = 300 \text{ K}$$

$$T_2 = 273 + 17 = 290 \text{ K}$$

Substituting these values in (i) and (ii) we get

$$N_1 = 10.87 \times 10^{24}$$

$$N_2 = 8.25 \times 10^{24}$$

∴ Number of molecules of oxygen ejected

$$= N_1 - N_2 = 2.62 \times 10^{24}$$

The molecular mass of oxygen = 32, i.e. one mole of oxygen (which contains 6.023×10^{23} molecules) has a mass of 32 g or 0.032 kg.

Hence 2.62×10^{24} molecules will have a mass of

$$\frac{0.032 \times 2.62 \times 10^{24}}{6.032 \times 10^{23}}$$

$$= 0.14 \text{ kg}$$

EXAMPLE 10.34 A narrow tube of length 1 metre is closed at one end and contains a 76.0 cm column of mercury which traps 15.0 cm column of air when held horizontally. What will happen if the tube is held vertically with the open end at the bottom?

Solution When held horizontally, the pressure of 15.0 cm column of air is atmospheric $= P_1$ (say) $= 76.0$ cm of Hg. When the tube is held vertically, let x cm length of mercury flow out so that the new length of the air column above mercury is now

$$15 + (100 - 76 - 15) + x = (24 + x) \text{ cm}$$

If the temperature remains constant, the pressure P_2 of this column of air is (from Boyle's law)

$$P_2 = \frac{15\, P_1}{24 + x} = \frac{15 \times 76}{24 + x} \text{ cm of Hg}$$

Now Pressure P_2 + pressure due to $(76 - x)$ cm of Hg

$$= \text{atmospheric pressure} = 76 \text{ cm of Hg}$$

i.e.

$$\frac{15 \times 76}{24 + x} + 76 - x = 76$$

which gives

$$x^2 + 24x - 15 \times 76 = 0$$

Solving this quadratic equation for the positive root of x gives

$$x = 23.8 \text{ cm}$$

Thus 23.8 cm length of mercury will flow out of the tube.

SUPPLEMENT

S.1 DERIVATION OF HOOKE'S LAW

Referring again to Fig. 10.4 we notice that there are two equal and opposite forces acting on the experimental wire—one due to the weight pulling the wire downwards and the other due to the reaction of the support pulling it upwards. Thus the tensile force consists of two equal and opposite forces tending to pull the atoms of the wire away from each other.

Let us suppose that due to the applied force F the separation between pairs of atoms increases by an amount, say, x. Each atom is in equilibrium (since the wire is not moving bodily) under the influence of two forces—a force f (which must be related to the applied force F) and a restoring force f_R due to the attraction by the neighbouring atoms. For small displacements x, the restoring force is proportional to the displacement, i.e.

$$f_R = kx = f_{\text{ext}} = f$$

where k is a constant which is a measure of how tightly this atom is bound to its neighbouring atoms. Thus

$$x = \frac{f}{k}$$

If the original inter-atomic separation is a, then the strain is given by

$$\text{strain} = \frac{\text{change in separation}}{\text{original separation}} = \frac{x}{a}$$

Therefore

$$\text{strain} = \frac{f}{ka} \tag{i}$$

To find the relation between f and the actual external applied force F, let us refer to Fig. 10.67. If A is the cross-sectional area of the wire, then the stress due to force F is

$$\text{Stress} = \frac{\text{applied force}}{\text{cross-sectional area}} = \frac{F}{A}$$

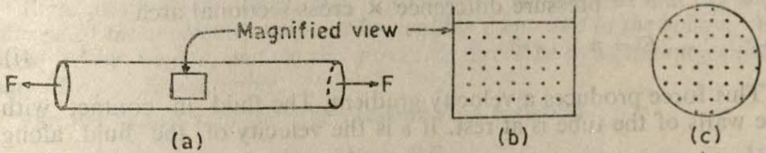


Fig. 10.67 (a) Wire under two equal and opposite forces (b) magnified view of a very small part of the wire; the dots represent atoms (c) the arrangement of atoms in the cross-section of the wire.

In any cross-section, the atoms are arranged on the corners of a square of side a [see Figs 10.67 (b) and (c)]. So the number of atoms in a cross-section is A/a^2 and the force f on each atom is

$$f = \frac{F}{A/a^2} = \frac{Fa^2}{A} = \text{stress} \times (a^2)$$

Substituting this in Eq. (i) we have

$$\text{Strain} = \frac{k}{a}\ (\text{stress})$$

or

$$\text{Stress} = \text{constant} \times \text{strain}$$

which is Hooke's law.

S.2 DERIVATION OF POISEIULLE'S FORMULA

Consider a fluid flowing steadily through a horizontal tube of a nar-
We bore. Due to viscosity a force is needed to maintain a steady flow.
rowwill deduce a relation between the rate of fluid flow and the force
need ed.

Consider a tube AB of length l and a uniform circular cross-section of a small radius r fixed horizontally on the side of a vessel containing a fluid up to height h as shown in Fig. 10.68. If ρ is the density of the fluid, then the presssure at $A = P_0 + h\rho g$ and that at $B = P_0$, the atmospheric pressure. The pressure difference between ends A and B is

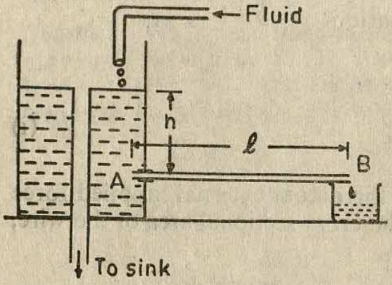


Fig. 10.68 Steady fluid flow under pressure through a tube

$$p = h\rho g$$

Under this pressure difference the fluid flows out of the tube. The force pushing the fluid out is given by

$$F = \text{pressure difference} \times \text{cross-sectional area}$$

or $$F = p \times \pi r^2 \qquad \text{(i)}$$

This force produces a velocity gradient. The fluid in contact with the walls of the tube is at rest. If v is the velocity of the fluid along the axis of the tube, the velocity gradient is nearly equal to $\frac{v}{r}$. The force F acts on the cylindrical area of the tube, this area is $2\pi rl$; i.e. $A = 2\pi rl$. From the definition (10.7) of viscosity we then have

$$F_v = -\eta(2\pi rl).\frac{v}{r} = -\eta 2\pi lv \qquad \text{(ii)}$$

where F_v is the viscous force. When the flow is steady, the external force just overcomes the viscous force so that $F_v = -F$ and using Eq. (i) in (ii) we get

$$v = \frac{pr^2}{2\eta l} \qquad \text{(iii)}$$

The rate of flow (Q) of the fluid is given by

$$Q = \frac{\pi r^2 l}{t}$$

where t is the time taken by the fluid to travel the length l of the tube, i.e. velocity (v) of the fluid is l/t. Therefore,

$$Q = \pi r^2 v \qquad \text{(iv)}$$

Using Eq. (iii) in Eq. (iv) we get

$$Q = \frac{\pi r^4 p}{2\eta l}$$

A more exact derivation involving the use of integral calculus, however, gives

$$Q = \frac{\pi r^4 p}{8\,\eta l}$$

which is Poiseiulle's formula for the steady flow of a fluid through a tube.

S.3 VAN DER WAALS' EQUATION OF STATE

The equation of state $PV = RT$ holds for an ideal gas. The behaviour of real gases shows departures from an ideal gas behaviour specially at high pressures. The model of an ideal gas is based on a number of assumptions which are listed in Sec. 10.21. Van der Waals modified the ideal gas equation $PV = RT$ by taking into account two of these assumptions which may not be valid. He argued that (*a*) *the volume of the molecules may not be neglible compared to the volume V occupied by the gas, and* (*b*) *the attractive forces between the molecules may also not be negligible.*

Van der Waals said that the pressure P in equation

$$PV = RT \tag{i}$$

is less than the true pressure by an amount p because of attractive forces between molecules. According to him, this pressure 'defect' p is proportional to the product of (a) the number of molecules striking a unit area of the wall per second and (b) the number of molecules per unit volume behind them exerting attractive forces. For a given volume, both these numbers are proportional to the density ρ of the gas so that pressure defect p is proportional to ρ^2. For a given mass of gas, $\rho \propto \frac{1}{V}$. Hence

$$p \propto \frac{1}{V^2}$$

or

$$p = \frac{a}{V^2} \tag{ii}$$

where a is a constant depending on the nature of the gas. Thus the true pressure of the gas is $P + p = P + \frac{a}{V^2}$.

He further argued that the volume V in Eq. (i) is not the true volume of the gas because the molecules themselves occupy a finite volume. The true volume of the gas, according to him. is $(V\text{-}b)$ where b is a factor depending on the actual volume of the molecules themselves.

Thus *van der Waal's equation for real gases is*

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \tag{iii}$$

At high pressures, when the molecules are too many and too close together, the correction factors a and b both become important. But at low pressures, when they are not too many and not too close together, a gas behaves like an ideal gas and obeys the equation $PV = RT$.

S.4 DISTRIBUTION OF MOLECULAR SPEEDS

So far we have used the term 'root-mean-square' speed of the molecules of a gas. It is some sort of average speed of the molecules. Not all molecules move with the same speed. What is the distribution of molecular speeds? What is the range of speeds for most molecules in a gas? Maxwell answered these questions on the basis of the advanced kinetic theory of gases. His results are shown in Fig. 10.69.

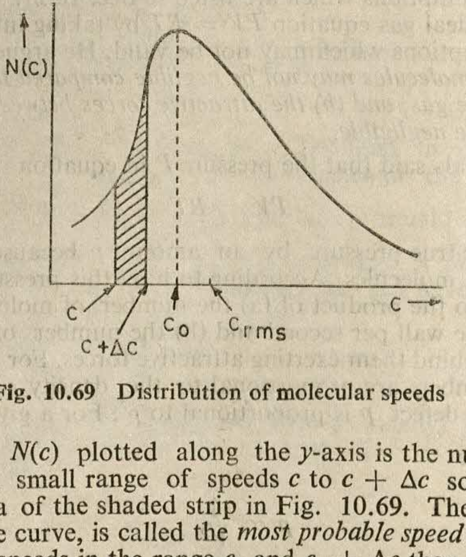


Fig. 10.69 Distribution of molecular speeds

The number $N(c)$ plotted along the y-axis is the number ΔN of molecules in a small range of speeds c to $c + \Delta c$ so that $\Delta N = N(c)\,\Delta c$ = area of the shaded strip in Fig. 10.69. The value c_0, the maximum of the curve, is called the *most probable speed* because more molecules have speeds in the range c_0 and $c_0 + \Delta c$ than in any other similar range Δc. This distribution of molecular speeds is called a *Maxwellian distribution.* The value c_{rms} is the root-mean-square speed. For such a distribution, c_0 and c_{rms} are related as

$$c_0 = \sqrt{\frac{2}{3}}\, c_{\text{rms}}$$

The wide distribution of molecular speeds in a gas has an interesting consequence. The average speeds of the molecules for gases, such as oxygen, nitrogen, carbon dioxide and water vapour, at moderate temperatures are of the order of (0.5 to 1.0) km s^{-1}, whereas for lighter gases, such as hydrogen and helium, they are (2-3) km s^{-1}. We have seen in Chapter 8 that the escape velocity for the earth is $v_e = \sqrt{2gR_e}$. **Molecules with velocities greater than v_e will escape from the earth. If** their fraction is sufficiently large, then the molecules of such gases will escape. Due to their higher speeds, the lighter gases (hydrogen and helium) have, therefore, escaped from the earth.

On the other hand, the escape velocity for the moon is much smaller. Consequently, even the heavier gases like oxygen, nitrogen, etc. have a chance to escape from the moon. The gravitational pull of the moon is too weak to hold these gases. Thus, in course of time, the gases that were present in the atmosphere of the moon, have escaped from it. This is believed to be the reason why the moon has no atmosphere.

S.5 COLLISIONS, MEAN FREE PATH AND DIFFUSION

During their random motion the molecules of a gas often come close to each other. When the distance between two molecules is comparable with the diameter *d* of a molecule (which is a few angstroms) the forces between them become very strong. As a result, their individual momenta before and after the encounter are different. When this happens a 'collision' is said to have occurred.

We can estimate the frequency of such collisions. Suppose the average speed of a molecule of diameter *d* is $\bar{c}$. In one second, this molecule sweeps out a volume $\pi d^2 \bar{c}$. If it finds any other molecules in this volume, it will suffer collisions with them. If n is the number density (i.e. number per unit volume) of the molecules then the number of molecules in this volume is $\pi d^2 \bar{c}\, n$. Thus

Number of collisions per second $= \pi\, d^2\, \bar{c}\, n$, i.e. collision frequency $(\nu_c) = \pi d^2 \bar{c} n$

Therefore, the average time between two collisions (called the collision period τ_c) is

$$\tau_c = \frac{1}{\nu_c} = \frac{1}{\pi d^2 \bar{c} n}$$

Hence the average distance a molecule travels before it suffers a collision with another molecule is

$$l_c = \bar{c}\, \tau_c = \frac{1}{\pi d^2 n}$$

This distance is called the *mean free path*. For air at 0° C and at atmospheric pressure the value of $l_c \simeq 3 \times 10^{-7}$ m, which is very short.

Diffusion

What is the effect of these collisions? Due to collisions, two gases diffuse into each other. The rate of diffusion depends on the collision frequency. The higher the average velocity $\bar{c}$, the higher is the collision frequency and the greater will be the diffusion rate. If r_A and r_B are the rates of diffusion of gases A and B, then

$$\frac{r_A}{r_B} = \frac{\bar{c}_A}{\bar{c}_B}$$

From kinetic theory the pressures of the gases are given by

$$P_A = \tfrac{1}{3}\rho_A \overline{c_A^2}$$

$$P_B = \tfrac{1}{3}\rho_B \overline{c_B^2}$$

In the steady state, when the rates of diffusion in each other are steady, $P_A = P_B$, i.e.

$$\rho_A \overline{c_A^2} = \rho_B \overline{c_B^2}$$

or $$\frac{\bar{c}_A}{\bar{c}_B} = \sqrt{\frac{\rho_B}{\rho_A}}$$

so that $$\frac{r_A}{r_B} = \sqrt{\frac{\rho_B}{\rho_A}}$$

This is *Graham's law of diffusion* which states that the rate of diffusion of a gas is inversely proportional to its $\sqrt{\text{density}}$. Thus, the denser the gas, the slower is the rate of diffusion.

SUMMARY

When a deforming force is applied to a body, it may suffer a change in length, volume or shape. The strain in the body may, therefore, be longitudinal strain, volume strain or shear strain. The corresponding moduli of elasticity are called Young's modulus, bulk modulus and shear modulus. Within the elastic limit, the stress developed in a solid is proportional to the strain produced in it. This is Hooke's law.

Fluids cannot withstand a tangential force and hence are not sheared. The forces acting on a fluid in equilibrium are always perpendicular to its surface. The force per unit area exerted normally on a surface is called pressure. The pressure in a fluid in equilibrium is the same everywhere. When a body is immersed in a fluid, it experiences an upward buoyant force equal to the weight of the fluid displaced.

The frictional force between the layers of a fluid in motion is called viscosity. When a body moves in a fluid, it experiences a retarding force which depends on its radius and velocity and the density and viscosity of the fluid. The total energy of a fluid flowing through a pipe always remains constant throughout the flow. This is Bernoulli's principle.

The free surface of a liquid at rest is under tension (called surface tension) which tends to contract the surface so as to have minimum surface area.

The quation of state of an ideal gas is $PV = RT$. For real gases, the equation of state is $\left(P + \frac{a}{V^2}\right)(V - b) = RT$.

EXERCISES

1. **Distinguish between crystalline and amorphous solids. Give two examples of each.**
2. **What do you understand by stress and strain?**
3. **What is elastic limit? State Hooke's law.**
4. **What is Young's modulus of a substance? How is it experimentally measured?**
5. **What is bulk modulus? What is the meaning of the statement, 'the bulk modulus of copper is 1.2×10^{11} N m^{-2}'?**
6. **What are shearing stress and shearing strain? Define shear modulus. How is the shear modulus of a substance measured? What is the use of this knowledge?**
7. **Show that when a wire is stretched within the elastic limit, the potential energy per unit volume stored in it is half the product of stress and strain.**
8. **(a) State and prove Pascal's law of fluid pressure. Give two applications.**
   (b) Deduce the pressure-depth formula.
9. **What is atmospheric pressure due to? How is it measured? What is its value at sea-level?**
10. **Discuss the variation of atmospheric pressure with altitude.**
11. **State and prove Archimedes' principle.**
12. **Distinguish between streamline and turbulent flow of a liquid.**
13. What is critical velocity of a fluid? Explain the significance of Reynold's number Establish a relation between them using dimensional considerations.
14. Enunciate Newton's law of viscous flow and deduce from it the definition of the coefficient of viscosity of a liquid. What are its units and dimensions?
15. Using dimensional considerations, deduce Poiseuille's formula for the rate of flow of a liquid through a capillary tube. How will you use this formula to determine the coefficient of viscosity of water?
16. State Stokes' law and deduce it from dimensional analysis.
17. What is terminal velocity? Using Stokes' law obtain an expression for it. How will you experimentally measure the coefficient of viscosity of a viscous liquid such as glycerine or castor oil?
18. Deduce the equation of continuity of an incompressible and non-viscous fluid flowing steadily through a pipe.
19. State and prove Bernoulli's theroem for flow of fluids. Discuss the special cases when (i) the fluid is stationary, (ii) the pressure at two points is the same and (iii) the mean horizontal level of the fluid flow is the same.
20. What is a venturimeter? How will you use it to determine the rate of flow of a liquid in a pipe?

21. What is the Pitot tube? How will you use it to determine the velocity of flow of a liquid.
22. Give two examples that illustrate Bernoulli's theorem.
23. On the basis of Bernoulli's principle how will you explain:
    - (i) The working of an atomizer
    - (ii) An object supported by a jet of air
    - (iii) The curving of a spinning ball
    - (iv) The lift of an aircraft wing.
24. Explain clearly what happens to the motion of a small spherical body when it is dropped from rest in a viscous liquid such as glycerine.
25. What do you mean by surface tension of a liquid? How do you explain this phenomenon in terms of molecular forces?
26. Describe three experiments that suggest that the surface of a liquid is in a state of tension.
27. What do you mean by surface energy of a liquid? Give an experiment to illustrate your answer.
28. Explain what determines the shape of a liquid meniscus, and define angle of contact. What is the meaning of the statement 'the angle of contact of mercury with glass is 135°'?
29. Deduce an expression for the excess pressure inside a spherical drop of a liquid in terms of its radius of curvature and surface tension. How is the expression modified in the case of a soap bubble?
30. What is capillarity? Give three examples of this phenomenon from everyday life.
31. Obtain an expression for the rise of a liquid in a capillary tube. What happens if the length of the capillary tube is less than that required for the rise of a liquid in it.
32. Give reasons for the following:
    - (i) Some insects can walk on the surface of water in a lake.
    - (ii) A small steel needle can be made to float on water although it is eight times more dense than water.
    - (iii) Moisture can reach the leaves through the stem of a tree.
    - (iv) Extremely tiny water drops act like ball-bearings and help smooth ice-skating.
    - (v) Water rises and mercury falls in a capillary tube.
    - (vi) A small drop of mercury is almost perfectly spherical whereas a large drop is flattened at the top.
    - (vii) Two glass plates stick together when a few drops of water are put between them.
    - (viii) A drop of aniline assumes a perfectly spherical shape when immersed in brine of the same density.
    - (ix) Although the surface tension of mercury is less than that of water, yet a steel paper-clip floats on mercury but sinks in water.
33. State the main assumptions of the kinetic theory of gases and give a qualitative explanation for the following:
    - (i) Evaporation is accompanied by cooling.
    - (ii) A gas exerts pressure on the walls of the container.

34. What is the model of an ideal gas in the kinetic theory? Obtain an expression for the pressure exerted by an ideal gas.
35. Show that the pressure exerted by an ideal gas is two-thirds of the average kinetic energy of the molecules in a unit volume of the gas.
36. What do you understand by the root-mean square speed of gas molecules? Give the kinetic interpretation of temperature.
37. Deduce the following laws from kinetic considerations:
    (i) Avogadro's law
    (ii) Boyle's law
38. Derive Hooke's law of elasticity from first principles.
39. Write a short note on Van der Waals' equation of state.
40. What do you understand by Maxwellian distribution of molecular speeds in a gas?
41. (a) What is meant by the term 'collision'?
    (b) What is the mean free path? Obtain an expression for it.
42. What is diffusion? How is it explained?

PROBLEMS

1. What force must be applied to a steel wire of length 6 m and diameter 1.6 mm to produce an extension of 1 mm? Young's modulus of steel = 2.0×10^{11} N m^{-2}.
2. What stress would cause a wire to increase in length by 0.1 per cent if the Young's modulus of the wire is 12×10^{10} N m^{-2} ? Calculate the force required to produce this stress if the diameter of the wire is 0.56 mm.
3. Determine the length of a copper wire, which when suspended vertically breaks under its own weight. Density of copper = 8600 kg m^{-3}, breaking stress for copper = 2.45×10^{8} N m^{-2} and $g = 9.8$ m s^{-2}.
4. Two wires each of length 1 m and radius 0.2 mm are tied together at both ends. A load of 25 kg hangs from the composite wire. If the Young's moduli are 12×10^{10} and 18×10^{10} N m^{-2}, find tension in each wire. Also calculate the total extension. Take $g = 10$ m s^{-2}.

5 The density of water at the surface of a lake is 1000 kg m^{-3}. Find the density at a depth of 400 m. Bulk modulus for water is 2×10^{9} N m^{-2} and $g = 9.8$ m s^{-2}.

6. A steel cable whose cross-sectional area is 7.5 cm^2 is used to support an elevator cab weighing 2.5×10^{4} N. If the stress in the cable is not to exceed 20% of the cable's elastic limit of 2.5×10^{8} Pa, find the maximum permissible upward acceleration. $g = 9.8$ m s^{-2}.
7. Determine the length of a lead wire which when suspended vertically breaks under its own weight.

   Breaking stress for lead = 2×10^{7} N m^{-2}
   Density of lead = 11300 kg m^{-3}, $g = 9.8$ m s^{-2}
8. A wire of length 1.20 m and diameter 2.6 mm is stretched between two rigid supports. If the temperature of the wire is decreased by 30°C, calculate the tension in the wire. Young's modulus of the material of the wire is 16×10^{10} N m^{-2} and its coefficient of linear expansion is 24×10^{-6} per °C.

9. An elevator cable can carry a total weight of 1000 kg and accelerate it upward at 2.20 m s^{-2}. Calculate the diameter (in cm) the cable must have if the safe working stress for the material of the cable is 1.2×10^8 N m^{-2}.
10. What is the pressure required to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100 °C? Young's modulus of steel is 20×10^{10} Nm^{-2} and the coefficient of linear expansion of steel is 11×10^{-6} per °C.
11. A copper wire and a steel wire, each of length 1.5 m and diameter 2 mm, are joined end-to-end to form a composite wire of length 3 m. The wire is loaded until its length becomes 3.003 m. Find the strains in the copper and steel wires and the force applied. Young's modulus of copper is 1.2×10^{11} Nm^{-2} and that of steel is 2.0×10^{11} Nm^{-2}.
12. The interior of a submarine located at a depth of 60 m in sea-water is maintained at sea-level atmospheric pressure. Find the force acting on a window 40 cm^2 in area. The density of sea-water is 1030 kg m^{-3}.
13. A block of brass has a mass of 1 kg and a density 8000 kg m^{-3}. It is suspended from a string. Find the tension in the string if the block is in air and if it is completely immersed in water. Neglect the buoyancy of the air.
14. What is the minimum area of a slab of ice 0.50 m thick floating on water that will hold up a car weighing 1000 kg? The density of ice is 920 kg m^{-3}.
15. A block of wood floats in water with 2/3 of its volume submerged. In oil it has 0.90 of its volume submerged. Find the densities of wood and oil.
16. A cubical block of wood 10 cm on a side and of density 0.5 g cm^{-3} floats in a jar of water. Oil of density 0.8 g cm^{-3} is poured on the water until the top of the oil layer is 4 cm below the top of the block. Calculate (a) the depth of the oil layer and (b) the gauge pressure at the lower face of the block.
17. A cubical block of wood 10 cm along each side floats at the interface between oil and water with its lower surface 2 cm below the interface. The heights of the oil and water columns are 10 cm each. The density of oil is 0.8 g cm^{-3}.
    (a) What is the mass of the block?
    (b) What is the pressure on the lower surface of the block?
18. A solid weighs 237.5 g in air and 212.5 g when totally immersed in a liquid of density 0.9 g cm^{-3}. Calculate (a) the density of the solid and (b) the density of a liquid in which the solid would float with one-fifth of its volume exposed above the liquid surface.
19. A gas bubble of diameter 1.0 cm rises steadily through a solution of density 2000 kg m^{-3} at the rate of 0.5 cm s^{-1}. Calculate the coefficient of viscosity of the solution. (Neglect density of the gas.)
20. A small spherical steel ball of diameter 1.0 mm is falling steadily at a speed of 5 cm s^{-1} in a solution of viscosity 30 N m^{-2} s. Calculate the retarding force experienced by the ball.
21. A sphere of diameter 2 mm and density 9000 kg m^{-3} is dropped from rest in a liquid of viscosity 0.04 N m^{-2} s. Assuming that the average acceleration is half the initial acceleration, calculate the time it takes to attain its terminal velocity.

22. At what speed will water emerge from a hole in a tank at which the gauge pressure is 2×10^5 N m^{-2}?
23. A tank is filled with water to a height of 5 m. A hole is made on one of its walls at a depth of 1 m below the water surface. (i) Find the distance from the foot of the wall at which the stream strikes the ground (ii) Can a hole be made at another depth so that the second stream strikes the ground at the same place? If so, at what depth?
24. A liquid drop of diameter 0.5 cm breaks up into 27 tiny droplets all of the same size. If the surface tension of the liquid is 0.07 N m^{-1}, calculate the resulting change in energy.
25. What would be the gauge pressure of an air bubble of 0.2 mm diameter situated just below the surface of water. The surface tension of water is 7.5×10^{-2} Nm^{-1}.
26. Find the pressure difference between the inside and outside of a soap bubble of 5 mm diameter. Surface tension of soap solution is 1.6 N m^{-1}.
27. A U-tube is such that the diameter of one limb is 0.4 mm and that of the other is 0.8 mm. When water is poured into it, it is found that the difference of water levels in the two limbs is 3.57 cm. If the angle of contact is assumed to be zero, calculate the surface tension of water.
28. The internal diameter of a tube of mercury barometer is 3.00 mm. Find the corrected reading of the barometer after allowing for the error due to surface tension, if the observed reading is 74.69 cm Hg. Surface tension of mercury = 4.8×10^{-1} N m^{-1}, angle of contact is 140°.
29. A soap bubble has a radius of 3 cm and another soap bubble 4 cm. If the two coalesce under isothermal conditions, calculate the radius of the bubble formed.
30. Calculate the work done against surface tension in blowing a soap bubble of 1 cm diameter, if the surface tension of soap solution is 2.0×10^{-2} Nm^{-1}.
31. Calculate the amount of work required to break up a drop of water of radius 0.5 cm into tiny droplets of water, each of radius 1 mm, assuming isothermal condition. Also determine the number of droplets so formed. (Surface tension of water = 7.0×10^{-2} N m^{-1}.)
32. An air bubble in a liquid of surface tension 1.0×10^{-3} N m^{-1} gradually grows from a radius of 1.0×10^{-4} m to 1.0×10^{-3} m in 9 μs. Calculate the average rate of change of pressure inside.
33. A capillary tube of diameter 0.4 mm is dipped vertically inside (i) water of surface tension 6.5×10^{-2} N m^{-1} and zero angle of contact and (ii) a liquid of density 800 kg m^{-3}, surface tension 5.0×10^{-2} N m^{-1}, angle of contact 30°. Calculate the height to which the liquid rises in the capillary in each case.
34. Calculate the root-mean square speed of oxygen molecules at STP. Density of oxygen at STP = 1.424 kg m^{-3}.
35. Calculate the root-mean square speed of hydrogen molecules at 300 K. Mass of hydrogen molecule = 3.34×10^{-27} kg, Boltzmann constant = 1.38×10^{-23} JK^{-1} per molecule.
36. At what temperature will the root-mean square speed of gas molecules be half its value at 400 K?

9. An elevator cable can carry a total weight of 1000 kg and accelerate it upward at 2.20 m s^{-2}. Calculate the diameter (in cm) the cable must have if the safe working stress for the material of the cable is 1.2×10^8 N m^{-2}.
10. What is the pressure required to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100 °C? Young's modulus of steel is 20×10^{10} Nm^{-2} and the coefficient of linear expansion of steel is 11×10^{-6} per °C.
11. A copper wire and a steel wire, each of length 1.5 m and diameter 2 mm, are joined end-to-end to form a composite wire of length 3 m. The wire is loaded until its length becomes 3.003 m. Find the strains in the copper and steel wires and the force applied. Young's modulus of copper is 1.2×10^{11} Nm^{-2} and that of steel is 2.0×10^{11} Nm^{-2}.
12. The interior of a submarine located at a depth of 60 m in sea-water is maintained at sea-level atmospheric pressure. Find the force acting on a window 40 cm^2 in area. The density of sea-water is 1030 kg m^{-3}.
13. A block of brass has a mass of 1 kg and a density 8000 kg m^{-3}. It is suspended from a string. Find the tension in the string if the block is in air and if it is completely immersed in water. Neglect the buoyancy of the air.
14. What is the minimum area of a slab of ice 0.50 m thick floating on water that will hold up a car weighing 1000 kg? The density of ice is 920 kg m^{-3}.
15. A block of wood floats in water with 2/3 of its volume submerged. In oil it has 0.90 of its volume submerged. Find the densities of wood and oil.
16. A cubical block of wood 10 cm on a side and of density 0.5 g cm^{-3} floats in a jar of water. Oil of density 0.8 g cm^{-3} is poured on the water until the top of the oil layer is 4 cm below the top of the block. Calculate (a) the depth of the oil layer and (b) the gauge pressure at the lower face of the block.
17. A cubical block of wood 10 cm along each side floats at the interface between oil and water with its lower surface 2 cm below the interface. The heights of the oil and water columns are 10 cm each. The density of oil is 0.8 g cm^{-3}.
    (a) What is the mass of the block?
    (b) What is the pressure on the lower surface of the block?
18. A solid weighs 237.5 g in air and 212.5 g when totally immersed in a liquid of density 0.9 g cm^{-3}. Calculate (a) the density of the solid and (b) the density of a liquid in which the solid would float with one-fifth of its volume exposed above the liquid surface.
19. A gas bubble of diameter 1.0 cm rises steadily through a solution of density 2000 kg m^{-3} at the rate of 0.5 cm s^{-1}. Calculate the coefficient of viscosity of the solution. (Neglect density of the gas.)
20. A small spherical steel ball of diameter 1.0 mm is falling steadily at a speed of 5 cm s^{-1} in a solution of viscosity 30 N m^{-2} s. Calculate the retarding force experienced by the ball.
21. A sphere of diameter 2 mm and density 9000 kg m^{-3} is dropped from rest in a liquid of viscosity 0.04 N m^{-2} s. Assuming that the average acceleration is half the initial acceleration, calculate the time it takes to attain its terminal velocity.

22. At what speed will water emerge from a hole in a tank at which the gauge pressure is 2×10^5 N m^{-2}?
23. A tank is filled with water to a height of 5 m. A hole is made on one of its walls at a depth of 1 m below the water surface. (i) Find the distance from the foot of the wall at which the stream strikes the ground (ii) Can a hole be made at another depth so that the second stream strikes the ground at the same place? If so, at what depth?
24. A liquid drop of diameter 0.5 cm breaks up into 27 tiny droplets all of the same size. If the surface tension of the liquid is 0.07 N m^{-1}, calculate the resulting change in energy.
25. What would be the gauge pressure of an air bubble of 0.2 mm diameter situated just below the surface of water. The surface tension of water is 7.5×10^{-2} Nm^{-1}.
26. Find the pressure difference between the inside and outside of a soap bubble of 5 mm diameter. Surface tension of soap solution is 1.6 N m^{-1}.
27. A U-tube is such that the diameter of one limb is 0.4 mm and that of the other is 0.8 mm. When water is poured into it, it is found that the difference of water levels in the two limbs is 3.57 cm. If the angle of contact is assumed to be zero, calculate the surface tension of water.
28. The internal diameter of a tube of mercury barometer is 3.00 mm. Find the corrected reading of the barometer after allowing for the error due to surface tension, if the observed reading is 74.69 cm Hg. Surface tension of mercury = 4.8×10^{-1} N m^{-1}, angle of contact is 140°.
29. A soap bubble has a radius of 3 cm and another soap bubble 4 cm. If the two coalesce under isothermal conditions, calculate the radius of the bubble formed.
30. Calculate the work done against surface tension in blowing a soap bubble of 1 cm diameter, if the surface tension of soap solution is 2.0×10^{-2} Nm^{-1}.
31. Calculate the amount of work required to break up a drop of water of radius 0.5 cm into tiny droplets of water, each of radius 1 mm, assuming isothermal condition. Also determine the number of droplets so formed. (Surface tension of water = 7.0×10^{-2} N m^{-1}.)
32. An air bubble in a liquid of surface tension 1.0×10^{-3} N m^{-1} gradually grows from a radius of 1.0×10^{-4} m to 1.0×10^{-3} m in 9 μs. Calculate the average rate of change of pressure inside.
33. A capillary tube of diameter 0.4 mm is dipped vertically inside (i) water of surface tension 6.5×10^{-2} N m^{-1} and zero angle of contact and (ii) a liquid of density 800 kg m^{-3}, surface tension 5.0×10^{-2} N m^{-1}, angle of contact 30°. Calculate the height to which the liquid rises in the capillary in each case.
34. Calculate the root-mean square speed of oxygen molecules at STP. Density of oxygen at STP = 1.424 kg m^{-3}.
35. Calculate the root-mean square speed of hydrogen molecules at 300 K. Mass of hydrogen molecule = 3.34×10^{-27} kg, Boltzmann constant = 1.38×10^{-23} JK^{-1} per molecule.
36. At what temperature will the root-mean square speed of gas molecules be half its value at 400 K?

9. An elevator cable can carry a total weight of 1000 kg and accelerate it upward at 2.20 m s^{-2}. Calculate the diameter (in cm) the cable must have if the safe working stress for the material of the cable is 1.2×10^8 N m^{-2}.
10. What is the pressure required to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100 °C? Young's modulus of steel is 20×10^{10} Nm^{-2} and the coefficient of linear expansion of steel is 11×10^{-6} per °C.
11. A copper wire and a steel wire, each of length 1.5 m and diameter 2 mm, are joined end-to-end to form a composite wire of length 3 m. The wire is loaded until its length becomes 3.003 m. Find the strains in the copper and steel wires and the force applied. Young's modulus of copper is 1.2×10^{11} Nm^{-2} and that of steel is 2.0×10^{11} Nm^{-2}.
12. The interior of a submarine located at a depth of 60 m in sea-water is maintained at sea-level atmospheric pressure. Find the force acting on a window 40 cm^2 in area. The density of sea-water is 1030 kg m^{-3}.
13. A block of brass has a mass of 1 kg and a density 8000 kg m^{-3}. It is suspended from a string. Find the tension in the string if the block is in air and if it is completely immersed in water. Neglect the buoyancy of the air.
14. What is the minimum area of a slab of ice 0.50 m thick floating on water that will hold up a car weighing 1000 kg? The density of ice is 920 kg m^{-3}.
15. A block of wood floats in water with 2/3 of its volume submerged. In oil it has 0.90 of its volume submerged. Find the densities of wood and oil.
16. A cubical block of wood 10 cm on a side and of density 0.5 g cm^{-3} floats in a jar of water. Oil of density 0.8 g cm^{-3} is poured on the water until the top of the oil layer is 4 cm below the top of the block. Calculate (a) the depth of the oil layer and (b) the gauge pressure at the lower face of the block.
17. A cubical block of wood 10 cm along each side floats at the interface between oil and water with its lower surface 2 cm below the interface. The heights of the oil and water columns are 10 cm each. The density of oil is 0.8 g cm^{-3}.
    (a) What is the mass of the block?
    (b) What is the pressure on the lower surface of the block?
18. A solid weighs 237.5 g in air and 212.5 g when totally immersed in a liquid of density 0.9 g cm^{-3}. Calculate (a) the density of the solid and (b) the density of a liquid in which the solid would float with one-fifth of its volume exposed above the liquid surface.
19. A gas bubble of diameter 1.0 cm rises steadily through a solution of density 2000 kg m^{-3} at the rate of 0.5 cm s^{-1}. Calculate the coefficient of viscosity of the solution. (Neglect density of the gas.)
20. A small spherical steel ball of diameter 1.0 mm is falling steadily at a speed of 5 cm s^{-1} in a solution of viscosity 30 N m^{-2} s. Calculate the retarding force experienced by the ball.
21. A sphere of diameter 2 mm and density 9000 kg m^{-3} is dropped from rest in a liquid of viscosity 0.04 N m^{-2} s. Assuming that the average acceleration is half the initial acceleration, calculate the time it takes to attain its terminal velocity.

22. At what speed will water emerge from a hole in a tank at which the gauge pressure is 2×10^5 N m^{-2}?
23. A tank is filled with water to a height of 5 m. A hole is made on one of its walls at a depth of 1 m below the water surface. (i) Find the distance from the foot of the wall at which the stream strikes the ground (ii) Can a hole be made at another depth so that the second stream strikes the ground at the same place? If so, at what depth?
24. A liquid drop of diameter 0.5 cm breaks up into 27 tiny droplets all of the same size. If the surface tension of the liquid is 0.07 N m^{-1}, calculate the resulting change in energy.
25. What would be the gauge pressure of an air bubble of 0.2 mm diameter situated just below the surface of water. The surface tension of water is 7.5×10^{-2} Nm^{-1}.
26. Find the pressure difference between the inside and outside of a soap bubble of 5 mm diameter. Surface tension of soap solution is 1.6 N m^{-1}.
27. A U-tube is such that the diameter of one limb is 0.4 mm and that of the other is 0.8 mm. When water is poured into it, it is found that the difference of water levels in the two limbs is 3.57 cm. If the angle of contact is assumed to be zero, calculate the surface tension of water.
28. The internal diameter of a tube of mercury barometer is 3.00 mm. Find the corrected reading of the barometer after allowing for the error due to surface tension, if the observed reading is 74.69 cm Hg. Surface tension of mercury = 4.8×10^{-1} N m^{-1}, angle of contact is 140°.
29. A soap bubble has a radius of 3 cm and another soap bubble 4 cm. If the two coalesce under isothermal conditions, calculate the radius of the bubble formed.
30. Calculate the work done against surface tension in blowing a soap bubble of 1 cm diameter, if the surface tension of soap solution is 2.0×10^{-2} Nm^{-1}.
31. Calculate the amount of work required to break up a drop of water of radius 0.5 cm into tiny droplets of water, each of radius 1 mm, assuming isothermal condition. Also determine the number of droplets so formed. (Surface tension of water = 7.0×10^{-2} N m^{-1}.)
32. An air bubble in a liquid of surface tension 1.0×10^{-3} N m^{-1} gradually grows from a radius of 1.0×10^{-4} m to 1.0×10^{-3} m in 9 μs. Calculate the average rate of change of pressure inside.
33. A capillary tube of diameter 0.4 mm is dipped vertically inside (i) water of surface tension 6.5×10^{-2} N m^{-1} and zero angle of contact and (ii) a liquid of density 800 kg m^{-3}, surface tension 5.0×10^{-2} N m^{-1}, angle of contact 30°. Calculate the height to which the liquid rises in the capillary in each case.
34. Calculate the root-mean square speed of oxygen molecules at STP. Density of oxygen at STP = 1.424 kg m^{-3}.
35. Calculate the root-mean square speed of hydrogen molecules at 300 K. Mass of hydrogen molecule = 3.34×10^{-27} kg, Boltzmann constant = 1.38×10^{-23} JK^{-1} per molecule.
36. At what temperature will the root-mean square speed of gas molecules be half its value at 400 K?

37. At what temperature will oxygen molecules have the same root-m square speed as that of hydrogen molecules at 173 K?
38. Calculate the rms velocity of oxygen molecules at 300 K. The density oxygen at STP = 1.424 kg m^{-3}.
39. Calculate the average translational kinetic energy of a molecule of a at 1000 K. Given R = 8.32 JK^{-1} mol^{-1}.
40. At what temperature, pressure remaining constant, will the rms speed gas molecules be half its value at 0 °C?
41. Helium gas occupies a volume of 0.04 m^3 at a pressure of 2 × 10^5 N and temperature 300 K. Calculate (i) the mass of helium (ii) the speed of its molecules and (iii) the rms speed when the gas is heated constant pressure to a temperature of 432 K.

11

Heat and Thermodynamics

This chapter is divided into three sections. Section A deals with heat, its measurement and its effects. Section B deals with thermodynamics which is a branch of science that deals with the relationship between heat and mechanical energy. In Section C we will discuss the modes of heat transfer.

SECTION A: HEAT

11.1 INTRODUCTION: HEAT AND TEMPERATURE

Heat is often confused with temperature, but the two are different concepts. Let us consider an example. If a red-hot piece of iron is dropped into water in a beaker, the temperature of water increases and that of the piece of iron decreases, until a thermal equilibrium is reached when they both have the same temperature. Hence, "something" is transferred from iron to water. This "something" that flows from a body at a higher temperature to one at a lower temperature, when placed in contact (or exposed to each other) is called heat.

Until the nineteenth century, scientists believed that heat was some kind of a fluid called *caloric* which flows like any actual fluid (liquid or gas). According to the caloric theory, every object has a certain amount of caloric in it. When caloric was added to an object, its temperature increased; and when caloric escaped from it, its temperature decreased. However, nobody could detect this caloric; so it was assumed to be odourless, tasteless, and invisible. Although caloric was such a mysterious substance, yet the caloric theory could successfully explain many aspects of heat.

One of the main troubles with the caloric theory was that it could not account for the heat generated by friction. By rubbing our hands together or by rubbing two pieces of metal together, we can generate heat indefinitely. If you rub the two metal pieces dipped in water in a beaker and if you have the patience and energy to continue rubbing, you may actually succeed in boiling water in the beaker. The amount of heat generated by friction seems limitless. This does not agree with the caloric theory which maintains that heat is a substance and therefore, every object can contain only a fixed amount of it.

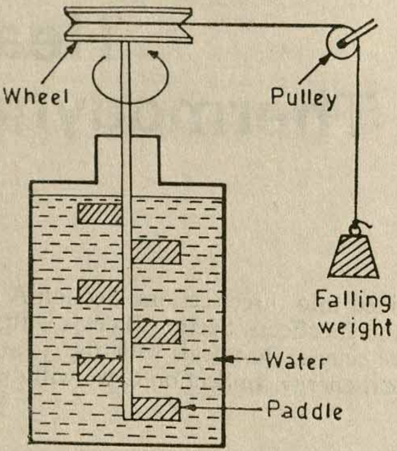


Fig. 11.1 Joule's paddle wheel experiment

It was Count Rumford (1753-1814) who rejected the idea that heat was a substance and said that heat could only be a kind of motion. It was an Englishman, James Prescott Joule (1818-1889), who performed the crucial experiment that settled the question. Joule was convinced that heat is a form of energy. His famous *paddle wheel experiment* is illustrated in Fig. 11.1. The slowly-falling weight does work on the paddle wheel, causing it to turn against the resistance (or friction) offered by the water. The potential energy lost be the falling weight is first converted into kinetic energy of the papple wheel which, in turn, produces heat by friction with the water. By measuring the rise in the temperature of water, Joule could calculate how much heat was produced by a particular amount of mechanical work and came to the conclusion that a fixed amount of mechanical work (or energy) always produced a fixed amount of heat. More accurate experiments have shown that 4.18 J of mechanical work produce 1 cal of heat.

11.2 MODERN CONCEPT OF HEAT

Heat is a form of energy called *thermal energy*. Thermal energy is the total energy (kinetic, vibrational and rotational) of all the individual molecules of which the given object is made. When an object is heated, its thermal energy increases, i.e. its molecules begin to move more violently. This is exactly what the paddle wheel in Fig. 11.1 does. It moves (or stirs) the moleculs of water and makes them move more and more violently (or vigorously), as a result of which the temperature of water rises. Thus, *temperature is a measurable manifestation of the thermal energy of a body*. To further clarify the distinction between temperature and thermal energy, consider an object, say, a hot brick and break it into two halves. Throw away one half. The remaining half will have the same temperature, but it will have only half the thermal energy, because it will have only half the ability to heat another object.

Convention Conventionally, the heat energy Q supplied to a body is taken to be positive ($+Q$) and the heat energy given out by a body is taken to be negative ($-Q$).

Unit of Heat Energy Till recently, the unit of heat energy was the calorie. *A calorie is defined as the amount of heat energy required to raise the temperature of 1 g of water through 1° C, more specifically, from 14.5 °C to 15.5 °C*. However, recently, it has been internationally agreed to express all energy changes in the energy unit, namely, the *joule* (J). A joule is defined as the amount of work done when a force of one newton acts through one metre parallel to itself. The relation between the two is:

$$1 \text{ cal} = 4.18 \text{ J}$$

Thus, if the amount of heat energy supplied to a system is 10 calories, then

$$Q = 41.8 \text{ J}$$

11.3 TEMPERATURE

In order to measure temperature quantitatively we use some property of matter that changes with temperature. *A device which utilizes a property of matter to measure temperature is known as a thermometer.* Several kinds of thermometers are used to measure temperature. Obviously each produces a different scale of temperature which can be related to one another. But there is one 'natural' scale of temperature called the *absolute scale* described in Chapter 10. This scale is based on the ideal gas thermometer discussed below.

Ideal Gas Thermometer

In Chapter 10 we have learnt that ideal or perfect gases and real gases (if their pressure is low and temperature is high) are governed by the ideal gas laws which give the relation between pressure, volume and temperature. The ideal gas equation

$$PV = nRT$$

states that if the volume of a gas is kept constant, the pressure will increase with increase in temperature. This is shown in Fig. 11.2.

When the straight-line graph is extrapolated to meet the temperature axis, it meets at a temperature of −273 °C (the exact value is −273.15 °C) which is called the absolute zero since at this temperature the pressure (or volume) of the gas goes to zero. This may imply that when the temperature is less than −273.15 °C, the volume becomes negative which is obviously not possible. This perhaps suggests that −273.15 °C is the lowest temperature man can achieve. Therefore, −273.15 °C is called the *absolute zero* of temperature. Although scientists have been able to reach within 0.00001 °C of obsolute zero, but the absolute zero itself cannot be reached.

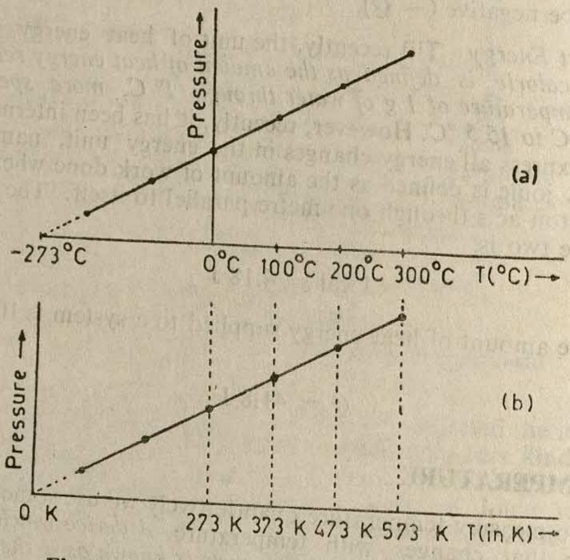


Fig. 11.2 (a) Pressure-temperature (°C) graph
(b) Pressure-temperature (K) graph

Absolute zero is the basis of the scale of temperature called the Kelvin (or absolute) scale. The zero of the scale is at —273.15 °C the size of the degree is the same as on Celsuis scale so that

$$K = {}^\circ\mathrm{C} + 273.15$$

Thus ice melts at 273.15 K and water boils at 373.15 K. The modern fixed-point temperature is the *triple point* of water. It is the temperature at which the three phases of water, namely, ice, liquid water and water vapour are equally stable and coexist. This temperature is defined to be 273.16 K. The triple point is suitable because it is unique, i.e. it occurs at one single temperature = 273.16 K and one single pressure which is about 0.46 cm of Hg (see Section 11.14).

Now suppose the pressure of an ideal gas at triple point T_{tr} (in absolute units) is p_{tr}. Then if the pressure is p at some other temperature T, this temperature T is

$$T = px \left(\frac{T_{tr}}{p_{tr}}\right)$$

provided, of course, the pressures p and p_{tr} are low.

Figure 11.3 shows a typical constant volume gas thermometer. It consists of a bulb B of glass, quartz or platinum which is connected by a capillary tube to a mercury manometer. The bulb B is immersed in the liquid whose temperature is to be measured. The gas in the bulb could be helium, hydrogen, nitrogen or air. The volume of the gas is kept constant by moving the reservoir R up or down till the mercury level in the left arm of the manometer coincides with a given fixed mark M, as shown in the diagram.

The pressure of the gas then is given by

$$p = h \rho g + P_0$$

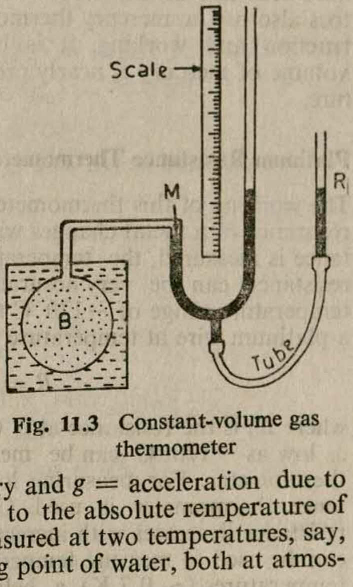


Fig. 11.3 Constant-volume gas thermometer

where P_0 = atmospheric pressure, h = difference of mercury levels in the manometer, ρ = density of mercury and g = acceleration due to gravity. This pressure is proportional to the absolute remperature of the gas. The pressure of the gas is measured at two temperatures, say, the melting point of ice and the boiling point of water, both at atmospheric pressure. We then have

$$p_0 = h_0 \rho g + P_0$$

$$p_{100} = h_{100} \rho g + P_0$$

Also

$$\frac{p_0}{p_{100}} = \frac{T_0}{T_{100}}$$

where T_0 = absolute temperature of mleting ice

and T_{100} = absolute temperature of boiling water.

If the $(T_{100} - T_0)$ is taken to be 100, we have

$$\frac{p_{100}}{p_0} = \frac{T_0 + 100}{T_0}$$

so that T_0 is determined. Some corrections have to be made in the measurement of temperature using a gas thermometer. The corrections are made to account for the expansion of the bulb containing the gas, for the fact that the gas in the connecting tubes is not at the same temperature as the bulb and for the fact that the gas used is not ideal. Furthermore, this thermometer is cumbersome to use.

Mercury Thermometers

Mercury thermometers are the ones most commonly used. You must have seen and used a mercury thermometer in your laboratory. Doctors also use a mercury thermometer. You are familiar with its construction and working. It is based on the fact that the change in volume of mercury is nearly proportional to the change in temperature.

Platinum Resistance Thermometer

The working of this thermometer is based on the principle that the resistance of a metal changes with temperature. If the change of resistance is measured, the temperature can be determined. The electrical resistance can be very accurately measured. It is known that over a temperature range of −170 °C to 200 °C, the electrical resistance R of a platinum wire at temperature t °C is given by

$$R = R_0 (1 + \alpha t)$$

where R_0 is the resistance at 0 °C and $\alpha = 0.004$ per °C. Temperatures as low as −170 °C can be measured fairly accurately by using this thermometer. For measuring lower temperatures, a germanium resistance thermometer is used. Germanium is a semiconductor whose resistance decreases with increase in temperature. This thermometer can be used to measure temperature up to about 4 K. For still lower temperatures (~ 0.7 K), a helium vapour pressure thermometer is used.

Pyrometers

Measurement of low temperatures is not too difficult. Very high temperatures are difficult to measure. Measurement of very high temperatures is called *pyrometry*. Liquid thermometers can measure temperatures up to about 300 °C, and electrical resistance thermometers up to about 1200 °C. Thermo-electric thermometers are used for measuring temperatures as high at 3000 °C.

For still higher temperatures, *pyrometers* are used. Various methods are used in pyrometers. One method is based on the fact that the colour of a hot metal filament depends on its temperature. Suppose we wish to measure the temperature of a very hot furnace. A lighted filament bulb is veiwed in the background of the furnace. If the colour of the radiation from the furnace matches that of the bulb, the filament of the bulb becomes invisible. The temperature of the filament is varied by passing a variable current through it till the filament 'disappears' so to speak. The temperature of the filament is calibrated in terms of current. This device is called the *disappearing filament pyrometer*.

EXAMPLE 11.1 A constant-volume gas thermometer using helium records a pressure of 20.0 kPa at the triple point of water and a pressure of 14.3 kPa at the temperature of solid carbon dioxide (called 'dry ice'). Find the temperature of dry ice.

Solution

$$\text{Triple point of water } (T_{tr}) = 273.16 \text{ K} \simeq 273 \text{ K}$$

$$\text{Pressure at triple point } (p_{tr}) = 20.0 \text{ kPa}$$

Let T be the absolute temperature of 'dry ice'.

$$\text{Pressure at temperature } T = (p) = 14.3 \text{ kPa}$$

Now

$$\frac{p_{tr}}{p} = \frac{T_{tr}}{T}$$

or

$$T = T_{tr} \times \frac{p}{p_{tr}}$$

$$= \frac{273 \times 14.3}{20.0} = 195.2 \text{ K}$$

$$\approx 195 \text{ K or } -78 \text{ °C}$$

EXAMPLE 11.2 The resistance R of the wire of an electrical resistance thermometer at temperature t °C is given by

$$R = R_0 (1 + \alpha t)$$

where R_0 is the value of R at 0° C and α is a constant of the wire. The resistance at the triple point of water is 101.6 Ω and that at 600.5 K is 165.5 Ω. What is temperature when the resistance is 123.4 Ω?

Solution

$$R_1 = 101.6 \ \Omega \text{ at } t_1 = \text{triple point of water} = 0 \text{ °C}$$

$$R_2 = 165.5 \ \Omega \text{ at } t_2 = 600.5 \text{ K} = 600.5 - 273 = 327.5 \text{ °C}$$

$$R_3 = 123.4 \ \Omega \text{ at } t_3 = ?$$

We have $\quad R_1 = R_0 (1 + \alpha t_1)$

or $\quad 101.6 = R_0 (1 + \alpha \times 0) = R_0$

$\therefore \quad R_0 = 101.6 \ \Omega$

Also $\quad R_2 = R_0 (1 + \alpha t_2)$

or $$165.6 = 101.6\,(1 + \alpha \times 327.5)$$

or $$\alpha = \frac{165.5 - 101.6}{101.6 \times 327.5} = 0.0019 \text{ per °C}$$

Now $$R_3 = (1 + \alpha\, t_3)$$

or $$t_3 = \frac{R_3 - R_0}{R_0\,\alpha} = \frac{123.4 - 101.6}{101.6 \times 0.0019}$$
$$= 112.9\ \text{°C} = 385.9\ \text{K}$$

11.4 THERMAL EXPANSION

When heat energy is supplied to a solid, its volume increases. This is known as *thermal expansion*. The thermal expansion of a gas has been made use of in the gas thermometer, and that of a liquid in a mercury thermometer. In both these examples, it turns out that the increase in volume is directly proportional to that in temperature, provided the temperature increase is not too large.

Coefficient of Volume Expansion

Suppose the volume of a substance increases from V to $V + \Delta V$ when its temperature is increased from T to $T + \Delta T$. The coefficient of volume expansion (α_v) of the substance is defined as the fractional increase ($\Delta V/V$) in volume V for a small increase (ΔT) in temperature, i.e.

$$\alpha_v = \frac{\Delta V}{V} \cdot \frac{1}{\Delta T} \tag{11.1}$$

Experiments tells us that the value of α_v is different for different substances. Even for a given substance the value of α_v depends, in general, on the temperature of the substance. As an example, we show in Fig. 11.4 how the measured value of the coefficient of volume ex-

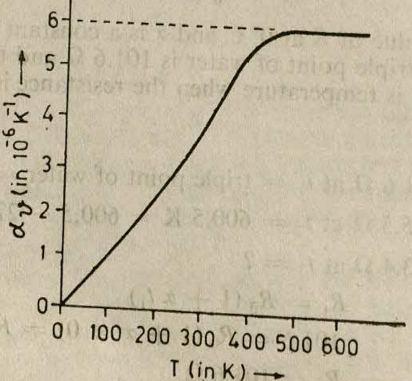


Fig. 11.4 Variation of coefficient of volume expansion of copper with temperature

pansion of copper varies with its temperature. Notice that the variation is linear up to about 100 °C, and also that the value of α_v becomes constant only at high temperatures.

Table 11.1 gives measured values of α_v for some common substances in the temperature range 0°-100 °C.

Table 11.1 Coefficients of volume expansion

Substance	α_v (K^{-1})	*Substance*	α_v (K^{-1})
Aluminium	7×10^{-5}	Glass (ordinary)	2.5×10^{-5}
Brass	6×10^{-5}	Glass (pyrex)	1×10^{-5}
Copper	5×10^{-5}	Rubber (hard)	2.4×10^{-4}
Silver	5.6×10^{-5}	Invar	2×10^{-6}
Steel	3×10^{-5}	Mercury	1.8×10^{-4}

Notice that the coefficients of volume expansion are generally quite small. Some substances, such as pyrex glass and invar (an alloy of iron and nickel), have a much smaller value of α_v than others. That is why a pyrex glass beaker does not crack when heated.

Gases also expand on heating. The value of α_v at canstant pressure for a perfect gas can be obtained from the perfect gas equation

$$PV = RT$$

If the temperature of the gas is increased from T to $T + \Delta T$, the volume increases from V to $V + \Delta V$, at constant pressure P. Then

$$P(V + \Delta V) = R(T + \Delta T)$$

or $$PV + P\Delta V = RT + R\Delta T$$

But $$PV = RT$$

$\therefore$ $$P\Delta V = R\Delta T$$

or $$RT\,\frac{\Delta V}{V} = R\Delta T$$

or $$\frac{\Delta V}{V} = \frac{\Delta T}{T}$$

$\therefore$ $$\alpha_v = \frac{1}{V}\cdot\frac{\Delta V}{\Delta T} = \frac{1}{T} \qquad (11.2)$$

Thus at 0 °C, $\alpha_v = \frac{1}{273} \simeq 3.7 \times 10^{-3}\ K^{-1}$ which is much higher than α_v for solids and liquids (see Table 11.1). This means that, for a given increase in temperature, gases expand much more than liquids and solids. From Eq. (11.2) we find that unlike solids and liquids, α_v decreases with increase in temperature of the gas.

Most substances expand when heated. Some liquids show an unusual behaviour. Water, for instance, contracts when heated from 0 °C to 4 °C, after which it shows expansion like other liquids.

Coefficient of Linear Expansion of a Solid

The increase in the volume of a substance on heating implies that all its dimensions (length, breadth, height, etc.) increase. In some cases the change in one of these dimensions is more relevant and easily observable. For example, if the solid is in the form of a long rod, the change in its length is easily observed. In such cases we define the *coefficient of linear expansion* (α_l) of the substance as the fractional increase ($\Delta L/L$) in length L divided by the increase (ΔT) in temperature, i.e.

$$\alpha_l = \frac{\Delta L}{L} \cdot \frac{1}{\Delta T} \tag{11.3}$$

Relation between α_v and α_l

Consider a cube of a substance, each side of which has a length L at temperature T. The original volume of the cube at temperature T is

$$V = L^3$$

Let us suppose that the temperature of the cube is increased from T to $T + \Delta T$. Let ΔL be the increase in the length of each side of the cube. It is given by [from Eq. (11.3)]

$$\Delta L = \alpha_l L \Delta T \tag{11.4}$$

The increase in volume of the cube is

$$\Delta V = (L + \Delta L)^3 - L^3$$

Now $(L + \Delta L)^3$ may be rewritten as

$$(L + \Delta L)^3 = L^3\left(1 + \frac{\Delta L}{L}\right)^3$$

If the increase in temperature (ΔT) is small, then the increase in length (ΔL) will also be small so that $\Delta L/L \ll 1$. Therefore, we expand $\left(1 + \frac{\Delta L}{L}\right)^3$ binomially and retain only the first order term in $\Delta L/L$. We then get

$$\left(1 + \frac{\Delta L}{L}\right)^3 \simeq 1 + \frac{3\Delta L}{L}$$

so that

$$(L + \Delta L)^3 = L^3\left(1 + \frac{3\Delta L}{L}\right) = L^3 + 3L^2\Delta L$$

Thus

$$\Delta V = L^3 + 3L^2\Delta L - L^3 = 3L^2\Delta L = 3L^3\,\frac{\Delta L}{L}$$

or

$$\Delta V = 3V\,\frac{\Delta L}{L}$$

or

$$\frac{\Delta V}{V} = 3\,\frac{\Delta L}{L}$$

or

$$\frac{\Delta V}{V}\cdot\frac{1}{\Delta T} = 3\,\frac{\Delta L}{L}\cdot\frac{1}{\Delta T}$$

Using Eqs (11.1) and (11.3) we have

$$\alpha_v = 3\,\alpha_l \qquad (11.5)$$

Thus *the coefficient of volume expansion of a solid is three times its coefficient of linear expansion.* This relation holds for a solid of any shape having a uniform cross-sectional area.

Although the thermal expansion of solids is quite small, yet it has some important consequences as in the following examples.

1. While laying railway tracks, gaps have to be left between successive lengths of rails to allow for expansion during summer or heat caused by friction between the wheels and the track.

2. Telegraph wires are given enough sag to allow for their contraction in winter.

3. In the construction of a bridge, the ends of the steel girders are not fixed but are placed on rollers so that they can expand or contract freely.

4. Cement floors or concrete roads are not laid out in one continuous piece which would expand or contract giving rise to cracks. To avoid this, the surface is laid in small pieces with gaps in between to allow for expansion in summer.

5. The length of the pendulum in a pendulum clock increases in summer and decreases in winter. Consequently the clock does not keep correct time throughout the year. To overcome this, compensated pendulums are used in clocks which are designed in such a way that the effective length of the pendulum remains the same throughout the year. Similarly, compensated balance wheels are used in watches to compensate for the change in its diameter due to the changes in temperature.

Although the thermal expansion of solids can be a source of trouble, expansion has very useful applications, the most important being the *thermostat* shown in Fig. 11.5.

A thermostat is a device that helps maintain a steady temperature. It consists of a *bimetallic strip* which consists of two thin strips of different materials (such as brass and invar) welded together along their lengths. On heating this combination bends into an arc. This happens because brass has a higher coefficient of expansion than invar.

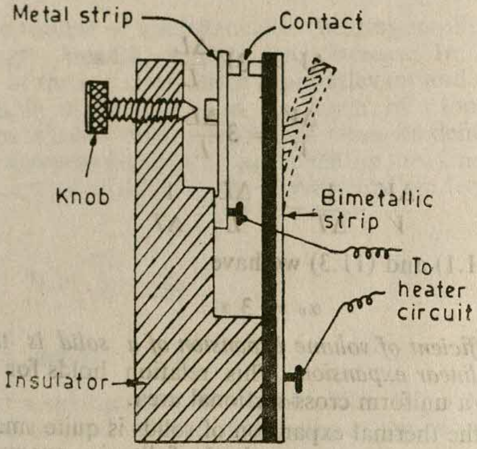


Fig. 11.5 A thermostat

Figure 11.5 shows a thermostat fitted with a bimetallic strip which may be used for controlling the temperature of a room heated by a heater. The heater circuit is completed through the bimetallic strip. When the temperature of the room reaches the upper required limit, the strip bends away, thus breaking the circuit at the contact points and switching off the heater. As the temperature of the room falls below the required limit, the bimetallic strip becomes straight again and the circuit is completed and the heater starts working again. If a lower temperature is to be maintained in the room, the knob is moved to the left thus increasing the gap between the metal strip and the bimetallic strip. Now the bimetallie strip has to bend more to make contact which can happen if it is cooled more.

EXAMPLE 11.3 A cylinder of diameter exactly 1 cm at 30 °C is to be slid into a hole in a steel plate. The hole has a diameter of 0.99970 cm at 30 °C. To what temperature must the plate be heated? For steel $\alpha_l = 1.1 \times 10^{-5}$ (°C^{-1}).

Solution The hole will expand in the same way as a circle of steel filling it would expand. The diameter of the hole needs to be changed by

$$\Delta L = 1 - 0.99970 = 0.00030 \text{ cm}$$

But $$\Delta L = \alpha_l L \Delta T$$

$$\therefore \quad \Delta T = \frac{\Delta L}{\alpha_l L} = \frac{0.00030}{1.1 \times 10^{-5} \times 0.99970} = 27.3 \text{ °C}$$

The plate must be raised to a temperature of 30 + 27.3 = 57.3 °C

Example 11.4 A steel scale measures the length of a copper rod as 80.00 cm when both are at 20 °C, the calibration temperature for the scale. What would the scale read for the length of the rod when both are at 40 °C? α_l for steel $= 11 \times 10^{-6}$ (°C^{-1}) and α_l for copper $= 17 \times 10^{-6}$ (°C)$^{-1}$.

Solution The length of 1 cm division of the steel scale at 40 °C is

$$(1 \text{ cm}) \times (1 + 11 \times 10^{-6} \times 20) = 1.00022 \text{ cm}$$

Length of the copper rod at 40 °C will be

$$(80) \times (1 + 17 \times 10^{-6} \times 20) = 80.0272 \text{ cm}$$

The number of cm read on the scale will be

$$\frac{80.0272}{1.00022} \text{ cm} = 80.0096 \text{ cm} \simeq 80.01 \text{ cm}$$

Example 11.5 A one litre flask contains some mercury. It is found that at different temperatures the volume of air inside the flask remains the same. What is the volume of mercury in the flask? Given the coefficient of linear expansion of glass $= 9 \times 10^{-6}$ (°C)$^{-1}$, the coefficient of volume expansion of mercury is 1.8×10^{-4} (°C)$^{-1}$.

Solution Let x be the volume of mercury in the flask. Since the volume of air in the flask remains constant, the volume expansion of the flask must be exactly equal to the expansion of mercury in the flask.

$$\therefore \quad V \alpha_g \Delta T = x \alpha_m \Delta T$$

where α_g and α_m are the volume expansivities of glass and mercury respectively.

$$\therefore \quad x = V \frac{\alpha_g}{\alpha_m}$$

Here $$\alpha_g = 3 \times 9 \times 10^{-6} = 27 \times 10^{-6} \text{ (°C)}^{-1}$$

$$\alpha_m = 18 \times 10^{-5} \text{ (°C)}^{-1}, \qquad V = 1000 \text{ cm}^3$$

$$\therefore \quad x = \frac{1000 \times 27 \times 10^{-6}}{18 \times 10^{-5}} \text{ cm}^3 = 150 \text{ cm}^3$$

Example 11.6 Mercury of 1.000 m^3 volume at 0 °C has a mass of 13,600 kg. Calculate its density when the temperature is raised to 100 °C. The coefficient of expansion of mercury $= 1.8 \times 10^{-4}$ °C.

Solution Given

$$V_0 = 1\ \text{m}^3$$
$$\Delta T = 100\ °\text{C}$$
$$\alpha_v = 1.8 \times 10^{-4}\ °\text{C}$$

Volume of mercury at 100 °C is given by

$$V_t = V_0 (1 + \alpha_v \Delta T)$$

or $V_{100} = 1\,(1 + 1.8 \times 10^{-4} \times 100) = 1 + 1.8 \times 10^{-2}$

$= 1 + 0.018 = 1.018\ \text{m}^3$

Mass of mercury = 13,600 kg

Since the mass cannot change, 13600 kg of mercury occupy a volume of 1.018 m^3 at 100 °C.

$\therefore$ Density of mercury at 100 °C $= \dfrac{\text{mass}}{\text{volume}} = \dfrac{13600}{1.018}$

$= 13{,}360\ \text{kg m}^{-3}$

Notice that the density of liquids that expand on heating, decreases with increase in temperature.

EXAMPLE 11.7 A brass wire 1.8 m long at 27 °C is held taut with negligible tension between two rigid supports. If the wire is cooled to a temperature of − 39 °C, what tension is developed in the wire? Diameter of the wire = 2.0 mm. Coefficient of linear expansion of brass = $2.0 \times 10^{-5}\ °\text{C}^{-1}$ and Young's modulus of brass = 0.91×10^{11} Nm^{-2}.

Solution Let L be the length of the wire at 27 °C and ΔL the contraction in length as the temperature falls to −39 °C so that $\Delta T = 27 - (-39) = 66$ °C. Now

$$\Delta L = \alpha_l L \Delta T$$

$\therefore$ Strain in the wire $\left(\dfrac{\Delta L}{L}\right) = \alpha_l \Delta T$

Since the wire is rigidily fixed at its ends, it cannot contract producing a tension F in the wire which is responsible for the strain. If A is the cross-sectional area of the wire, we have

$$\text{Young's modulus } Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{\Delta L/L}$$

or

$$F = \frac{YA\Delta L}{L}$$

Now $\frac{\Delta L}{L} = \alpha_l \Delta T$ and $A = \pi d^2/4$, where $d =$ diameter of the wire. Hence

$$F = \tfrac{1}{4}\,(\pi d^2 Y \alpha_l \Delta T)$$

Substituting for d ($= 2.0$ mm $= 2.0 \times 10^{-3}$ m) and other quantities in this we get

$$F = 3.77 \times 10^2\,\text{N} \simeq 3.8 \times 10^2\,\text{N}$$

11.5 SPECIFIC HEAT

We know that heat is a form of energy. We have defined a *calorie* as the amount of heat energy required to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C. The reason why these two specical temperatures are chosen to define a calorie is that experimentally it is found that the heat energy required to raise the temperature of a given mass of water by 1 °C depends on the temperature. It is a little more for 1 °C difference from 65 °C to 66 °C than from 14.5 °C to 15.5 °C. Experiments also show that different substances need different amounts of heat for a given mass (e.g. 1 kg) and the same temperature rise.

Specific Heat

The amount of heat energy ΔQ needed to raise the temperature of a mass m of a of substance by an amount ΔT depends on m as well as ΔT, i.e.

$$\Delta Q \propto m\Delta T$$

or

$$\Delta Q = cm\Delta T \qquad (11.6)$$

where c is a constant called the *specific heat* of the substance. If $m =$ 1 unit and $\Delta T = 1$ unit, then $c = Q$. Hence the specific heat of a substance is the amount of heat required to raise the temperature of a unit mass of the substance through a unit degree. A commonly used unit of c is cal g^{-1} °C^{-1}. In the SI system c is expressed in J kg^{-1} K^{-1}. The two units are related as

$$1\ \text{cal g}^{-1}\ {}^\circ\text{C}^{-1} = 4.18\ \text{J}\ g^{-1}\ {}^\circ\text{C}^{-1}$$

$$(\because\ 1\ \text{cal} = 4.18\ \text{J})$$

or

$$1\ \text{cal g}^{-1}\ {}^\circ\text{C}^{-1} = 4180\ \text{J kg}^{-1}\ {}^\circ\text{C}^{-1}$$

Since the size of a degree on the celsius scale is equal to that on the kelvin scale, a temperature *difference* of, say, 1 °C is equal to a temperature difference of 1 K. Thus

$$1\ \text{cal g}^{-1}\ {}^\circ\text{C}^{-1} = 4180\ \text{J kg}^{-1}\ \text{K}^{-1}$$

With our definition of a calorie, the specific heat of water = 1 cal g^{-1} °C^{-1} or 4180 J kg^{-1} K^{-1}. In fact, this is the value of c for water at

15 °C. As stated earlier, the value of c for water is different (though only slightly) at different temperatures. Figure 11.6 shows how the specific heat of water changes with temperature.

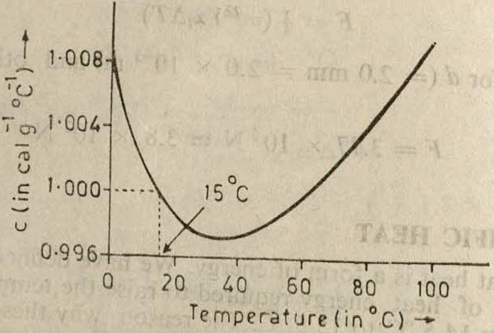


Fig. 11.6 Variation of specific heat of water with temperature

Notice the peculiar behaviour of water again, its specific heat decreases with temperature up to about 40 °C after which it increases with temperature. For most other substances the specific heat generally increases with temperature and assumes a constant value at high temperatures.

Molar Specific Heat

The molar specific heat C of a substance is defined as the amount of heat energy required to raise the temperature of 1 mole of the substance through 1 K. It is expressed in cal $\text{mol}^{-1}\ \text{K}^{-1}$ or J $\text{mol}^{-1}\ \text{K}^{-1}$.

Specific heat (c) and molar specific heat (c) are related as

$$c = \frac{C}{m} \tag{11.7}$$

where m is the number of kilograms per mole in the substance.

Table 11.2 gives the specific heats of some solids at room temperature (20 °C) and atmospheric pressure.

Table 11.2 Specific heats of some solids at 20 °C and atmospheric pressure

Substance	*Specific heat*		*Molar Specific heat*	
	cal $g^{-1}\ K^{-1}$	J $kg^{-1}\ K^{-1}$	cal $mol^{-1}\ K^{-1}$	J $mol^{-1}\ K^{-1}$
Aluminium	0.215	900.0	5.82	24.4
Carbon	0.121	506.5	1.46	6.1
Copper	0.0923	386.4	5.85	24.5
Lead	0.0305	127.7	6.32	26.5
Silver	0.0564	236.1	6.09	25.5
Tungsten	0.0321	134.4	5.95	24.9

11.6 SPECIFIC HEATS OF A GAS

When heat energy is supplied to a body, it is spent in two ways. A part of the heat energy is used up in increasing the temperature of the body. From the kinetic interpretation of temperature, this implies an increase in the internal energy of the molecules whose average velocity increases as a result of the increase in temperature. The body may also expand on heating and another part of the heat energy supplied is used up to do work during expansion against external pressure. In the case of solids and liquids the increase in volume by heating is usually very small and, therefore, the heat energy supplied is almost entirely utilised in only bringing about an increase in temperature. But in the case of gases, the increase in volume on heating is considerable since the coefficient of expansion for gases is much higher than that for liquids or solids. Thus, if a gas is heated and is allowed to expand, the heat energy will be required not only to increase its temperature but also to do work in overcoming external pressure during expansion. This is a statement of the law of conservation of energy (including heat energy) called the first law thermodynamics which we will study in Section B.

Let us consider two different experiments with a gas. Suppose we have a gas enclosed in a cylinder fitted with a piston, and the piston is suddenly pushed in, thus compressing the gas. The temperature of the gas rises, although no heat is supplied to it, only work has been done on it. Now the specific heat is defined as

$$\text{Specific heat} = \frac{\text{Heat energy supplied } (\Delta Q)}{\text{Mass } (M) \times \text{ Rise in temperature } (\Delta T)}$$

In this case $\Delta Q = 0$, but $\Delta T \neq 0$, and hence the specific heat of the gas is zero. On the other hand, a gas may be allowed to expand isothermally (i.e. at constant temperature). If the piston is raised slowly, the volume of the enclosed gas increases slowly and the gas is cooled slowly. If the necessary heat energy is supplied slowly from outside, its temperature can be kept constant, i.e. $\Delta T = 0$, but $\Delta Q \neq 0$. It is clear that, under these conditions, the specific heat of the gas becomes infinite. Thus, we can conclude that the specific heat of a gas can have any value ranging from zero to infinity depending on the manner in which it is being heated. It is, therefore, meaningless to speak about the specific heat of a gas unless the conditions under which it is heated are clearly specified.

Two Specific Heats of a Gas

Two specific heats of a gas are of special significance, namely, the specific heat at constant volume (C_v) and that at constant pressure (C_p). They are defined as: *The molar specific heat of a gas at constant volume* (C_v) *is the amount of heat energy required to raise the temperature of 1 mole of the gas through 1 K when its volume is kept constant. The molar specific heat of a gas at constant pressure* (C_p) *is the* *amount of heat energy required to raise the temperature of 1 mole of the gas through 1 K when its pressure is kept constant.*

C_p is greater than C_v: It is clear that if heat energy is supplied to a gas keeping its volume constant, the gas does no work against external pressure and all the energy supplied is used up in raising the temperature of the gas. But if a gas is heated at constant pressure, its volume increase. Heat energy, in this case, is required not only to increase the temperature but also to do work against the external pressure. Hence more heat energy, is required to heat 1 mole of a gas through 1 K when it is heated at constant pressure than when it is heated at constant volume. In other words, C_p is greater than C_v.

Relation between C_p and C_v

Consider one mole of an ideal gas enclosed in a cylinder fitted with a frictionless piston as shown in Fig. 11.7a.

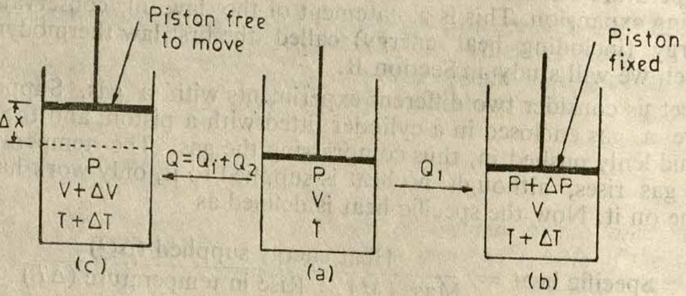


Fig. 11.7 The relation between C_p and C_v

Let P, V and T be the pressure, volume and absolute temperature of the gas. The gas is heated at constant volume until its temperature rises to $T + \Delta T$. The piston is held in position and not allowed to move while the gas is being heated so that its volume remains constant (Fig. 11.7b).

Let Q_1 be the heat energy supplied. Then, evidently, from the definition of C_v, we have

$$Q_1 = 1 \times C_v \times \Delta T = C_v \Delta T \tag{11.8}$$

Since the temperature of the gas has increased from T to $T + \Delta T$, it follows from Charles' law that the pressure will also increase. Let the new pressure be $P + \Delta P$.

The gas is then cooled so that its temperature falls from $T + \Delta T$ to T. Consequently, the pressure falls from $P + \Delta P$ to P. The gas is heated once again, but this time the piston is not held in position and the gas is allowed to expand. The gas keeps expanding and being warmed. Heating is stopped after the temperature has risen from T to

$T + \Delta T$. The piston is allowed to move through a distance Δx in the process of expansion.

Let the heat energy supplied be Q. It is obvious that $Q > Q_1$, since now apart from raising the temperature to $T + \Delta T$, the gas has to do work in expanding against external pressure. Let the new volume be $V + \Delta V$. Since the piston is frictionless and free to move, the pressure remains constant at P. Hence

$$Q = Q_1 + W \tag{11.9}$$

where W = work done by the gas in expanding against external pressure. To calculate W, let A be the area of the piston, and then the force exerted by the gas on the piston is $P \times A$.

∴ Work done W by the gas in pushing the piston through a distance Δx

$$\begin{aligned} &= \text{Force} \times \text{Distance} \\ &= P \times A \times \Delta x \\ &= P\Delta V \end{aligned}$$

But $A \times \Delta x$ is the increase in volume of the gas and is equal to ΔV.

Hence $$W = P \cdot \Delta V \tag{11.10}$$

Now by definition $$Q = 1 \times C_p \times \Delta T = C_p \Delta T \tag{11.11}$$

Substituting Eqs (11.8), (11.10) and (11.11) in Eq. (11.9) we have

$$C_p \cdot \Delta T = C_v \, \Delta T + P. \, \Delta V \tag{11.12}$$

Now equations of the state of the gases in Figs 11.7a and c are

$$PV = RT \tag{11.13}$$

and $$P \times (V + \Delta V) = R(T + \Delta T) \tag{11.14}$$

Subtracting Eq. (11.13) from Eq. (11.14), we have

$$P \cdot \Delta V = R. \, \Delta T \tag{11.15}$$

Substituting for $P \cdot \Delta V$ from Eq. (11.15) in Eq. (11.12) we get

$$C_p \Delta T = C_v \, \Delta T + R \cdot \Delta T$$

or $$C_p = C_v + R$$

or $$C_p - C_v = R \tag{11.16}$$

Equation (11.16) tells us that C_p is greater than C_v by an amount equal to the universal gas constant R ($= 8.315$ J mol^{-1} K^{-1}). It may be recalled that the relation holds only for an ideal gas.

C_p and C_v in Terms of R

In Chapter 10 we have seen that the kinetic energy of 1 mole of an ideal gas is given by

$$E = \frac{3}{2} RT$$

If the gas is not allowed to expand, the heat energy supplied to it is completely used up in increasing the kinetic energy of the molecules of the gas.

Thus, if the volume is kept constant, the gain in kinetic energy for a 1 K rise in temperature is the same as the amount of heat energy gained by the gas for a 1 K rise in temperature, Thus, from the definition of C_v, we have

$$C_v = \frac{3}{2} R (T + 1) - \frac{3}{2} RT$$

or

$$C_v = \frac{3}{2} R = 1.5R \qquad (11.17)$$

Since $C_p = C_v + R$, we have

$$C_p = \frac{3}{2} R + R = \frac{5}{2} R = 2.5R \qquad (11.18)$$

Now $R = 8.32 \text{ J mol}^{-1} \text{ K}^{-1}$, we get

$$C_v = 1.5 \times 8.32 = 12.48 \text{ J mol}^{-1} \text{ K}^{-1}$$

and

$$C_p = 2.5 \times 8.32 = 20.80 \text{ J mol}^{-1} \text{ K}^{-1}$$

Equations (11.16) to (11.18) are theoretical results based on the perfect gas model. Do they agree with experimental results for real gases? Table 11.3 gives the measured values of C_p/R and C_v/R for some gases.

Table 11.3 C_p/R, C_v/R **and** $(C_p\text{-}C_v)/R$ **for different gases.**

Type of gas	*Gas*	$\frac{C_p}{R}$	$\frac{C_v}{R}$	$\frac{1}{R}(C_p - C_v)$
1. Monoatomic	He	2.5	1.5	1.00
	A	2.5	1.5	1.00
2. Diatomic	H_2	3.45	2.45	1.00
	O_2	3.53	2.53	1.00
	N_2	3.49	2.49	1.00
2. Polyatomic	CO_2	4.43	3.42	1.01
	SO_2	4.85	3.77	1.08
	C_2H_6	6.21	3.34	1.03

The study of this table reveals that our theoretical result (11.16) holds for all gases. This implies that our assumptions about the extra work done under constant pressure and about the perfect gas are both correct. But our results (11.17) and (11.18) hold only for monoatomic gases. For diatomic and polyatomic gases the experimental values of C_p and C_v are considerably higher than the theoretical values. The departure becomes very marked as the number of atoms in a molecule of the gas increases as in C_2H_6. What is the cause of this discrepancy? Perhaps something is wrong with or missing in our theory when it is applied to complicated molecules. The reason is not hard to find.

Since the measured values of C_p and C_v are higher than the theoretical values for diatomic and polyatomic molecules, it implies that it needs more energy to heat such molecules. This means that these molecules have internal motions with associated energy. This can be understood as follows:

The molecules of monoatomic gases like helium, argon, etc. consist of a single atom. Since the atom is a tiny particle, such a molecule may be regarded as a point mass. It is capable of only translatory motion.

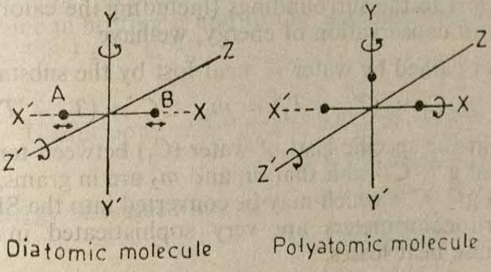


Fig. 11.8

The molecules of diatomic gases, such as hydrogen, oxygen, nitrogen, etc. consist of two atoms (which are bound to each other by a force) a fixed distance apart from each other. A diatomic molecule may be considered to have the shape of a dumb-bell, i.e. two mass points A and B fixed to the two ends of a spring as shown in Fig. 11.8. It has three degrees of freedom corresponding to the translation of its centre of mass. Besides, the molecule can also rotate about any two axes YY′ and ZZ′ joining A and B. YY′ lies in the plane of the paper and ZZ′ is perpendicular to this plane. Since the atoms are supposed to be mass points, the third rotation about the line joining the atoms A and B is not possible. The diatomic molecule can, however, vibrate along this line.

Experiments have shown that, at ordinary temperatures, the vibratory motion does not occur in a diatomic molecule. Thus a diatomic molecule has translatory as well as rotatory motions. A polyatomic molecule has translatory, rotatory and vibratory motions. Since energy is associated with each of these motions, it follows that it requires more energy to heat polyatomic molecules than diatomic molecules.

The equation $E = \frac{3}{2} RT$ which we used to obtain Eq. (11.17) and Eq. (11.18) needs to be modified to include these internal motions in the case of more complex molecules.

Measurement of Specific Heat: The Calorimeter

Measurement and comparison of heat is carried out with the help of a vessel called the *calorimeter*. It consists of a vessel, usually made of copper, with a thermometer in it and is very well insulated from the surroundings.

To determine the specific heat of a substance, we compare the heat needed to increase its temperature with a substance of known specific heat, such as water whose specific heat, by definition, is 1 cal g^{-1} $°C^{-1}$ or 4180 J kg^{-1} K^{-1}. Suppose we have a mass m_1 of water at temperature T_1 in the calorimeter. To this we add a known mass m_2 of the substance (of unknown specific heat C) at a known temperature T_2 greater than T_1. After some time both water and substance attain the common temperature T_3 (which is greater than T_1 but less than T_2). If on heat is lost to the surroundings (including the calorimeter), then from the law of conservation of energy, we have

$$\text{Heat gained by water} = \text{heat lost by the substance}$$

i.e. $$m_1 \times C_1 \times (T_3 - T_1) = m_2 \times C \times (T_2 - T_3)$$

Assuming that the specific heat of water (C_1) between temperatures T_1 and T_3 is 1 cal g^{-1} $°C^{-1}$ and that m_1 and m_2 are in grams, then C is obtained in cal g^{-1} $°C^{-1}$ which may be converted into the SI units, J kg^{-1} K^{-1}. Modern calorimeters are very sophisticated in design which would minimise heat losses.

EXAMPLE 11.8 One mole of oxygen is heated from 0 °C, at constant pressure till its volume increases by 10%. (a) Find the quantity of heat required. The specific heat of oxygen, under these conditions, is 0.22 cal g^{-1} K^{-1}. (a) If the same quantity of heat is supplied to the gas at constant volume, what will be the final temperature?

Solution (a) For an enclosed gas at constant pressure, $V \propto T$ (Charles' law).

Hence, to increase the volume by 10%, the absolute temperature must be increased by 10%. The initial temperature of oxygen is $T = 273$ K.

The final temperature must be $1.1 \times T = 300.3$ K.

$$\therefore \qquad \text{Rise in temperature} = 27.3 \text{ K}$$

Quantity of heat supplied is given by

$$\begin{aligned} Q &= (\text{mass of 1 mole}) \times (\text{specific heat}) \times (\text{rise in temp.}) \\ &= 32 \text{ g} \times 0.22 \text{ cal g}^{-1} \text{K}^{-1} \times 27.3 \text{ K} \\ &= 192.2 \text{ cal} = 807.24 \text{ J} \end{aligned}$$

(b) C_p = molar specific heat at constant pressure

$$= 0.22 \times 32 \text{ cal K}^{-1}\text{ mol}^{-1}$$

$$= 0.22 \times 32 \times 4.2 = 29.568 \text{ J K}^{-1}\text{ mol}^{-1}$$

But

$$C_v = C_p - R$$

$$= 29.568 - 8.315$$

$$= 21.253 \text{ J K}^{-1}\text{ mol}^{-1}$$

When 807.24 J of heat energy is supplied to 1 mole of oxygen at 0 °C, at constant volume, the rise in temperature is given by

$$\Delta T = \frac{Q}{C_v} = \frac{807.24}{21.253} = 38 \text{ K}$$

$$\therefore \quad \text{Final temperature} = 38\ ^\circ\text{C}$$

EXAMPLE 11.9 A 10 kW drilling machine is used to drill a hole in a small copper block of mass 7.5 kg. How much is the rise in the temperature of the block in 3 minutes, assuming that 40% of power is lost in the machine in heating it and to the surroundings. Specific heat of copper = 0.386 J $\text{g}^{-1}\ ^\circ\text{C}^{-1}$.

Solution

$$\text{Useful power available} = 60\% \text{ of } 10 \text{ kW}$$

$$= 6 \text{ kW} = 6 \times 10^3 \text{ W}$$

$$\therefore \quad \text{Energy consumed in 3 min} = 6 \times 10^3 \times 3 \times 60$$

$$= 1.08 \times 10^6 \text{ J}$$

$$\text{Specific heat of copper} = 0.386 \text{ J g}^{-1}\ ^\circ\text{C}^{-1}$$

$$= 386 \text{ J kg}^{-1}\ ^\circ\text{C}^{-1}$$

$$\text{Mass of copper block} = 7.5 \text{ kg}$$

$$\therefore \quad \text{Rise in temperature} = \frac{1.08 \times 10^6 \text{ J}}{7.5 \text{ kg} \times 386 \text{ J kg}^{-1}\ ^\circ\text{C}^{-1}} = 373\ ^\circ\text{C}$$

SECTION B: THERMODYNAMICS

11.7 INTRODUCTION

Thermodynamics is the branch of science that deals with the relation between heat and mechanical energy. It is mainly concerned with the transformation of heat into mechanical work and vice versa. It is common experience that mechanical work can be converted into heat. For example, we can generate heat by rubbing our hands or two pieces of metal together. Heat can also be converted into mechanical work, as in the steam engine. The steam engine is typical of a large class of engines known as heat engines which perform work at the expense of heat. In the earlier forms of heat engines, the efficiency was as low as 5 per cent, i.e. only 5 per cent of heat could be converted into work. In the modern heat engines, an efficiency of upto 55 per cent has been achieved, i.e. at most only 55 per cent of the heat supplied is converted into work. The question, therefore, naturally arises whether the low efficiency is due to bad designing of the engine or whether there is something in the very nature of things which forbids the conversion of the whole of heat into mechanical work. Attempts to answer questions such as this form the origin of thermodynamics. By theoretical considerations based on a few laws, it is possible to predict what part of heat can be transformed into work. The structure of thermodynamics rests upon two simple laws known as the First and Second Laws of Thermodynamics. These laws, which are generalised summaries of experience and observation, are the subject-matter of this section.

Although thermodynamics had its origin in considerations of the efficiency of heat engines, its use and scope, at the present time, is by no means restricted to this field alone. The principles of thermodynamics are very general and exact and have been successfully applied not only to problems in physics and engineering but also to those in chemistry, such as chemical reactions, chemical equilibria, electrochemistry, etc.

The study of any new branch of science must start by isolating a limited region of space or a finite portion of matter from its surroundings. The isolated portion (e.g. gas enclosed in a cylinder) on which the attention is focused is called the *system* and everything external to the system, that may have an effect on its behaviour, is known as the *surroundings*. Having chosen a system, the next step is to describe it in terms of the physical quantities that will help discuss the behaviour of the system and its interactions with the surroundings. There are two approaches that may be adopted, the *microscopic* approach or the *macroscopic* approach. The kinetic theory discussed in Chapter 10 adopts the macroscopic approach. In the kinetic theory, all the thermal phenomena are interpreted in terms of the disorderly motion of atoms or molecules. Thermodynamics, on the other hand, adopts the microscopic approach. It deals with the gross (or macro) characteristics of the system, taking no account of the atomic constitution of matter. It is indeed remarkable that highly accurate results can be obtained with the help of a few laws of thermodynamics. Thermodynamics is an exact science.

11.8 DEFINITION OF SOME TERMS USED IN THERMODYNAMICS

Some of the terms used in the study of thermodynamics are as follows.

Thermodynamic System

So far the word 'system' has been used in a broad sense to include any object or material. For thermodynamic considerations, *a system is any material or quantity of matter* (*that is influenced by heat*) *enclosed by a boundary surface.* The boundary walls may be of two types: (i) *diathermic* that allow transmission of heat, and (ii) *adiabatic* that do not permit heat flow. For example, a liquid contained in a metallic container can be easily heated as the metal boundary is diathermic. But the same liquid contained in a vacuum flask cannot be heated by a burner; the double walled vacuum flask has an adiabatic boundary. A system wholly enclosed in an adiabatic boundary, so that there is no transfer of matter or heat to the surroundings, is called an *isolated* system. A system wholly enclosed in a diathermic boundary, so that there is an exchange of heat with the surroundings but no exchange of matter, is called a *closed* system.

Thermodynamic State and Thermodynamic Coordinates

The thermodynamic state of a system can be described in terms of the properties of the system, such as its composition, volume, pressure, temperature, etc. These are called the *thermodynamic coordinates* or *parameters* of the system. In the case of gases, the parameters are volume, pressure, temperature, etc. In the case of liquids, these parameters can be density, surface tension, temperature, etc. and in the case of solids they could be elasticity, area, volume, temperature, etc. These coordinates describe the thermodynamic state of the system.

Equation of State

If the system is homogeneous (i.e. it has the same constitution everywhere), then it is usually enough to specify only three parameters, namely, volume, pressure and temperature. The mathematical relasionship between these parameters is known as the *equation of state* of the system. The exact relationship between these parameters is not known for solids, liquids and non-homogeneous substances. In the case of ideal gases, the equation of state is

$$PV = RT, \qquad \text{for one mole}$$

and

$$PV = nRT, \qquad \text{for } n \text{ moles}$$

where P, V and T are respectively the pressure, volume and temperature of the gas and R is the universal gas constant. In such a case, if only two parameters are specified, the third is automatically fixed. Hence, the thermodynamic state of a simple homogeneous system is described by specifying any two of the three parameters.

11.9 ZEROTH LAW OF THERMODYNAMICS: THERMAL EQUILIBRIUM AND CONCEPT OF TEMPERATURE

A thermodynamic system is said to be in a state of thermal equilibrium if its thermodynamic coordinates remain constant. Consider a gas enclosed in a cylinder fitted with a piston. Let P and V be the pressure and volume of the enclosed gas at the temperature of the surroundings. These values of P and V will remain constant as long as the external conditions remain unchanged. We say that the gas is in thermal equilibrium with the surroundings. Now consider two systems A and B separated by an adiabatic (non-conducting) wall. The state P_1, V_1 for system A and the state P_2, V_2 for system B will co-exist as separate equilibrium states with any attainable constant values of the four quantities (see Fig. 11.9a). If the two systems are separated by a diathermic (heat conducting) wall, the values of P_1, V_1 and P_2, V_2 will change spontaneously until an equilibrium state of the combined system is attained (see Fig. 11.9b). We say that systems A and B are in thermal equilibrium with each other.

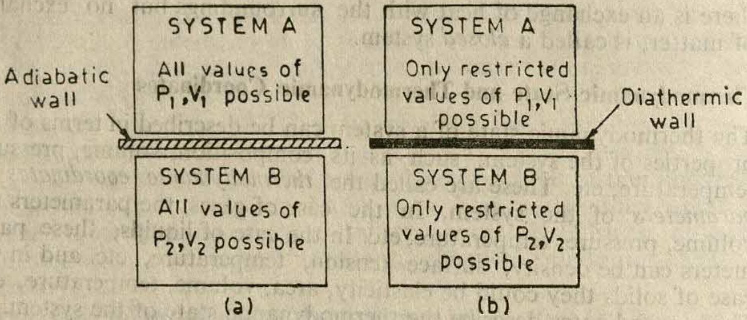

SYSTEM A
Adiabatic wall
All values of P_1, V_1 possible
SYSTEM B
All values of P_2, V_2 possible
(a)
SYSTEM A
Only restricted values of P_1, V_1 possible
Diathermic wall
SYSTEM B
Only restricted values of P_2, V_2 possible
(b)

Fig. 11.9 Properties of adiabatic and diathermic walls

Now imagine two systems A and B separated from each other by an adiabatic wall but each system is in contact with a third system C through diathermic walls, as shown in Fig. 11.10a. Systems A and B will separately attain thermal equilibrium with system C. If

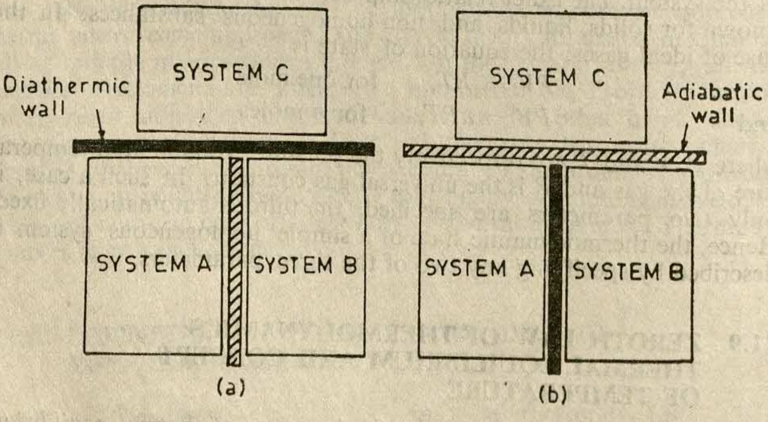


Fig. 11.10 Thermal equilibrium

now the adiabatic wall between A and B is replaced by a diathermic wall (Fig. 11.10b), it is found experimentally that no further change occurs in systems A and B. In other words, systems A and B are also in thermal equilibrium with each other. These experimental facts may be stated in the form of a postulate known as the *Zeroth law of thermodynamics which states that if two systems A and B are separately in thermal equilibrium with a third system C, then the systems A and B are in thermal equilibrium with each other.* The physical quantity that determines whether or not a system is in thermal equilibrium with another system is called *temperature.* If a number of systems are in thermal equilibrium we say that they have a common property, namely, temperature. It means that if two systems are not in thermal equilibrium, they are at different temperatures. For example, if a thermometer (i.e. system C) shows the same reading in two systems A and B, then it will show the same reading even when systems A and B are brought in contact.

11.10 ISOTHERMAL AND ADIABATIC PROCESSES

If a system is perfectly conducting to the surroundings and the temperature remains constant throughout the process, the process is called *isothermal*. During an isothermal process, the temperature of the system remains constant, but it can absorb heat from or lose heat to the surroundings. The process has to be extremely slow to be isothermal.

During an *adiabatic* process, the system is completely insulated from the surroundings. It can neither give heat to nor take heat from the surroundings. Sudden processes are adiabatic such as the bursting of a cycle tube, the compression stroke in an internal combustion engine, etc.

11.11 INTERNAL ENERGY

Every system (solid, liquid or gas) possesses a certain amount of energy. This energy is called the *internal energy* and is usually denoted by the symbol U. The internal energy of a solid, liquid or gas consists of two parts (i) kinetic energy due to the motion (translational, rotational and vibrational) of the molecules, and (ii) potential energy due to the configuration (separation) of the molecules.

There are no experimental methods for measuring the internal energy of a system in a particular state. This is no handicap, since, in actual practice, we are only concerned with changes in the internal energy as the state of the system undergoes a change. The change ΔU in U can be easily measured. Consider, for example, the case of expansion of a gas enclosed in a cylinder. When the gas expands, it does external work W. If the piston and the walls of the cylinder are thermally insulated from the surroundings, the gas cannot take in heat from the surroundings. Hence its internal energy diminishes by an amount equal to the work done W. A decrease in internal energy implies a fall in temperature. On the other hand, if the gas is adiabatically compressed, its temperature rises. Consequently, the internal energy increases by an amount equal to the work done on the gas in compressing it.

The internal energy of a homogeneous system depends on its thermodynamic state, i.e. on its thermodynamic coordinates P, V and T. Each definite state of the system possesses a definite quantity of internal energy. A change in the internal energy can occur only if a transfer of energy between the system and the surroundings is permitted. This can take place if (i) some work is performed on or by the system and (ii) some heat is absorbed or given out by the system.

It may be mentioned that a system placed in a gravitational or electric field may acquire extra energy. For instance, a mole of water at the top of a hill has more potential energy than that on the ground. This extra energy is not regarded as a part of internal energy. It is external energy.

11.12 FIRST LAW OF THERMODYNAMICS

The experimental investigations by Rumford, Joule and others clearly established the relation between mechanical work W and heat Q as

$$W = JQ$$

where J is Joule's mechanical equivalent of heat. If W is measured in joules and Q in calories, J will be measured in joules per calorie. The value of J is 4.18 J cal^{-1}. The fact that heat, like mechanical work, is a form of energy, forms the experimental basis of the first law of thermodynamics.

Consider a gas inside a cylinder fitted with a movable frictionless piston (Fig. 11.11). All the walls of the cylinder, except the bottom, are thermally insulated. Let the bottom be brought in contact with a hot body such as a burner. When the gas is heated, the whole of the heat is not converted into work. A part of the heat energy is used up in raising the temperature of the gas (i.e. in increasing the internal energy of the gas) and the rest is used up in moving the piston (i.e. in doing external work).

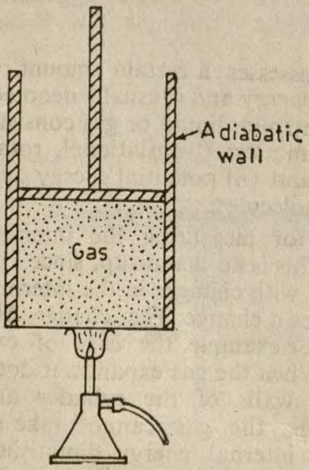


Fig. 11.11 Absorption of heat by a gas

If ΔQ is the heat energy supplied to the gas and if ΔW is the work done by it, then, from the law of conservation of energy, the increase in the internal energy ΔU must be equal to $(\Delta Q - \Delta W)$ or

$$\Delta Q = \Delta U + \Delta W \qquad (11.19)$$

Here all quantities are measered in energy units, i.e. joules. This equation is the mathematical statement of the *first law of thermodynamics*, which may be stated in words as '*if heat is supplied to a system which is capable of doing work, then the quantity of heat energy absorbed by the system will be equal to the sum of the increase in the internal energy of the system and the external work done by it*'. It must be remembered that, according to the accepted convention, ΔQ is positive if heat is supplied to the system and negative is heat is taken out of it and that ΔW is positive when work is done by the system, and negative when work is done on the system.

This law may also be understood as follows: consider a system in a state A having internal energy U_A. It absorbs heat ΔQ and changes to state B. The total energy of the system in state A is $U_A + \Delta Q$. In the transformation the system does work, say, ΔW. If U_B is the internal energy of the system in state B, then the net energy in state B is $U_B + \Delta W$. From the law of conservation of energy, we have

$$U_A + \Delta Q = U_B + \Delta W$$

or

$$\Delta Q = (U_B - U_A) + \Delta W$$

or

$$\Delta Q = \Delta U + \Delta W$$

where $\Delta U = U_B - U_A$ is the change in internal energy of the system. Thus, we conclude that the first law of thermodynamics is not a new law, but a special case of the general law of conservation of energy. It may be called the law of conservation of energy applied to the case of transformation of heat energy into mechanical energy.

11.13 APPLICATIONS OF THE FIRST LAW OF THERMODYNAMICS

Measurement of the Change in the Internal Energy of a System Undergoing a Phase Change

We will apply the first law of thermodynamics to a system undergoing a change of phase. Let us consider the process of boiling. We know that when a liquid is heated, it changes into vapour. Consider the process of the vaporization of a mass m of a liquid at a constant temperature (called the boiling point) and at a constant pressure P. Let V_1 be the volume occupied by the mass m of the liquid and V_2 the volume of the vapour. The work done by the liquid is given by

$$W = \int_{V_1}^{V_2} P dV = P \int_{V_1}^{V_2} dV \qquad (\because \ P \text{ is constant})$$

$$= P(V_2 - V_1)$$

Let the latent heat of vaporization be L. (It is the heat needed to change a unit mass of a liquid to vapour phase at constant temperature and pressure.) Evidently, the heat Q absorbed by the liquid is given by

$$Q = mL$$

Now, the first law of thermodynamics states that

$$\Delta Q = \Delta U + \Delta W$$

Integrating we have

$$Q = (U_2 - U_1) + W$$

$$\therefore \quad U_2 - U_1 = Q - W$$

$$= mL - P(V_2 - V_1)$$

Knowing m, L, P, V_1 and V_2, the gain $(U_2 - U_1)$ in the internal energy can be calculated. Similarly, the first law can be applied to calculate the change in the internal energy during a melting process.

Relation between the Two Specific Heats of a Gas

In Sec. 11.6 we deduced the relation between the two specific heats C_p and C_v of an ideal gas. Our deduction was based on the kinetic theory of gases. We will now obtain the same relation by using the first law of thermodynamics. Consider n moles of an ideal gas. If the temperature of the gas is raised by ΔT at constant volume, then the heat energy ΔQ required is, by definition, equal to $nC_v\,\Delta T$, where C_v is the molar specific heat at constant volume. Since, there is no change in volume (i.e. $\Delta V = 0$), the work done $\Delta W = P\,\Delta V$ will be zero. Applying the first law of thermodynamics

$$\Delta Q = \Delta U + \Delta W$$

to this process, we have

$$nC_v\Delta T = \Delta U \tag{11.20}$$

The same gas can also be heated at constant pressure. The heat energy required to change the temperature of n moles of the gas by the same amount ΔT at constant pressure is evidently equal to $nC_p\ \Delta T$, where C_p is the molar specific heat at constant pressure. If the gas expands by an amount ΔV, the work done ΔW in this process is

$$\Delta W = P\Delta V$$

Applying the first law to this process, we get

$$nC_p\Delta T = \Delta U + P\Delta V \tag{11.21}$$

We now make the assumption that the internal energy of the gas depends only on its temperature. This assumption holds good only for an ideal gas. This assumption cannot be made for real gases. Thus, the change ΔU in internal energy depends only on the change ΔT in temperature. Since ΔT is the same in the two processes, ΔU in Eq. (11.20) is the same as in Eq. (11.21). Substituting for ΔU from Eq. (11.20) in Eq. (11.21), we get

$$nC_p\,\Delta T = nC_v\,\Delta T + P\,\Delta V \tag{11.22}$$

Now, for n moles of an ideal gas

$$PV = nRT$$

Differentiating this equation, keeping P constant, we have

$$P\,\Delta V = nR\Delta T$$

Substituting for $P\,\Delta V = nR\Delta T$ in Eq. (11.22), we get

$$nC_p\,\Delta T = nC_v\,\Delta T + nR\Delta T$$

or

$$C_p - C_v = R$$

We note that C_p, C_v and R are measured in the same units, i.e. J mol^{-1} K^{-1}. See supplement SI at the end of the chapter for P-V relation in an adiabatic change.

Example 11.10 At atmospheric pressure 1 g of water, having a volume of 1 cm^3, becomes 1671 cm^3 of steam at 100 °C. What is the change in the internal energy? Latent heat of vaporisation of water is 2.256×10^6 J kg^{-1}.

Solution We have seen above that the change in internal energy is

$$\Delta U = mL - P(V_2 - V_1)$$

where $m = 1\text{ g} = 1 \times 10^{-3}\text{ kg}$

P = atmospheric pressure $= 1.013 \times 10^5\text{ Nm}^{-2}$

$L = 2.256 \times 10^6\text{ J kg}^{-1}$

$V_2 = 1671\text{ cm}^3 = 1671 \times 10^{-6}\text{ m}^3$

$V_1 = 1\text{ cm}^3 = 1 \times 10^{-6}\text{ m}^3$

Substituting these values we have

$$\Delta U = 1 \times 10^{-3} \times 2.256 \times 10^6 - (1.013 \times 10^5) \times (1670) \times 10^{-6}$$

$$= 2256 - 169 = 2087\text{ J}$$

11.14 PHASES AND PHASE DIAGRAMS

We know that substances change their state due to absorption or evolution of heat. On heating, solids generally change to liquids and liquids to vapours. The three states of a substance, namely, solid, liquid and gas are called its *phases*. *Whenever a substance can occur in several forms which are homogeneous, physically distinct and mechanically separable from each other, these forms are called phases of the same substance.* Phase changes generally involve either absorption or evolution of heat. The temperature at which phase change takes place is called the *transition temperature*.

Isotherms

We know that the state variables P, V and T that describe the thermodynamic state of a system are not independent. For example, if we increase the pressure (P) on a substance, at a fixed temperature, its volume (V) generally decreases. Thus there is relation between P and V. This relation must depend upon the temperture (T) of the substance. A gas at a high temperature, can be easily compressed but a solid at a low temperature cannot be. As stated earlier the relation between P, V and T that a substance obeys is called its *equation of state*. This relation is different for different substances and for different phases of the same substance. The simplest of these relations is that for a perfect gas, namely

$$PV = n\,RT$$

A perfect gas is supposed to obey this relation under all conditions. If compressed, at a given temperature, it is expected to remain a gas under all pressures; the volume would continue to decrease with increase in pressure in accordance with Boyle's law.

All real gases show deviations from this ideal behaviour. In fact, all gases can be liquefied under pressure. Experimental determination of these deviations are carried out by studying the *isotherms* of gases at various temperatures. At a given temperature, the volume of a given mass of a gas is measured under different pressures. The graph of pressure against volume is a curve which is called the *isotherm* of the gas at that temperature. An isotherm is a graphical description of the state of the system at that temperature. One can gain useful information about the system by studying its isotherms at different temperatures.

Figure 11.12 shows a set of isotherms for a mole of H_2O at various temperature. The temperature for each P-V curve (or isotherm) is in-

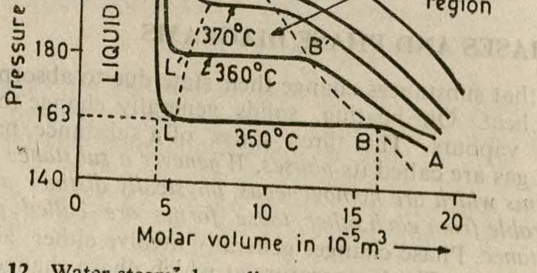


Fig. 11.12 Water-steam phase diagram for one mole of H_2O near the critical point C.

dicated on each isotherm. The temperature range chosen is 350-390 °C. In this temperature range, water exists in two phases—liquid and vapour (or gas) under the indicated pressures in the range 140 to 240 atmospheres. The isotherm is then called a phase diagram of the substance. Figure 11.12 shows a portion of the water-steam phase diagram near the critical temperature.

The lowest isotherm ABLM corresponds to 350 °C. The portion AB represents the gaseous state of H_2O which is more compressible. Here the gas approximately obeys Boyle's law. Liquefaction begins at B and the volume decreases rapidly as the gas is converted into liquid whose density is much larger. Portion BL represents the condition when the gas and liquid co-exist. The pressure at which the liquid and its vapour are in equilibrium is called the vapour pressure of the liquid. At 350 °C its value is 163 atmospheres. At L, the gas is completely liquefied and steep curve LM represents the isotherm of the liquid phase. This portion of the curve is almost parallel to the pressure-axis because the liquid is practically incompressible.

The isotherm corresponding to 360 °C is similar to that at 350 °C but the horizontal portion $B' L'$ is much smaller than BL. As the temperature is raised this horizontal portion of the isotherm becomes smaller and smaller until (as indicated by the dotted curve) this portion becomes a point C at 374.1 °C. At this critical point, the gas and the liquid have the same density and there is no physical difference between them. At this critical temperature $T_c = 374.1$ °C, the pressure is $P_c = 216$ atm. Above this temperature H_2O cannot be liquefied even at a pressure of 400 atmospheres, whereas just below 374.1 °C, a pressure of 216 atmospheres is sufficient to liquefy. It turns out that all gases exhibit this behaviour, the critical temperature (and the corresponding pressure) depending on the nature of the gas. *The maximum temperature at which a gas can be liquefied by the application of pressure alone is called its critical temperature.* Above this temperature the substance cannot exist in the liquid phase. The pressure required to produce liquefaction at the critical temperature is called *critical pressure*. It is obvious that this is the highest vapour pressure a liquid can exert. It may be recalled that Fig. 11.12 shows only two phases—liquid and gas—which exist at the indicated temperature-range considered. If water is cooled to 0 °C, it freezes to ice. This third phase (solid) is not shown.

Pressure-Temperature Phase Diagram

The pressure-temperature phase diagram is a very useful and a compact graphical representation of the different phases of a substance. In a *P-T* diagram each phase corresponds to a region or area. The boundary between two areas is a curve on which the two phases coexist. Figure 11.13 shows the *P-T* phase diagram of a typical substance indicating the three phase changes, namely, sublimation, melting and vaporization.

The melting temperature and the corresponding pressure (remember that a substance has a definite melting point under a definite pressure) are indicated by a definite point in a *P-T* diagram. At this point, the solid and liquid phases co-exist in equilibrium. Joining all such points at which the two phases (solid and liquid) are in equilibrium, we obtain a smooth curve which is called the melting curve. In Fig. 11.13. TB is the melting curve. The vaporization curve (TC) and the sublimation curve (TA) can be defined in a similar way.

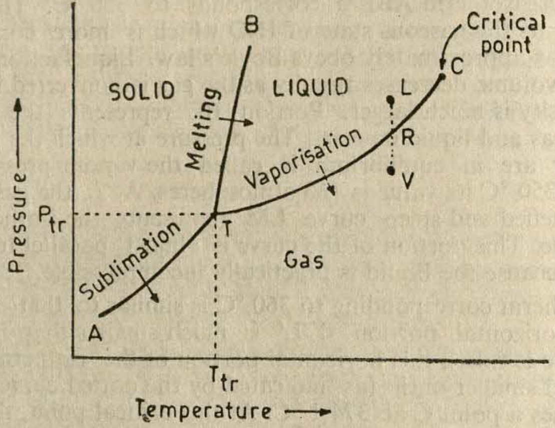


Fig. 11.13 The pressure-temperature phase diagram of a typical substance. *T* is the triple point

The point T where the three curves intersect is called the *triple point* of the substance.

The isotherms of Fig. 11.12 correspond to the region of the *P-T* diagram close to the critical point C. The isotherm ABLM of Fig. 11.12 appears as a vertical constant temperature line LV which cuts the curve TC at point R. This point gives the vapour pressure at which the vapour is in equilibrium with the liquid at a given temperature. Under this condition of temperature and pressure, the two phases (liquid and vapour) can coexist in equilibrium. Consider a point V for which the temperature is the same as at R but the pressure is less than the equilibrium value. The vapour, at this temperature and pressure, must be unsaturated. If any liquid is also present it will evaporate resulting in an increase in the vapour pressure to the saturation value. Thus the liquid and the vapour cannot exist in equilibrium under conditions specified by point V. Hence vapour is the only stable state at point V. This is true for all the points below the curves TC and TA.

Now consider a point L above the curve TC for which the temperature is the same as for point R. Here the pressure is greater than the saturation (equilibrium) vapour pressure. If any vapour is present at this temperature and pressure, it must be in a supersaturated state and hence have a tendency to condense and form the liquid. It follows that only the liquid phase can exist in a stable state at point L. This will be true for all points lying between the curves TC and TB. Similarly, the only stable phase which can exist between the curves TB and TA is the solid phase.

It follows from the *P-T* diagram that, at the triple point T, all the three phases (solid, liquid and gas) coexist in equilibrium with one another. At the normal melting point, the solid and the liquid are in equilibrium at the atmospheric pressure; while at the triple point (T_{tr}) they are in equilibrium at the pressure (P_{tr}) corresponding to that point. Since the rate of change of melting point with pressure is small, the *triple point dlffers slightly from the normal melting point.*

In the case of water, the triple point temperature is 0.01 °C and the pressure is 4.6 mm of Hg. On the Absolute scale, the triple point of water is 273.16 K. This explains the difference of 0.01 K in the normal melting point of ice and the triple point temperature. Figure 11.14 shows the *P-T* diagram in the case of water. Notice that the melting curve has a negative slope.

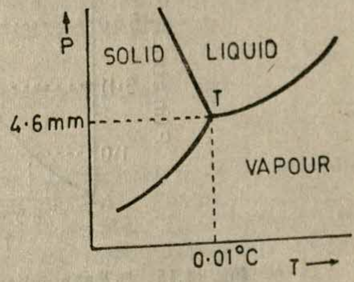


Fig. 11.14 *P-T* diagram for water

For evaporation and boiling, refer to Supplement S2 at the end of this chapter.

EXAMPLE 11.11 Prove that the sublimation curve, the melting curve and the vaporization curve meet at a point on a *P-T* phase diagram

Proof: Let us suppose that the three curves do not meet at a point as shown in Fig. 11.15 which shows the three pairs of curves meeting at points T, T′ and T″ respectively.

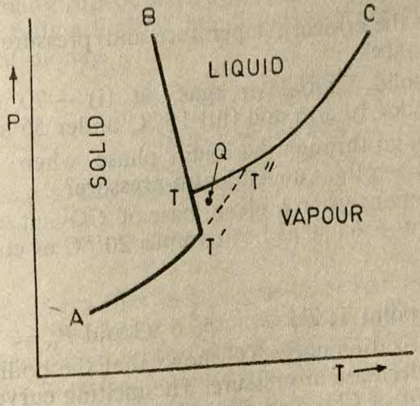


Fig. 11.15

The points T, T′ and T″ form a triangle. Consider a point Q in this triangle. Since the point Q lies above the sublimation curve AT″, it must represent the solid state. But Q also lies below the vaporization curve TC. Hence it must also correspond to the vapour state. These results are contradictory to each other. Similarly point Q corresponds simultaneously to a solid and liquid state. This is again contradictory. These contradictions do not arise if and only if the three curves meet at a point.

EXAMPLE 11.12 Answer the following questions based on the *P-T* phase diagram of carbon dioxide shown in Fig. 11.16.

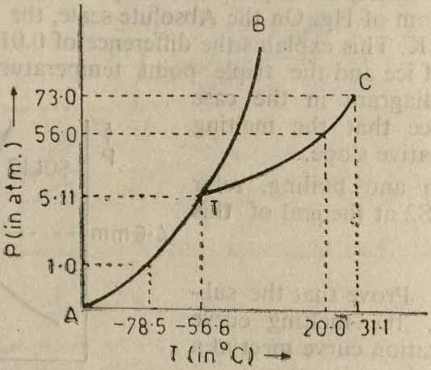


Fig. 11.16 *P-T* phase diagram of CO_2 (not drawn to scale)

(a) At what temperature and pressure can the solid, liquid and vapour phases of CO_2 coexist in equilibrium?

(b) What is the effect of decrease of pressure on melting and boiling of CO_2?

(c) What are the critical temperature and pressure for CO_2? What is their significance?

(d) Is CO_2 solid, liquid or gas at (i) – 70 °C under 1 atm, (ii) – 60 °C under 10 atm and (iii) 15 °C under 56 atm?

(e) Does CO_2 go through the liquid phase when it is compressed isothermally at – 60 °C under 1 atm pressure?

(f) What happens when a given mass of CO_2 at 10 atm pressure and temperature – 65 °C is heated upto 20 °C at constant pressure?

Solution

(a) At triple point T; $T_{tr} = -56.6$ °C and $P_{tr} = 5.11$ atm.

(b) The vaporization curve TC shows that the boiling point of CO_2 decreases with decrease in pressure. The melting curve TB shows that the melting point of CO_2 also decreases with decrease in pressure.

(c) $T_c = 31.1$ °C and $P_c = 73.0$ atm. Above $T_c = 31.1$ °C, CO_2 will not liquefy under any pressure, however high.

(d) (i) This point lies between the sublimation and vaporization curves AT and TC. Hence at this point CO_2 is in vapour phase.

(ii) This point lies between the sublimation and melting curves AT and TB. Hence CO_2 is a solid at this point.

(iii) This point lies between the melting and vaporization curves TB and TC. Hence CO_2 is a liquid at this point.

(e) This point is to the left of the triple point T and below the sublimation curve AT. When compressed at this constant temperature, the point will move perpendicular to temperature axis into the solid phase which occupies the region to the left of curves AT and TB. Hence CO_2 will condense directly to solid.

(f) This point lies to the left of melting curve TB and above the triple point T. Hence CO_2 is a solid at − 65 °C and 10 atm pressure. When it is heated at constant pressure, this point will move perpendicular to the pressure-axis into the region between curves TB and TC and finally to the right of the curve TC. Hence solid CO_2 will change into liquid phase and finally into the vapour phase.

11.15 HEAT ENGINES AND SECOND LAW OF THERMODYNAMICS

The first law of thermodynamics is a generalization of the law of conservation of energy to include heat energy. It tells us that heat and mechanical work are mutually interconvertible. Experimental investigations have revealed that 4.18 J of work produce one calorie of heat. The first law does not tell us whether this conversion can take place at all and if so, how much of one form of energy is converted into another. In everyday life, we come across a particular type of exchange, namely, the exchange of money for goods. The rate at which money can be exchanged for goods is fixed by the price schedule, i.e. so many rupees can buy so much, say, sugar. But an individual can buy only a limited quantity of sugar, even if he has unlimited amount of money.

MONEY ↔ SUGAR

HEAT ↔ WORK

It is pertinent to ask whether there is any limitation (rationing) in the mutual exchange of heat and work. The second law of thermodynamics, which we will state later, will provide an answer to this question.

Experience tells us that there seems to be no restriction on the conversion of work into heat. Unlimited amount of mechanical work can be converted into heat by friction. If you rub two metal pieces together and if you have the patience and the energy, you can produce any amount of heat. There are, however, severe restrictions on the reverse process—the conversion of heat into mechanical work. There is some kind of a 'rationing' in nature. A producer of sugar can sell any quantity of sugar in the market, but a consumer can buy only a limited quantity of it.

What are the restrictions imposed by nature on the conversion of heat into work? To answer this question, we again resort to our experience and observation. Our experience tells us that heat is not converted into work all by itself. Some external agency called *engine* is required. The engine must have a medium called the *working substance* which will absorb heat from some source and convert it into work. The first law tells us that if no heat is supplied to the engine it cannot do any work. Is this the only restriction; i.e. do we need just an engine, a working substance and a source of heat? The answer is no. If this were the only restriction, we could run a heat engine without the fuel problem. The atmosphere and the oceans have a huge store of internal energy which could be utilized to do useful work. For example we could drive a steamship by extracting heat from the ocean or run a thermal power plant by extracting heat from the surroundings. There is nothing in the first law that excludes this possibility. If such machines are made, there would be no energy crisis. Yet, the fact is that no engineer has been able to construct an engine which can convert heat from a single reservoir (*source*) into work, without rejecting some heat to a second reservoir (called the *sink*) at a lower temperature. This negative experience forms the basis of the second law of thermodynamics which asserts that it is futile to make any attempt to construct a machine which utilizes heat from a single reservoir. Thus, the first law denies the possibility of destroying or creating energy and the second law (to be stated later) denies the possibility of utilizing heat from a single body for the purpose of converting it into work.

Now the question arises as to what kind of process (isothermal or adiabatic) should be employed in a heat engine. Suppose we have an engine, e.g. a cylinder with a piston containing n moles of an ideal gas at a temperature T equal to that of the surroundings. Let the gas expand isothermally from volume V_1 to V_2. Since there is no change in the internal energy (as T is constant), the heat absorbed is completely converted into work.

But, when this has been achieved, the engine has suffered a change in volume from V_1 to V_2. To obtain more work, the volume has to be increased continuously, which will clearly be an absurd exercise, as the size of the engine is limited. Thus, to ensure a continuous production of work, the working substance of the engine must be brought back to its original state, i.e. the engine must operate *in a cycle*. If the gas in the above experiment is brought back to its original volume V_1 from V_2 we must do work on the gas, so that it is compressed. This work is the same as we had obtained. The net result is that an engine, working in a cycle, at the same temperature as the surroundings, would produce no useful work. Thus, under isothermal conditions, no engine can produce useful work.

The above observations lead us to the conclusion that three conditions must be fulfilled to utilize heat for useful work.

1. A device called engine with a working substance is essential.
2. The engine must work in a reversible cyclic process.

3. The engine must operate between two temperatures. It will absorb heat from a hot body (called source), convert a part of it into useful work and reject the rest to a cold body (called sink) (see Fig. 11.17).

Efficiency of a Heat Engine

The thermal efficiency of a heat engine is defined as:

$$= \frac{\text{Useful work done}}{\text{Heat absorbed (in energy units)}}$$

$$= \frac{W}{Q}$$

If Q_1 = heat absorbed from the source in a cycle, and Q_2 = heat rejected to the sink in that cycle, then the heat absorbed by the engine is $(Q_1 - Q_2)$.

From the first law, no part of this heat is used up in increasing the internal energy since change in internal energy(ΔU) is zero as the system returns to the initial state after the completion of the cycle. Thus $(Q_1 - Q_2)$ heat is converted into useful work, i.e.

$$W = Q_1 - Q_2$$

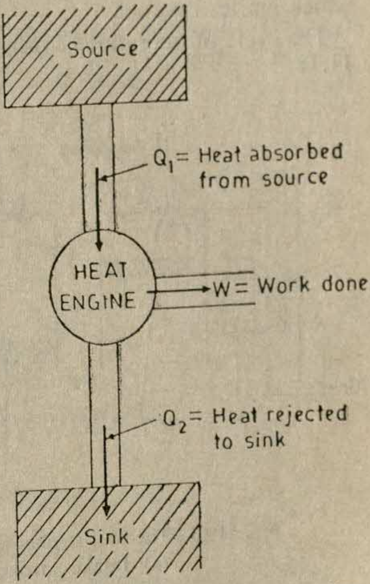

Source
Q_1 = Heat absorbed from source
HEAT ENGINE
W = Work done
Q_2 = Heat rejected to sink
Sink

Fig. 11.17 Symbolic representation of a heat engine

Hence, the efficiency of the engine is given by

$$\eta = \frac{\text{Work done}}{\text{Heat absorbed}} = \frac{Q_1 - Q_2}{Q_1}$$

or

$$\eta = 1 - \frac{Q_2}{Q_1} \qquad (11.23)$$

Since $Q_2 < Q_1$, the value of η is always less than unity. For a given value of Q_1, the smaller the value of Q_2, the higher will be the thermal efficiency of the engine.

Internal Combustion Heat Engine: The Otto Engine In the internal combustion engine, heat energy is generated inside a cylindrical vessel (called *cylinder*) by the combustion (burning) of the fuel. The engines of scooters, cars and motor cycles use a mixture of petrol and air as fuel. The fuel used in the engines of buses, trucks and other heavy vehicles in a mixture of diesel oil and air.

Figure 11.18 shows the working of an internal combustion engine. It consists of a cylinder C with a snugly fitting movable piston P. The piston is connected via a crankshaft to the wheels as shown. There are two valves V_1 and V_2 known as the inlet and exhaust valves respectively. The fuel (petrol vapour + air) enters through V_1 and burnt gases are ejected through V_2. A spark plug produces an electric spark which ignites the fuel.

The cycle of operations consists of four strokes illustrated in Fig. 11.18.

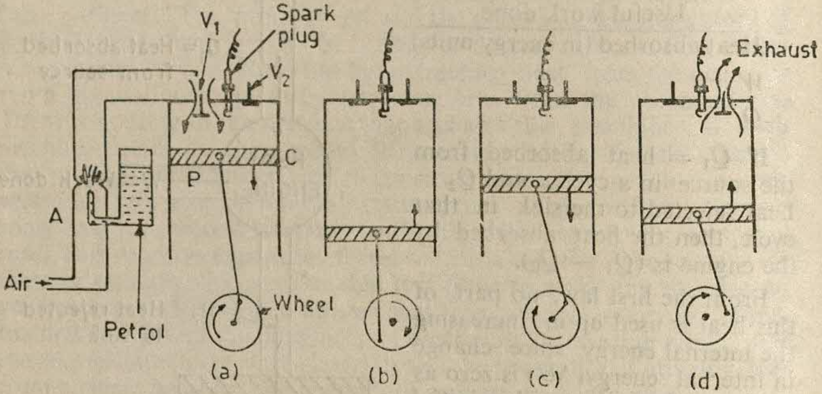


Fig. 11.18 The four strokes of an internal combustion engine. (a) Intake stroke, (b) compression stroke, (c) power stroke and (d) exhaust.

The Charging or Intake Stroke In this stroke the piston moves downwards and the inlet valve is open, thus sucking in air past the nozzle of the carburettor A. Because of the high velocity of air, its pressure is reduced. As a result, petrol vapour from the jet of the carburettor is sprayed into the air stream. This mixture enters the cylinder C through the inlet value V_1 (Fig. 11.18a).

The Compression Stroke In the second stroke, both the valves are closed and the piston moves upwards. The mixture is thus adiabatically compressed and its temperture and pressure rise considerably (Fig. 11.18b).

The Power Stroke Just at the end of the compression stroke when the piston has reached the top (maximum compression) an electric spark is produced inside the cylinder which ignites the mixture. The resulting products of combustion attain a very high temperature and pressure due to a chemical reaction. As a result the piston is driven downwards while both the valves remain closed (Fig. 11.18c). This motion of the engine is the power stroke of the engine. The engine does work in this stroke.

The Exhaust Stroke At the end of the power stroke, the gases have fully expanded and the exhaust valve V_2 now opens. The combustion products are driven out by the upward motion of the piston. When the piston reaches the top, the exhaust valve V_2 closes and a fresh cycle starts all over again (Fig. 11.18d).

We can now discuss the efficiency of the engine by representing the four stages on the *P-V* phase diagram of the working substance (air petrol mixture). The cycle is illustrated in Fig. 11.19.

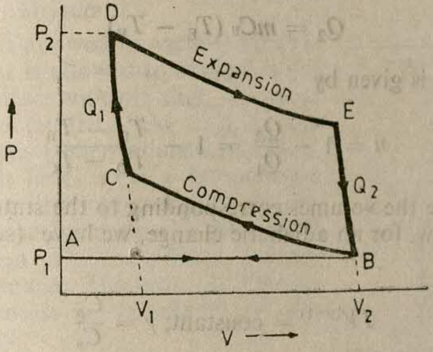


Fig. 11.19 The *P-V* diagram of the Otto engine

The various parts of the cycle as interpreted as follows:

(i) The part AB represents the intake of the mixture at atmospheric pressure P_1. This is the intake stroke.

(ii) The part BC represents adiabatic compression of the mixture. This is the compression stroke.

(iii) The part CD represents the result of the explosion of the mixture. Both temperature and pressure rise considerably at almost constant volume. During this part the heat energy from combustion is supplied to the engine.

(iv) The part DE represents adiabatic expansion of the mixture during the power stroke. Due to sudden expansion the temperature of the mixture falls.

(v) The part EB represents the drop in temperature and pressure at constant volume brought about by the opening of the exhaust valve. The pressure falls to atmospheric pressure.

(vi) The part BA represents exhaust at constant pressure.

We can now work out the efficiency of the engine. Let Q_1 be the heat supplied to the engine during the ignition stage CD. Out of this the exhaust gases take away some heat, say, Q_2, during the exhaust stage so that $(Q_1 - Q_2)$ is the useful work done per cycle of the engine. Thus the efficiency of the engine is given by [see Eq. (11.23)]

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} \tag{11.24}$$

We assume that the mixture behaves as a perfect gas. Let T_D and T_C be the temperatures for the states represented by D and C in Fig. 11.19. Then the heat Q_1 absorbed during the stage CD is given by

$$Q_1 = m\, C_v\, (T_D - T_C)$$

where m is the number of moles of the gas and C_v its molar specific heat at constant volume. Similarly the heat Q_2 rejected during the stage EB is given by

$$Q_2 = mC_v\, (T_E - T_B)$$

Hence efficiency is given by

$$n = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_E - T_B}{T_D - T_C} \qquad (11.25)$$

Let V_1 and V_2 be the volumes corresponding to the states CD and EB respectively. Now, for an adiabatic change, we have (see supplement SI)

$$TV^{(\gamma-1)} = \text{constant};\ \gamma = \frac{C_p}{C_v}$$

$$\therefore \qquad T_D\, V_1^{\gamma-1} = T_E\, V_2^{\gamma-1}$$

and $$T_C\, V_1^{\gamma-1} = T_B\, V_2^{\gamma-1}$$

Hence $$(T_D - T_C)\, V_1^{\gamma-1} = (T_E - T_B)\, V_2^{\gamma-1}$$

or $$\frac{T_E - T_B}{T_D - T_C} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Using this in Eq. (11.25) we get

$$1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1} \qquad (11.26)$$

where $\rho = \frac{V_2}{V_1}$ is called the compression ratio.

In a typical engine the value of $\rho \simeq 8$. Taking $\gamma = 1.4$, this implies that $\eta \simeq 0.55$ or 55%. In an actual engine the efficiency is, in fact, much smaller than 55% due to turbulence, conduction and many other losses.

The Refrigerator or Heat Pump

We have seen that in a heat engine, the working substance extracts some heat Q_1 from the source at a high temperature, converts a part of it into work W and rejects the rest Q_2 to the sink at lower temperature. One can also proceed in the opposite way where the working substance takes heat Q_2 from a sink at a lower temperature. Then work W is done on it and a higher amount of heat $Q_1 = Q_2 + W$ is rejected to the source at a higher temperature (see Fig. 11.20).

Such a device which, by doing work on it, is made to extract heat from a colder body is called a refrigerator and the working substance the refrigerant.

First the gas is allowed to expand adiabatically. Then it is allowed to expand isothermally in contact with the sink, the heat Q_2 being extracted from the sink in the process. The gas is then adiabatically compressed until it heats up to a temperature equal to that of the source. Finally the gas is compressed isothermally till the initial state is reached with the cylinder in contact with the source. The heat Q_1 generated in the process is transferred to the source. Clearly

$$Q_1 = Q_2 + W$$

where W is the work done on the gas in this cycle. This equation implies that work has to be done on the working substance in order to transfer heat from a cold body to a hot body.

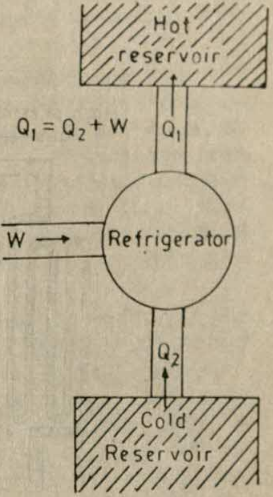


Fig. 11.20 Schematic representation of a refrigerator

In household refrigerators, the work is done with the help of an electric motor. Freon is commonly used as the working substance. It absorbs heat from the things kept inside the refrigerator and rejects it to the surrounding air at a higher temperature.

The working of a commonly used refrigerator is schematically illustrated in Fig. (11.21). The gas is compressed to a high pressure by means of a compressor P. The warm, high-pressure gas is cooled by passing it through spiral pipe C exposed to the surrounding air. The gas gets converted into liquid. The high-pressure liquid is made to pass through a throttling valve T towards the low pressure side. Due to throttling, the temperature is lowered and a part of the liquid is converted into vapour. At B this mixture of liquid and vapour extracts heat from the surrounding material (eatables, etc). The material is cooled and the liquid is converted into vapour at low pressure. The gas is sucked by the compressor P. A repeated operation of the machine can cause substantial cooling.

Second Law of Thermodynamics

In the description of the internal combustion engine, we have seen that the useful work done $(Q_1 - Q_2)$ is less than the heat Q_1 taken in during the first part of the power stroke because some heat Q_2 is given out by the exhaust gases. If Q_2 were zero, the efficiency of the engine would be $=1$ or 100%. This has never been realized. Similarly if, in a refrigerator, the heat given out at the higher temperature were equal to (not less than) the heat absorbed at the lower temperature, the efficiency of the refrigerator would be 100%. Such a perfect refrigerator is never realized. These impossibilities form the basis of the *second law of thermodynamics*. There are several equivalent statements of this law. We give below three of them.

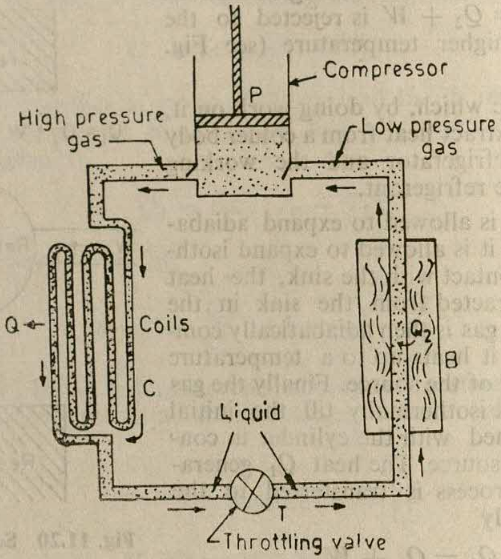


Fig. 11.21 The working of a refrigerator

(*a*) *Kelvin's Statement* *It is impossible to obtain a continuous supply of energy by cooling a body below the coldest of its surroundings*. This statement can be understood from the fact that a heat engine cannot work if the temperature of the source is the same as that of the sink, for then the efficiency of the engine is zero. It is, therefore, impossible to obtain any work from the engine if the source cools down to a temperature lower than that of the coldest body in the surroundings.

(*b*) *Clausius' Statement* *It is impossible for a self-acting machine, unaided by any external agency, to transfer heat from a body at a lower temperature to a body at a higher temperature, or heat cannot by itself* (*i.e. without the performance of work by an external agency*) *pass from a cold to a hot body*. Viewed in the opposite way, this statement is equivalent to Kelvin's. It is easily understood from the fact that no one has been able to construct a refrigerator that works without performing work on the working substance.

(c) *Planck's Statement* *It is impossible to construct an engine which, operating in a cycle, will produce on effcet other than extracting heat from a reservoir and performing an equivalent amount of work.* This statement implies that the working substance, working in a cycle, cannot convert all the heat it extracts into work, it has to reject some heat to the sink.

All these statements of the second law are equivalent, since one can be derived from the other. The second law cannot be proved directly, since it is negative in form. It is universally accepted as a true law, as no exception has yet been found. No other law of physics has been applied to so wide a range of phenomena as the second law of thermodynamics. It has been applied to the study of change of state, solutions, osmotic pressure, electric cells. thermo-electricity, chemical reactions, etc. In every case, the predictions of the law have been confirmed experimentally.

It is important to note the distinction between the first and the second laws of thermodynamics. The first law asserts that heat and work are only different forms of energy and are mutually interconvertible. It does not tell us how and to what extent one can be converted into the other. It only tells us that energy (including heat) is conserved. The second law imposes a restriction on the first law by asserting that when heat is converted into work, only a partial conversion is possible, even under the most ideal conditions and that any such conversion must require a hot body and a cold body. To summarize, *the first law denies the possibility of destroying or creating energy including heat energy and the second law denies the possibility of using heat from a single body for the purpose of converting it into work.*

SECTION C: TRANSFER OF HEAT

11.16 INTRODUCTION

If one end of a long metal rod is heated, the other end also becomes hot after some time. If a liquid in a vessel is heated by placing a burner below the vesssel, we find that the top surface of the liquid also becomes warm after some time. We receive the heat from the sun directly. These three examples suggest three different methods by which heat can be transferred from one place to another. The process of heat transfer in the first example is called *conduction*. In the second example heat is transferred by *convection* and in the third by *radiation*. We will study these processes in more detail.

11.17 CONDUCTION

When two bodies or two different parts of the same body have different temperatures and are brought into thermal contact, they exchage heat energy and tend to equalize the temperature.

The mechanism of heat transfer can be understood in terms of the thermal motion of the atoms and molecules in them. We know that temperature is a manifestation of the thermal motion of molecules and is related to the average kinetic energy per molecule. Suppose two adjacent parts A and B of a body are at different temperatures T_A and T_B with $T_A > T_B$. This means that the average energy of a molecule in part A is greater than that in part B. When the more energetic molecules in part A collide with the less energetic molecules in part B, they transfer some heat energy to the latter. The parts A and B exchange energy by collisions between atoms or molecules. The net effect of a large number of such collisions is to equalize their energies on an average. Energy is thus transferred from A to B via collisions till T_A becomes equal to T_B. The converse of this is also true. If two parts of a body are to be kept at different temperatures, a steady flow of heat energy must be maintained. Or, if two parts of a body are at different temperatures, there is a steady flow of heat energy from one part to the other. *This process of transfer of heat from one place to the next via collisions is called conduction.*

Thermal Conductivity

Consider a piece of a substance (in the form of a uniform rectangular plate or a cylindrical rod with flat end faces) in which one flat end A is kept at a temperature T_A and the other flat end B at a lower temperature T_B (Fig. 11.22). The sides of the rod are completely insulated so that no heat is lost by the rod.

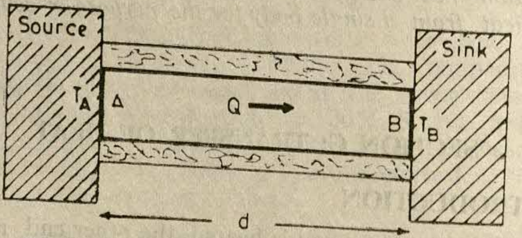


Fig. 11.22 Heat transfer by conduction in a bar

If a steady temperature difference $(T_A - T_B)$ is to be maintained between the ends A and B of the rod, heat must be supplied at A at a certain steady rate and the same must be taken out at the other end B. (If heat is not supplied At A, T_A will become equal to T_B by conduction; and if heat is not taken out at B, T_A will keep on increasing). Suppose an amount of heat Q flows through the rod in time t so that Q/t is the rate of heat flow. Then it is reasonable to expect that

(i) Q/t will be proportional to the area S of the cross-section of the rod. The larger the cross-sectional area, the greater will be the number of molecules which conduct heat via collisions.

(ii) Q/t will be proportional to $(T_A - T_B)$. The greater the temperature difference to be maintained, the greater must be the rate of heat flow.

(iii) Q/t will be inversely proportional to d, the distance between ends A and B. The smaller the distance d, the greater must be the rate of heat flow to maintain a given temperature difference.

Thus we find that

$$\frac{Q}{t} \propto \frac{S(T_A - T_B)}{d}$$

or

$$\frac{Q}{t} = \frac{kS(T_A - T_B)}{d} \tag{11.27}$$

where k is a constant of proportionality called the *thermal conductivity* of the substance. It is a measure of how quickly heat energy can conduct (flow) through the substance. In the SI system, k is expressed in $\text{Js}^{-1}\,\text{m}^{-1}\,{}^\circ\text{C}^{-1}$ or $\text{Js}^{-1}\,\text{m}^{-1}\,\text{K}^{-1}$ or $\text{Wm}^{-1}\,\text{K}^{-1}$.

Thermal conductivities of a few common substances are given in Table 11.4.

Table 11.4 Thermal conductivity of some common materials

Material	*Thermal conductivity* (*in* $\text{Js}^{-1}\,\text{m}^{-1}\,\text{K}^{-1}$)
Copper	400
Aluminium	240
Iron	80
Glass	0.8
Water	0.6
Insulating material	0.04
Air	0.02

The value of thermal conductivity of a substance gives us a measure of its ability to retain or lose heat. This knowledge is very useful in the design of buildings for industry and homes especially in cold countries.

EXAMPLE 11.13 A slab of stone of area 3600 cm^2 and thickness 10 cm is exposed on the lower surface to steam at 100 °C. A block of ice at 0°C rests on the upper surface of the slab. In one hour 4.8 kg of ice is melted. Calculate the thermal conductivity of stone. The latent heat of fusion of ice $= 3.35 \times 10^5\ \text{J kg}^{-1}$. Assume that heat loss from sides is negligible.

Solution The amount of heat flowing through the stone in 1 hour,

$$Q = 4.8 \times 3.35 \times 10^5 = 1.608 \times 10^6 \text{ J}$$

Time $(t) = 1$ hour $= 60 \times 60$ s

Area $(S) = 3600 \text{ cm}^2 = 3.6 \times 10^{-1} \text{ m}^2$

Temperature diff. $= (T_A - T_B) = 100$ °C

Thickness $(d) = 10$ cm $= 0.1$ m

Now we know that thermal conductivity k is given by

$$k = \frac{Q \times d}{tS\,(T_A - T_B)}$$

$$= \frac{1.608 \times 10^6 \times 0.1}{(60 \times 60) \times (3.6 \times 10^{-1}) \times 100}$$

$$= 1.2 \text{ Js}^{-1} \text{ m}^{-1} (°\text{C})^{-1}$$

EXAMPLE 11.14 An ice-box is made of wood 1.75 cm thick, lined inside with cork 2 cm thick. If the temperature of the inner surface of the cork is steady at 0 °C and that of the outer surface is steady at 12 °C, what is the temprature of the interface? The thermal canductivity of wood is 5 times that of cork.

Solution

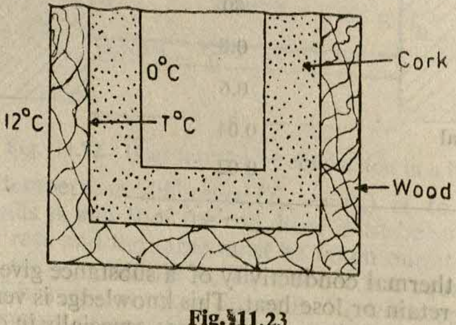


Fig. 11.23

Let T be the temperature of the interface in the steady state. Since the rate of flow of heat through wood and cork must be the same in the steady state, we have

$$\frac{k_w S\,(T_w - T)}{d_w} = \frac{k_c S\,(T - T_c)}{d_c}$$

where S is the area of the interface. Thus we have

$$\frac{T_w - T}{T - T_c} = \frac{d_w}{d_c}\frac{k_c}{k_w}$$

or $$\frac{12 - T}{T - 0} = \frac{1.75}{2} \times \frac{1}{5}$$

which gives $$T = 10.2\ °\text{C}$$

11.18 CONVECTION

Fluids (liquids and gases) are heated mainly by a process called *convection* in which buoyancy and gravity play an important role. When a fluid is heated from below, the hot portion at the bottom expands. Since the mass of a given portion remains unchanged, its density decreases. According to Archimedes' principle this lighter portion will rise upwards. The denser fluid from above will take its place by moving downwards. Thus *convection* current is set up in the fluid. This movement of the liquid can be seen by colouring it, say, with potassuim permanganate crystals at the bottom of the fluid (Fig. 11.24).

In convection heat is transferred by the physical movement of matter (fluid). No such large-scale movement of matter is involved in conduction where heat energy is transported by molecules via collisions in their *local regions.*

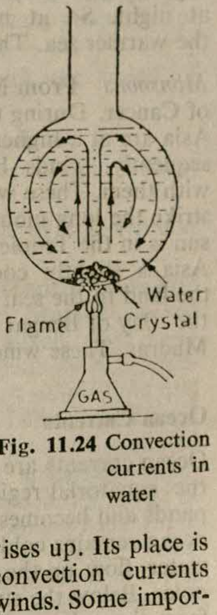


Fig. 11.24 Convection currents in water

Wind Systems in the Atmosphere

Winds are an important illustration of the phenomenon of convection currents in nature. Some parts of the earth are heated by the sun in the day-time. The air above these places expands and rises up. Its place is taken up by heavier air from cold places. Thus convection currents are set up in the atmosphere which constitute the winds. Some important wind systems are as follows.

Trade Winds The surface of the earth near the equator is heated by the sun. The air in contact with the surface of the earth becomes warmer and rises upwards. The air from the colder polar regions rushes in towards the equatorial regions, thus producing northward wind in northern hemisphere and southward wind in southern hemisphere. Due to the rotation of the earth about its axis from west to east, the actual direction of the wind in the northern hemisphere is northeast and in the southern hemisphere, southwest. These winds are called *trade winds*. In ancient times, the traders used these winds to find the direction of motion of their sailing vessels.

Land and Sea Breezes At places near the sea, especially in summer time, one notices a breeze that generally blows from the sea towards the land during the day, while at night the direction of the breeze is from the land towards the sea. These land and sea breezes are the local convection currents.

Water has a high specific heat. Therefore, it is less readily heated by the sun than land. In other words, during the day the land is heated by the sun to a higher temperature than water in the sea. Air over the land is therefore heated, expands and rises while the cooler air from the sea blows towards land to take its place which constitutes the *sea breeze.*

At night the land is no longer heated by the sun. It therefore cools. But land cools much faster than water (due to the high specific heat capacity of water). By comparison the sea is much warmer than land at night. So at night a current of air blows from the colder land to the warmer sea. This constitutes the *land breeze.*

Monsoons From May to October (summer) the sun is in the Tropic of Cancer. During this time large parts of the earth situated in Central Asia are at a higher temperature than the surrounding oceans. Consequently, winds blow from the ocean to the land carrying moisture with them. These winds cause widespread rains in India after they strike the long mountain ranges. From November to April (winter) the sun is in the Tropic of Capricorn. During this time the land in Central Asia is slightly cooler than the sea. Consequently, winds blow from the land to the sea. These winds take up moisture as they pass over the Bay of Bengal and cause rainfall in the coastal regions such as Madras. These winds which bring rain are called *monsoons.*

Ocean Currents

Ocean currents are convection currents in the body of the ocean. In the equatorial regions of the ocean, water is heated by the sun, it expands and becomes lighter. But the water in the polar regions of the ocean remains cold and so is heavier. Consequently, currents of warm water flow on the surface of the ocean from the equator towards the poles. Below the surface of the ocean, currents of cold water flow from the poles to the equator. These currents control the temperature of the ocean.

11.19 RADIATION

All bodies emit heat radiation from their surfaces by virtue of their temperature. This radiation is called *radiant energy*. The heat that we receive from the sun is transferred to us by a process which, unlike conduction or convection, does not require the help of a medium in the intervening space. It is well-known that most of the space between the sun and the earth is a vacuum and there is almost no trace of matter in that space and yet we receive heat from the sun. The heat travels in straight lines as does light. This is evident from the fact that the shape of a tree coincides with its shadow. Because heat and light travel with the same speed, they are both simultaneously cut off in an eclipse of the sun. Since light is propagated by waves of some kind, it is reasonable to expect that heat is also propagated by similar waves. Thus, heat is propagated by wave motion and we say it is radiated. We shall use the term *radiant energy* or *thermal radiation* to denote heat radiated by an object.

Nature and Properties of Thermal Radiation

At the beginning of the nineteenth century, scientists began to investigate the nature and properties of thermal radiation. These invstigations revealed striking similarities between thermal radiations and visible light. We shall now briefly describe some important properties of thermal radiation established by these investigations:

1. Thermal radiations travel through vacuum with the speed of light, i.e. $3 \times 10^8 \text{ ms}^{-1}$.
2. Like light, thermal radiations travel in straight lines.
3. Thermal radiations obey the same laws of reflection and refraction as light does.
4. Thermal radiations exhibit the phenomena of interference, diffraction and polarization, as does light.

All these observations show that the nature of thermal radiation and light is the same. In the year 1863, James Clerk Maxwell proved that light is due to electric and magnetic vibrations which travel together in space. In other words, *light is an electromagnetic wave*. Electromagnetic waves causing a visible sensation are called visible light and have wavelengths in the range 3500 Å (violet) to 7800 Å (red). (Å stands for Angstrom and 1 Å = 10^{-8} cm.) *Infrared* rays are so called because they occur beyond the red end of the visible spectrum. Infrared rays are electromagnetic waves with a wavelength in the range of about 8000 Å to 0.04 cm. Infrared rays thus have a longer wavelength than the rays in the visible spectrum. Waves with still longer wavelengths are called *microwaves* and *radiowaves*. Electromagnetic waves whose wavelengths are shorter than for violet rays are called *ultraviolet* rays. Still shorter wavelength waves are called *x-rays* and *γ-rays*. All these electromagnetic waves travel with a velocity c in a vacuum, where $c = 3 \times 10^8 \text{ m s}^{-1}$. The frequency ν of these waves is related to wavelength by the relation

$$c = \nu\lambda$$

Although thermal radiation and light produce different sensations, it is clear that the only difference between them is their wavelength (or frequency). Both are forms of energy and they both travel in space as electromagnetic waves with the same velocity. Thermal radiations are simply infrared rays. The radiations emitted by hot bodies are, therefore, also known as infrared radiations. Table 11.5 shows the wavelength range of the known electromagnetic radiations. We notice that the visible light is only a very small part of the total electromagnetic spectrum.

Table 11.5 Electromagnetic spectrum

Name of radiation	*Range of wavelengths*
Radiowaves	30 cm to 600 m
Mircowaves	1 mm to 30 cm
Infrared	8×10^{-5} cm to 0.04 cm
Visible light	3.8×10^{-5} cm to 7.8×10^{-5} cm
Ultraviolet	1.4×10^{-5} to 3.8×10^{-5} cm
x-rays	10^{-9} cm to 10^{-6} cm
γ-rays	Below 10^{-9} cm

Electromagnetic waves thus have a wide range of wavelengths. Although they are identical in nature, their interaction with matter or their physiological action on living bodies depends on their wavelengths. Thermal radiations (i.e. infrared rays with wavelengths ranging from 8×10^{-5} cm to about 0.04 cm) produce heat when they are absorbed by a body. This is the property that distinguishes thermal radiation from the rest of the electromagnetic waves.

Distribution of Radiant Energy in the Electromagnetic Spectrum

When a body is heated, it emits radiation. The nature (i.e. wavelength) of the radiation depends upon the temperature. At a low temperature, a body emits radiation which is principally of long wavelength. At a high temperature, the proportion of shorter-wavelength radiation increases. Furthermore, the amount of the emitted radiation is different for different wavelengths. It is of interest to see how the energy of total radiation from a hot body is distributed among different wavelengths at various temperatures. In other words, we would like to find out how E_λ for a body depends upon λ and T, where E_λ stands for radiant energy at wavelength λ.

The solution of this problem is obtained if we observe the colour of the radiation emitted by a body as its temperature is gradually raised. Thus, for instance, when a blackened platinum wire is heated, it appears dull-red at about 500 °C, changes to cherry-red at 900 °C, becomes orange-red at 1100 °C, yellow at 1300 °C and finally white at about 1600 °C. This shows that, as the temperature is increased, the radiation becomes richer in shorter wavelengths. In other words, the amount of emission shifts towards the shorter wavelength side as the temperature is raised.

Experiments have shown that the energy carried by radiation with different wavelengths varies with wavelength as shown in Fig. 11.25.

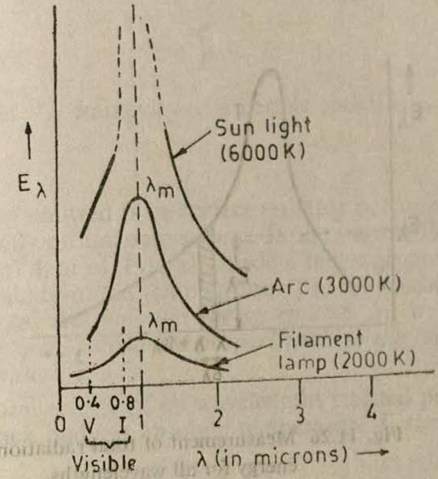


Fig. 11.25 Distribution of energy E_λ emitted from a hot body as a function of wavelength of the radiation for various temperatures. (1 micron = 10^{-6} m)

A study of the curves in Fig. 11.25 reveals the following interesting facts:

1. At a given temperature, the energy is not uniformly distributed in the radiation spectrum of a body, i.e. energy for different wavelengths is different.

2. At a given temperature, the energy of emission has maximum value for a particular wavelength λ_m. With increase in temperature λ_m decreases. The points on the dashed line represent λ_m at various temperatures. The actual measurements show that the product $\lambda_m\ T$ remains constant and is about 2.892×10^{-3} m K.

3. The maximum of each curve shifts towards shorter wavelength as the temperature is raised.

4. For all wavelengths an increase in temperature causes an increase in energy emission.

5. The area under each curve represents the total energy emitted for the complete spectrum at a particular temperature. By definition, $E_\lambda \delta\lambda$ is the energy emitted per second between λ and $\lambda + \delta\lambda$. It is clear from Fig. 11.26 that $E_\lambda\ \delta\lambda$ around λ is measured by the area of the shaded strip. The sum of various such elementary areas measures the total energy for all wavelengths. Actual measurement of area under each curve (i.e. for each temperature) shows that the ratio,

$$\frac{\text{area under curve at temperature } T}{T^4}$$

remains constant implying that the total emitted energy varies as the fourth power of absolute temperature. Thus Stefan's law stated below is experimentally verified.

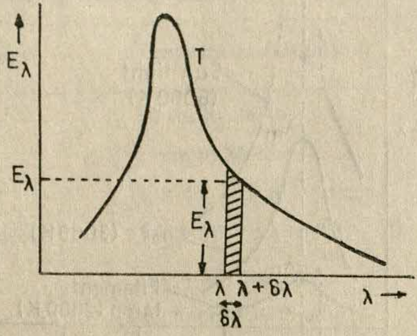


Fig. 11.26 Measurement of total radiation energy for all wavelengths

6. At a temperature 2000 K, for example, there is very little radiant energy in the visible spectrum V-R, but the energy increases as longer (infrared) wavelengths are reached and then decreases. At 6000 K, which is the temperature of sun's surface, the maximum energy occurs in the wavelengths of the visible spectrum.

Emissive Power and Emissivity

Different sources of heat radiation emit different amounts of energy even when they are at the same temperature. The amount of heat energy emitted (or radiated) per second by a body depends upon (i) the area of its surface, (ii) the difference of temperature between surface and the surroundings, and (iii) the nature of its surface. The strength of emission is estimated by a quantity known as *emissive power* which is defined as *the amount of heat energy emitted per second from unit area of the radiating surface* and is usually denoted by e. Defined this way, the emissive power of a given surface depends only on the nature of the surface and its temperature. The SI unit of e is joules per second per square metre ($Js^{-1}\ m^{-2}$) or watts per square metre ($W\ m^{-2}$). In the special case of a perfect radiator, the emissive power is denoted by E. Obviously, at the same temperature, the emissive power E of a perfect radiator has a maximum value, i.e. at the same temperature,

$$E > e$$

The emissivity (denoted by ϵ) of a body is defined as the ratio of its emissive power to that of a perfect radiator (called a black body) at the same temperature, i.e.

$$\epsilon = \frac{e}{E}$$

It follows that the emissivity of a perfect radiator is unity.

Stefan's Law

The total energy emitted by a perfect radiator per unit area per unit time depends only on its temperature. In the year 1879, Stefan, using the experimental data of Tyndall, made a lucky guess at the law relating the total radiation and temperature. The law known after him as *Stefan's law states that the total energy emitted by a unit area of a perfect radiator per second is proportional to the fourth power of its absolute temperature.*

If Q is the total energy of all wavelengths emitted per unit area per second by a black body at absolute temperature T, then

$$Q \propto T^4$$

or

$$Q = \sigma T^4 \tag{11.28}$$

For a body of emissivity ϵ, we have

$$Q = \sigma \epsilon T^4 \tag{11.29}$$

where σ is a constant known as Stefan's constant. In the SI system of units, σ is measured in W m^{-2} K^{-4}. The accepted experimental value of σ is

$$\sigma = 5.67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$$

In 1884, Boltzmann gave a rigorous theoretical proof of Stefan's law based on thermodynamical considerations. Therefore, the law is frequently called the Stefan-Boltzmann Law. In 1897, Lummer and Pringsheim experimentally verified the law and obtained the value of Stefan's constant.

It may be mentioned that Stefan's law is not a law of cooling and does not refer to the net loss of heat by a body after exchange with the surroundings. The law holds for an isolated body. It simply deals with the amount of heat energy emitted by a body by virtue of its temperature irrespective of what it receives from the surroundings. Furthermore, the law refers to total radiation, applying to the whole range of wavelengths, without being limited to any particular wavelength.

However, Stefan's law can be extended to include the net loss of energy by a body after exchange of heat with the surroundings. If a body at temperature T is surrounded by another body (which may be an enclosure to be called surroundings) at a lower temperature T_0, then the body loses heat by emission of radiation and also gains heat from the surroundings. The net loss of energy per unit area per second is given by

$$Q = \sigma (T^4 - T_0^4) \tag{11.30}$$

EXAMPLE 11.14 The tungsten filament of an electric lamp has a length of 0.25 m and a diameter 6×10^{-5} m. The power rating of the lamp is 100 W. If the emissivity of the filament is 0.8, estimate the steady temperature of the filament. Stefan's constant $= 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Solution

$$\text{Area of the filament} = 2\pi \times (\text{radius}) \times (\text{length})$$

or $$A = 2\pi \times (3 \times 10^{-5}) \times 0.25 = 4.71 \times 10^{-5}\ \text{m}^2$$

Now $Q = \sigma\epsilon T^4$, where Q is the energy radiated by second per unit area at absolute temperature T. Therefore, the energy radiated per second (or power radiated) from the filament of area A is

$$P = A\epsilon\sigma T^4$$

When the temperature is steady,

$$\begin{aligned}\text{Power radiated from filament} &= \text{power received}\\ &= 100\ \text{W}\end{aligned}$$

$\therefore$ $$A\epsilon\sigma T^4 = 100\ \text{W}$$

Now $$A = 4.71 \times 10^{-5}\ \text{m}^2, \qquad \epsilon = 0.8$$

and $$\sigma = 5.67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$$

Substituting these values we have

$$\begin{aligned}T &= \left(\frac{100}{4.71 \times 10^{-5} \times 0.8 \times 5.67 \times 10^{-8}}\right)^{1/4}\\ &= 2616\ \text{K}\end{aligned}$$

EXAMPLE 11.15 A man, the surface area of whose skin is 2 m^2, is sitting in a room where the air temperature is 20 °C. If his skin temperature is 28 °C, find the rate at which his body loses heat. The emissivity of his skin = 0.97.

Solution

$$\text{Absolute room temperature } (T_0) = 20 + 273 = 293\ \text{K}$$

$$\text{Absolute skin temperature } (T) = 28 + 273 = 301\ \text{K}$$

Now $$\begin{aligned}\text{Rate of heat loss} &= \sigma\epsilon A\,(T^4 - T_0^4)\\ &= 5.67 \times 10^{-8} \times 0.97 \times 2\\ &\quad \times \{(301)^4 - (293)^4\}\\ &= \mathbf{92.2\ W}\end{aligned}$$

This is a surprisingly high rate of heat loss even at a comfortable room temperature of 20 °C. The reason why we do not feel uncomfortable is that our body generates energy at a rate of about 120 W. So our body loses a lot of heat by radiation even at 20 °C. It is for this reason that people living in Arctic regions use a special clothing made with an additional thin shiny metallic layer next to the skin to reflect the body's radiation.

EXAMPLE 11.16 The solar constant (which is the energy received per unit area per second on the earth from the sun) is about 1400 W m^{-2}. Estimate the surface temperature of the sun, given, that the sun's radius = 7×10^8 m, the distance of the earth from the sun = 1.5×10^{11} m and Stefan's constant = 5.7×10^{-8} W m^{-2} K^{-4}.

Solution Suppose T is the absolute temperature of the surface of the sun. Then total energy radiated per second from sun's surface (assuming its emissivity to be unity)

$$= A\sigma T^4$$

or

$$E = 4\pi R_s^2 \sigma T^4$$

where R_s = radius of the sun; its surface area = $4\pi R_s^2$.

This energy falls, all around a sphere of radius R_0, where R_0 is the radius of the earth's circular orbit around the sun. The area of this sphere = $4\pi R_0^2$. Therefore, the energy falling per second per unit area of this sphere

$$= \frac{E}{4\pi R_0^2}$$

$$= \frac{4\pi R_s^2 \sigma T^4}{4\pi R_0^2}$$

$$= \left(\frac{R_s}{R_0}\right)^2 \sigma T^4$$

So

$$\left(\frac{R_s}{R_0}\right)^2 \sigma T^4 = 1400$$

or

$$T^4 = \frac{1400}{5.7 \times 10^{-8}} \times \left(\frac{1.5 \times 10^{11}}{7 \times 10^8}\right)^2$$

which gives $T \simeq 5800$ K

Radiative Transfer

The main modes of the transfer of heat in very large systems, such as the atmosphere, the sun and the solar system, are radiation and convection. Radiation (or radiative transfer) is the dominant mode of heat transfer in interstellar space and the galaxy.

SUPPLEMENT

S.1 Pressure-Volume Relation in an Adiabatic Change

In an adiabatic process, heat is not allowed to enter or leave the system, i.e $\Delta Q = 0$. The first law of thermodynamics, for an adiabatic change, reads

$$0 = \Delta U + P\Delta V$$

Since $\Delta U = nC_v\,\Delta T$ [Eq. (11.20)], we have

$$nC_v\,\Delta T + P\Delta V = 0 \tag{i}$$

Now $PV = nRT$ which on differentiation becomes

$$P\Delta V + V\Delta P = nR\Delta T$$

giving
$$\Delta T = \frac{P\Delta V + V\Delta P}{nR}$$

Substituting for ΔT in Eq. (i), we get

$$nC_v\left(\frac{P\Delta V + V\Delta P}{nR}\right) + P\Delta V = 0$$

or
$$C_v\,(P\Delta V + V\Delta P) + RP\Delta V = 0$$

or
$$C_v\,(P\Delta V + V\Delta P) + (C_p - C_v)\,P\Delta V = 0$$

($\because$ $R = C_p - C_v$)

or
$$C_vV\Delta P + C_p\,P\Delta V = 0$$

or
$$\frac{\Delta P}{P} + \gamma\,\frac{\Delta V}{V} = 0 \tag{ii}$$

where $\gamma = \dfrac{C_p}{C_v}$ is the ratio of the two specific heats of the gas.

Since γ is practically constant, Eq. (ii) can be integrated to give

$$\int \frac{\Delta P}{P} + \gamma \int \frac{\Delta V}{V} = 0$$

or
$$\log P + \gamma \log V = \text{constant}$$

or
$$\log\,(PV^{\gamma}) = \text{constant}$$

or
$$PV^{\gamma} = \text{constant} \tag{iii}$$

Thus, for an adiabatic change in an ideal gas PV^{γ} = constant, whereas for an isothermal change in an ideal gas, PV = constant. Since $PV = nRT$, Eq. (iii) may be written in terms of T and V as

$$PV^{\gamma} = \frac{nRT}{V}\cdot V^{\gamma} = \text{constant}$$

or
$$TV^{(\gamma-1)} = \text{constant} \tag{iv}$$

($\because$ n and R are constant)

S.2 Evaporation and Boiling

Evaporation and boiling are two very different phenomena. Evaporation takes place from the surface of a liquid at all temperatures, so that, at any temperature, there is always some amount of vapour present above the liquid surface. Boiling, on the other hand, takes place at a definite temperature depending on pressure.

These processes can be understood in terms of the *P-T* phase diagram. Figure 11.27 shows the liquid-vapour part of the *P-T* phase diagram of water.

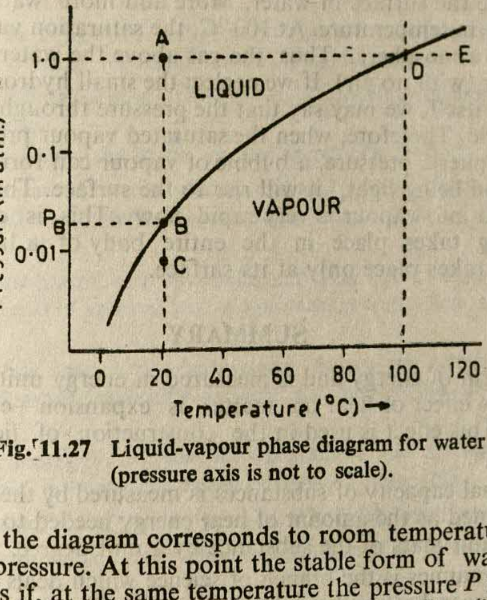


Fig. 11.27 Liquid-vapour phase diagram for water (pressure axis is not to scale).

Point A in the diagram corresponds to room temperature 20 °C at atmospheric pressure. At this point the stable form of water is liquid. What happens if, at the same temperature the pressure P is lower than P_B, the pressure corresponding to point B? If $P < P_B$, (i.e. at point C), the vapour is the stable form. The value of P is determined by the amount of water vapour present and this depend upon the temperature and other conditions. P is called the *partial pressure* due to water vapour. On a rainy day $P \simeq P_B$, whereas on a hot summer day $P \ll P_B$. Notice that P_B is the maximum pressure at which the vapour co-exists with its liquid; this is called the *saturation vapour pressure* at that temperature. The ratio P/P_B determines the *relative humidity* at that temperature. Thus if, at a particular temperature, P is less than P_B (i.e. we are at point C of the *P-T* diagram), then water will evaporate from the surface of the liquid, and the evaporation will continue till P equals P_B at that temperature. This explains evaporation. Evaporation takes place from the surface because the molecules at the surface have less attractive energy (see section on surface tension in Chapter 10) and also because it requires less energy to escape from the surface than from within the body of the liquid. In order to evaporate, the liquid takes the necessary latent heat from itself which results in the cooling of the liquid. Thus evaporation is accompanied by cooling.

Let us now consider boiling. Measurements show that at 100 °C the saturated vapour pressure of water is 1 atmosphere, i.e. water boils at 100 °C under 1 atmosphere pressure. Suppose water is heated at atmospheric pressure. This is represented by the line ADE in Fig. 11.27. Below 100 °C, the liquid is the stable from of water and above 100 °C, the vapour is the stable from. Now suppose we take some water in a vessel and open it to the atmosphere while it is being heated. As the water temperature increases, so does the pressure of water vapour above the surface of water. More and more water evaporates with increase in temperature. At 100 °C, the saturation vapour pressure of water is 1 atmosphere. Thus the gas above the water surface is all water vapour (with no air). If we neglect the small hydrostatic pressure of the liquid itself, we may say that the pressure throughout the liquid is atmospheric. Therefore, when the saturated vapour pressure is equal to the atmospheric pressure, a bubble of vapour can form anywhere in the liquid and being light, it will rise to the surface. Thus the conversion of liquid into vapour is very rapid now. This is called *boiling*. Thus boiling takes place in the entire body of a liquid, whereas evaporation takes place only at its surface.

SUMMARY

Heat is a form of energy and is measured in energy units, joule. The most obvious effect of heat on matter is expansion—called thermal expansion. This effect is used in the construction of liquid and gas thermometers.

The thermal capacity of substances is measured by their specific heat which is defined as the amount of heat energy needed to raise the temrature of a unit mass of the substance by a unit degree.

Thermodynamics is the branch of science which deals with the relationship between heat and mechanical energy. The first law of thermodynamics is an energy conservation law applied to the case of transformation heat energy into mechanical energy. The second law of thermodynamics denies the possibility of converting the entire heat energy into mechanical energy.

The transfer of heat takes place by three processes called conduction, convection and radiation. In conduction, the molecules move about a fixed location and transfer heat by collisions. In convection, the transfer of heat takes place by transport of matter. Radiation involves is transfer of heat by electromagnetic waves without the help of any material medium.

EXERCISES

1. How will you show that heat is a form of energy?
2. Describe the working of a constant-volume gas thermometer for the measurement of temperature.

3. Explain the principle of (a) resistance thermometers and (b) pyrometers.
4. Define the coefficients of cubical and linear expansion. Obtain the relation between them.
5. What do you understand by 'specific heat' of a substance. State its units.
6. Define the specific heats of a gas at constant pressure (C_p) and at constant volume (C_v). Deduce the relation between them, clearly explaining why C_p is greater than C_v.
7. In the context of thermodynamics, what do you understand by the terms: thermodynamic system, thermodynamic state and thermodynamic coordinates?
8. What is the meaning of the statement 'system A is in thermal equilibrium with system B'? State the zeroth law of thermodynamics.
9. Distinguish clearly between isothermal and adiabatic processes. Give an example of each.
10. What do you understand by the internal energy of a system?
11. State the first law of thermodynamics and apply it to calculate the change in the internal energy during (i) the boiling process and (ii) the melting process.
12. Apply the first law of thermodynamics and obtain the relation between the specific heats of an ideal gas at constant pressure and at constant volume.
13. Using the first law of thermodynamics, deduce the relation between pressure and volume changes in an ideal gas undergoing an adiabatic change.
14 Give one example from everyday life which shows that (i) work can be converted into heat and (ii) heat can be converted into work.
15. State the second law of thermodynamics. What is its significance?
16. Describe the principle, construction and working of an internal combustion engine.
17. Describe the principle, construction and working of a refrigerator.
18. State the three modes of heat transfer. Give one example of each.
19. Define thermal conductivity and state its SI units.
20. What are convection currents? What role do they play in relation to trade winds, land and sea breezes and monsoons?
21. What is the nature of heat radiations?
22. In what respect is thermal radiation different from light?
23. State and explain Stefan's law of heat radiation.
24. Draw labelled curves showing the spectral distribution of energy radiated from (a) a filament lamp, (b) an arc and (c) the sun. What conclusions will you draw from these curves?

PROBLEMS

(Take Stefan's constant = 5.67×10^{-8} Wm^{-2} K^{-4} wherever necessary.)

1. A mercury barometer has a quartz bulb of volume of 0.300 cm^3 and a stem of bore 0.0100 cm. By how much will the mercury meniscus in the stem move if the temperature changes from 30 °C to 45 °C? Coefficient of volume expansion of mercury = 1.82×10^{-4} (°C)$^{-1}$. Ignore the small expansion of the quartz bulb.
2. A glass flask of volume 250 cm^3 is just filled with mercury at 20 °C. How much mercury will overflow when the temperature of the system is raised to 100 °C? The coefficients of linear expansion of glass = 4×10^{-6} (°C)$^{-1}$ and coefficient of cubical expansion of mercury = 1.8×10^{-4} (°C)$^{-1}$.
3. A sphere of diameter 7 cm and mass 266.5 g floats in a liquid bath. The temperature of the bath is raised and when the temperature is 35 °C, the sphere begins to sink. If the density of liquid is 1527 kgm^{-3}, find the coefficient of volume expansion of the liquid. Neglect the expansion of the sphere.
4. An aluminium container of mass 100 g contains 200 g of ice at — 20 °C. Heat is added to the system at the rate of 100 calories per second. Find the temperature of the system after 4 minutes. Specific heats of ice and aluminium respectively are 0.5 and 0.2 cal g^{-1} (°C)$^{-1}$ and latent heat of fusion of ice is 80 cal g^{-1}.
5. The temperatures of equal masses of three different liquids A, B and C are 12 °C, 19 °C, and 28 °C respectively. The temperature when A and B are mixed is 16 °C, and is 23 °C when B and C are mixed. What is the temperature when A and C are mixed?
6. 5 g of water at 30 °C is mixed with 5 g of ice at — 20 °C in a vessel of negligible heat capacity. Find the final temperature and composition of the mixture. Specific heat of ice = 2100 J kg^{-1} K^{-1} and latent heat of fusion of ice = 3.36×10^5 J kg^{-1}.
7. A lead bullet strikes a target with a velocity of 240 m s^{-1}. If the bullet is stopped within the target, calculate the rise in its temperature, assuming that the heat developed is equally shared between it and the target. Specific heat of lead is 0.03 cal g^{-1} °C^{-1}.
8. Calculate the rate of flow of heat through 2 m^2 of a brick wall 12 cm thick if the temperature on one side is 8 °C and on the other side is 32 °C. Thermal conductivity of brick = 0.13 Wm^{-1} K^{-1}.
9. A bar of copper of length 75 cm and a bar of steel of length 125 cm are joined together end-to-end. Both are of circular cross-section with diameter 2 cm. The free ends of the copper and steel bars are maintained at 100 and 0 °C respectively. (a) What is the temperature of the copper-steel junction? (b) Calculate the heat transmitted per second across the junction. Thermal conductivities of copper and steel respectively are 386.4 and 46.2 Js^{-1} m^{-1} K^{-1}.
10. A metallic sphere of diameter 2 cm is coated with lamp black and is maintained at a temperature of 927 °C. Calculate the rate at which energy is radiated from its surface, assuming that it is a perfect radiator.

11. The temperature of the filament of a 40 W tungsten lamp is 2170 °C and the effective surface area of the filament is 0.66 cm^2. Estimate the value of Stefan's constant assuming that the energy radiated is 0.25 of that from a perfect radiator under similar conditions. Neglect the effect due to radiation from the glass envelope.
12. A metal ball of 14 cm diameter and mass 10 kg is heated to a temperature of 227 °C and suspended in a box whose walls are at a temperature of 27 °C. Calculate the net rate of loss of heat by the ball. What will be its temperature at the end of 5 minutes? Specific heat of the metal = 420 $\text{J kg}^{-1}\text{ K}^{-1}$.
13. Each square metre of the sun's surface radiates energy at the rate of $6.3 \times 10^7 \text{ W m}^{-2}$. Assuming that Stefan's law applies to the radiation, calculate the temperature of the sun's surface.

12

Oscillations

12.1 INTRODUCTION

The world is full of things that move. The motion of physical systems can be classified into two broad categories—translatory and vibratory or oscillatory. If the position of a body varies linearly with time, its motion is translatory, e.g. a train moving on a straight track or a ball rolling on the ground. A motion that repeats itself in equal intervals of time is called periodic motion, e.g. the motion of the hands of a clock. If a particle in periodic motion moves back and forth over the same path, its motion is called vibratory or oscillatory. Some examples of oscillatory motion are the oscillation of the hands of a walking person, the balance wheel of a watch, the bob of a pendulum clock, a mass attached to a spring, the prongs of a tuning fork, the piston of an automobile engine, etc. Oscillations may be very complex, such as those of a piano string or those of the earth during an earthquake. It may be remarked that mechanical systems are not the only ones that can oscillate. The atoms in a solid vibrate. The electrons in a radiating or receiving antenna are in oscillation. A tuned circuit in a radio can oscillate electromagnetically. Radio waves, microwaves and visible light are simply oscillating electric and magnetic fields. Thus, the study of oscillations is essential for the understanding of various physical systems—mechanical, acoustical, electrical and atomic.

In this chapter, we will study the simplest and smoothest type of oscillatory motion, namely, *simple harmonic motion* (usually designated as SHM). Among the various types of oscillatory motions, SHM is of central importance for three basic reasons: (i) the oscillation of all physical systems is simple harmonic or a close approximation to it, if the oscillation is not too violent, (ii) the study of SHM is essential for the understanding of wave motion, and (iii) we will learn in higher classes that complex oscillations, e.g. those of a piano string, can be expressed as a combination of harmonic motions.

12.2 WHAT CAUSES A SYSTEM TO OSCILLATE

The oscillation of a physical system results from two basic properties of the system, namely, *elasticity* and *inertia*. Consider a body in equilibrium so that forces on it balance. Let us displace it from its position of equilibrium (by doing work on it, i.e. applying a force) by a distance x. When it is released a restoring force is called into play whose tendency is to 'restore' x to its original value which is zero, by imparting to it an appropriate negative velocity $\frac{dx}{dt}$. Inertia, on the other hand, tries to oppose any change in velocity. When the body reaches its equilibrium position ($x = 0$), the negative velocity is maximum which produces a negative displacement. The body then overshoots its position of equilibrium. The restoring force now becomes positive (i.e. it helps increase x) and must now overcome the inertia of the negative velocity. Thus, the velocity keeps on decreasing and ultimately becomes zero when x is negative. This process of the restoring force trying to bring x to zero by imparting a velocity and inertia preserving the velocity and making x overshoot, repeats itself and the body oscillates.

12.3 SIMPLE HARMONIC MOTION

We shall first discuss the general dynamics of an SHM and later analyse a few specific examples of it. We know that work has to be done on a system to displace it from its position of equilibrium. The restoring force F obviously depends on the work done to give a displacement x. Thus, F is some general function of x. For systems oscillating violently (large x) the dependence of F on x is very complex. We shall, however, not deal with such systems but focus our attention on systems in which the moving part always stays close to its mean position (small x). This is called *small oscillation approximation*. For such systems, the restoring force is proportional to the displacement and opposes it increase. In other words,

$$F = -Kx \tag{12.1}$$

The negative sign indicates that F opposes increase in x. K, the constant of proportionality, is called the force constant. The MKS unit for K is Nm^{-1}. The magnitude of K depends on the elastic properties of the system under study. In the specific examples of SHM we shall compute the value of K for each case. For example, in the system consisting of a mass and a spring, K will depend on the stiffness or strength of the spring. Relation (12.1) is a statement of Hooke's law for elastic forces. The general definition of *SHM is the motion in which the restoring force is proportional to the displacement from the mean position and opposes its increase*. We shall see that in such a motion the displacement varies harmonically with time.

Equation of Motion

Under the influence of a restoring force $F = (-Kx)$ a body acquires a velocity $\frac{dx}{dt}$ and hence an acceleration $\frac{d^2x}{dt^2}$. If m is the mass of the body, then from Newton's law (Force = mass × acceleration), the acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{\text{force}}{\text{mass}} = -\frac{K}{m}x \qquad (12.2)$$

The physical statement corresponding to this equation is that the acceleration is proportional (and opposite) to the displacement. In order to determine what type of motion is represented by Eq. (12.2) we need to solve this differential equation, i.e. obtain an expression of displacement x as a function of time.

Solution The usual procedure to solve any differential equation is to guess a solution and see if it works. Equation (12.2) relates a function $x(t)$ (not yet known) to its second derivative $\frac{d^2x}{dt^2}$. To satisfy Eq. (12.2) we have to look for a function $x(t)$ whose second derivative, except for a negative constant factor $\left(-\frac{K}{m}\right)$, is the same as the function $x(t)$ itself. Our knowledge of calculus tells us that sine and cosine functions have just this property, since,

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and $$\frac{d^2}{d\theta^2}(\sin\theta) = \frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

and $$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

and $$\frac{d^2}{d\theta^2}(\cos\theta) = \frac{d}{d\theta}(-\sin\theta) = -\cos\theta$$

One can immediately verify that functions $a\sin\theta$ and $b\cos\theta$, where a and b are constants, also obey this property. Since angle θ, measured in radians, must depend on time t, we set $\theta = \omega t$ where ω is a constant to be measured in radians/second.

Thus, let us try, as a solution of Eq. (12.2)

$$x(t) = a\sin\omega t$$

Differentiating twice with respect to t we get,

$$\frac{d^2x}{dt^2} = -a\omega^2\sin\omega t$$

Substitution in Eq. (12.2) gives

$$-a\omega^2\sin\omega t = -\frac{K}{m}a\sin\omega t$$

Therefore, if we choose the constant such that

$$\omega = \sqrt{\frac{K}{m}} \tag{12.3}$$

then $x(t) = a \sin \omega t$, is indeed, a solution of Eq. (12.2). We can similarly verify that $x(t) = b \cos \omega t$ is also a solution of Eq. (12.2). We can go a step further and say that

$$x(t) = a \sin \omega t + b \cos \omega t \tag{12.4}$$

is the general solution of Eq. (12.2) because

$$\frac{dx}{dt} = a\,\omega \cos \omega t - b\,\omega \sin \omega t$$

and $$\frac{d^2x}{dt^2} = -\,\omega^2 (a \sin \omega t + b \cos \omega t) = -\,\omega^2 x(t)$$

which is indeed Eq. (12.2) with the constant ω given by Eq. (12.3). The solution (12.4) has been written in various alternative forms in physics literature. We shall now write it in the more conventional form. Instead of writing $x(t)$ in terms of constants a and b we will write it in terms of two other constants A and ϕ which are related to a and b as

$$a = A \cos \phi$$

and $$b = A \sin \phi$$

which give $A = (a^2 + b^2)^{1/2}$ and $\tan \phi = \frac{b}{a}$. In terms of constants A and ϕ, Eq. (12.4) becomes

$$x(t) = A \sin \omega t \cos \phi + A \cos \omega t \sin \phi$$

or $$x(t) = A \sin(\omega t + \phi) \tag{12.5}$$

Some authors prefer to write the solution as

$$x(t) = A \cos(\omega t + \delta) \tag{12.6}$$

which is equivalent to Eq. (12.5) since, if constant $\phi = \delta + \frac{\pi}{2}$, the two expressions become identical.

You will recognize that Eqs (12.4), (12.5) and (12.6) are completely equivalent (but alternative) ways of writing the same expression. Any one of them can be used to describe a simple harmonic motion. We will be using Eq. (12.5) as a mathematical representation of SHM.

Meaning of ω

The *smallest* time interval after which a motion repeats itself is called the *time period* (or simply *period*) of the motion. It is usually denoted by the symbol T. If the time t in Eq. (12.5) is increased to a value $t^* = t + \frac{2\pi}{\omega}$, the function becomes

$$\begin{aligned} x(t^*) &= A\sin(\omega t^* + \phi) \\ &= A\sin\left\{\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right\} \\ &= A\sin(\omega t + \phi + 2\pi) \\ &= A\sin(\omega t + \phi) \\ &= x(t) \end{aligned}$$

i.e. the value of the function at time $t^*\left(= t + \frac{2\pi}{\omega}\right)$ is equal to the value the function had at an earlier time t. In other words, the function repeats itself after a time interval $2\pi/\omega$. Therefore the period T is given by

$$T = \frac{2\pi}{\omega}$$

or

$$\omega = \frac{2\pi}{T} \tag{12.7}$$

Thus we find that the constant ω is related to the period (T) of motion by Eq. (12.7). We would obtain the same result if we had used the cosine function [Eq. (12.6)] since

$$\begin{aligned} x(t^*) &= A\cos\left\{\omega\left(t + \frac{2\pi}{\omega}\right) + \delta\right\} \\ &= A\cos(\omega t + \delta + 2\pi) \\ &= A\cos(\omega t + \delta) \\ &= x(t) \end{aligned}$$

This shows that functions like sin ωt and cos ωt are periodic having a period $T = 2\pi/\omega$. The constants ϕ and δ do not depend upon time and hence do not affect the periodicity. In terms of period T, these functions are $\sin\left(\frac{2\pi t}{T}\right)$ and $\cos\left(\frac{2\pi t}{T}\right)$.

Notice that these functions repeat also at times intervals which are multiples of T, i.e. at intervals $2T$, $3T$, . . . etc. But T is the smallest time interval after which these functions repeat themselves, i.e. the period of these functions is T.

12.4 PERIODIC AND HARMONIC FUNCTIONS

Before we proceed with the study of SHM, it is useful to learn more about periodic functions and to distinguish them from harmonic functions. We have seen above that functions such as

$$f(t) = \sin \omega t = \sin \frac{2\pi t}{T}$$

and
$$g(t) = \cos \omega t = \cos \frac{2\pi t}{T}$$

are periodic functions having a period T. Figure 12.1 shows how these functions vary with time.

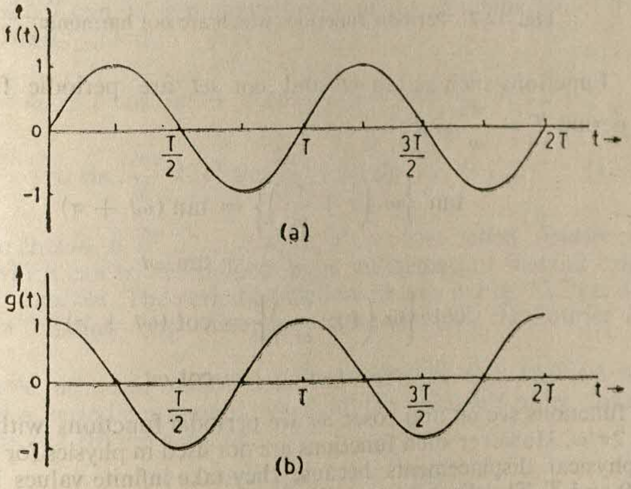


Fig. 12.1 (a) Graph of $f(t)$ against t (b) Graph of $g(t)$ against t

Notice that these functions oscillate between a maximum value + 1 and minimum value − 1 passing through zero in between. *Functions which are represented by a sine or a cosine curve are called simple harmonic (or sinusoidal) functions.*

All harmonic functions are also periodic functions. But the converse is not necessarily true. *All periodic functions are not harmonic functions.* Figure 12.2 shows some periodic functions which are not harmonic. They repeat themselves in a period T but cannot be represented by a single sine or cosine function, such as those shown in Fig. 12.2.

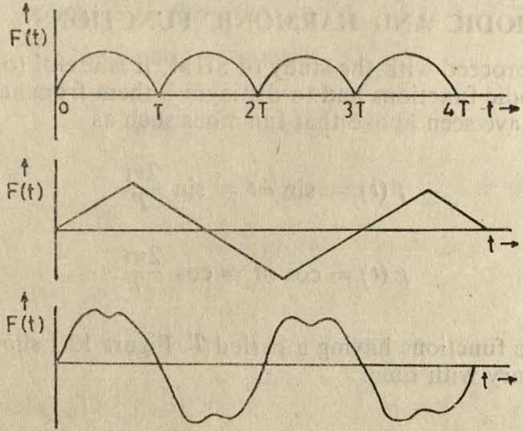


Fig. 12.2 Periodic functions which are not harmonic

Note Functions such as tan ωt and cot ωt are periodic functions with period $T = \frac{\pi}{\omega}$ since

$$\tan\left\{\omega\left(t + \frac{\pi}{\omega}\right)\right\} = \tan(\omega t + \pi)$$

$$= \tan \omega t$$

and
$$\cot\left\{\omega\left(t + \frac{\pi}{\omega}\right)\right\} = \cot(\omega t + \pi)$$

$$= \cot \omega t$$

Also functions sec ωt and cosec ωt are periodic functions with period $T = 2\pi/\omega$. However such functions are not used in physics for describing physical displacements because they take infinite values between $t = 0$ and T. The displacement function describing a physical phenomenon must always remain finite.

In higher classes you will learn that any periodic (non-harmonic) function such as those shown in Fig. 12.2 can be constructed from two infinite sets of periodic harmonic functions. These sets are

$$f_1(t) = 0 + \sin\frac{2\pi t}{T} + \sin\frac{4\pi t}{T} + \ldots \tag{12.8}$$

and
$$f_2(t) = 1 + \cos\frac{2\pi t}{T} + \cos\frac{4\pi t}{T} + \ldots \tag{12.9}$$

The functions $f_1(t)$ and $f_2(t)$ are periodic with period T, i.e.

$$f_1(t + T) = f_1(t)$$

$$f_2(t + T) = f_2(t)$$

This can be understood as follows: In Eq. (12.8) the harmonic terms $\sin\frac{2\pi t}{T}$, $\sin\frac{4\pi t}{T}$, $\sin\frac{6\pi t}{T}$, . . . etc. have periods T, $\frac{T}{2}$, $\frac{T}{3}$, . . . respectively. This means that in time T, the term $\sin\frac{2\pi t}{T}$ completes one cycle, the term $\sin\frac{4\pi t}{T}$ completes two cycles, the term $\frac{6\pi t}{T}$ completes three cycles and so on. Thus at time $(t + T)$ the function f_1 is an exact repetition of its value at time t, i.e. the function $f_1(t)$ is periodic with period T. Similarly the function $f_2(t)$ given by Eq. (12.9) is also periodic with period T; the constant function 1 in this equation is periodic for any value of T and hence will not alter the periodicity of $f_2(t)$.

J.B.J. Fourier (1768-1830), the great French mathematician, proved an important theorem which states that any periodic function $F(t)$ with periodic T can be represented by a unique combination of functions $f_1(t)$ and $f_2(t)$ as

$$F(t) = b_0 + b\cos\frac{2\pi t}{T} + b'\cos\frac{4\pi t}{T} + b''\cos\frac{6\pi t}{T} + \ldots$$

$$+ a\sin\frac{2\pi t}{T} + a'\sin\frac{4\pi t}{T} + a''\sin\frac{6\pi t}{T} + \ldots \qquad (12.10)$$

The coefficients $b_0, b, b', b'', \ldots, a, a', a''$ . . . are called *Fourier coefficients* which can be determined by a mathematical method called the *Fourier analysis*. The periodic functions shown in Fig. 12.2 are described by a function $F(t)$ with appropriate values of the Fourier coefficients.

The simple harmonic motion described above in Sec. 12.3 is a special case of a periodic motion in which all the constants other than a and b in Eq. (12.10) are zero, i.e.

$$F(t) = a\sin\frac{2\pi t}{T} + b\cos\frac{2\pi t}{T}$$

$$= a\sin\omega t + b\cos\omega t$$

which is the displacement function $x(t)$ given by Eq. (12.4) obtained earlier.

EXAMPLE 12.1 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic and (c) non-periodic motion? Find the period for each periodic motion. Here k is a positive real constant.

(i) $\sin kt + \cos kt$, (ii) $\sin\pi t + 2\cos 2\pi t + 3\sin 3\pi t$, (iii) $\cos\left(2kt + \frac{\pi}{3}\right)$, (iv) $\cos kt + 2\sin^2 kt$, (v) e^{-kt} and (vi) $\log kt$.

Solution (i) $\sin kt + \cos kt$ represents simple harmonic motion since we can write it as $\sqrt{2}\sin\left(kt + \frac{\pi}{4}\right)$ or $\sqrt{2}\cos\left(kt - \frac{\pi}{4}\right)$. The coefficient of t in the argument of the sine or cosine function $= \frac{2\pi}{T}$, where T is the period. Hence $k = \frac{2\pi}{T}$ or $T = \frac{2\pi}{k}$, i.e. the period of the function is $\frac{2\pi}{k}$.

(ii) Each term in $(\sin \pi t + 2\cos 2\pi t + 3\sin 3\pi t)$ represents SHM. The period T of the term $\sin \pi t$ is given by $\pi = \frac{2\pi}{T}$ or $T = 2$ s. The period of the term $2\cos 2\pi t$ is 1 s, i.e. $T/2$ and the period of the term $3\sin 3\pi t$ is 2/3 s, i.e. $T/3$. The sum of the three terms, however, does not represent SHM; it represents a periodic motion. By the time the first term completes one cycle, the second term completes two cycles and the third term completes three cycles. This shows that the sum represents a periodic motion with period $T = 2$ s.

(iii) $\cos\left(2kt + \frac{\pi}{3}\right)$ represents an SHM whose period T is given by

$$2k = \frac{2\pi}{T}$$

or

$$T = \frac{\pi}{k}$$

(iv) $\cos kt + 2\sin^2 kt = \cos kt + (1 - \cos 2kt) = 1 + \cos kt - \cos 2kt$. The period of $\cos kt$ is $T = \frac{2\pi}{k}$ and that of $\cos 2kt$ is $\frac{\pi}{k} = \frac{T}{2}$. These two terms together have a period $T = \frac{2\pi}{k}$ as explained in (ii). The other term 1 is a constant independent of t and hence does not affect the period of the sum. Hence $(\cos kt + 2\sin^2 kt)$ represents a periodic motion with period $T = \frac{2\pi}{k}$.

(v) e^{-kt} decreases monotonically to zero at $t \to \infty$. It is an exponential function with a negative exponent of e where $e \simeq 2.71828$. It is non-periodic.

(vi) The function $\log kt$ also increases monotonically with time. Therefore, like e^{kt}, it never repeats itself and is a non-periodic function.

12.5 CHARACTERISTICS OF AN SHM

We have seen that Eq. (12.5) is a solution of Eq. (12.2) with

$$\omega = \sqrt{\frac{K}{m}}$$

Since $x(t)$ given by Eq. (12.5) varies harmonically with time, the motion described by Eq. (12.5) is said to be SHM and Eq. (12.2) is called the differential equation of SHM. The three constants A, ω and ϕ characterize an SHM, i.e. they distinguish one SHM from another.

The three characteristics of SHM are:

Amplitude

The amplitude of an SHM is the maximum (positive or negative) value of the displacement from the mean position. Since the maximum and minimum values of any sine function are respectively $+1$ and -1, the maximum and minimum values of x in Eq. (12.5) are respectively $+A$ and $-A$. The quantity A is called the amplitude of SHM.

Time Period

The smallest time interval during which the oscillation repeates itself is called the time period (or simply, *period*) *of the oscillation.* If the time t in Eq. (12.5) is increased by $\frac{2\pi}{\omega}$, the function becomes

$$
\begin{aligned}
x(t) &= A \sin \left\{\omega \left(t + \frac{2\pi}{\omega}\right) + \phi\right\} \\
&= A \sin(\omega t + \phi + 2\pi) \\
&= A \sin(\omega t + \phi)
\end{aligned}
$$

In other words, the function repeats itself after a time $\frac{2\pi}{\omega}$. Therefore the period T is given by

$$T = \sqrt{\frac{2\pi}{\omega}} = 2\pi \sqrt{\frac{m}{K}} \tag{12.11}$$

The frequency of SHM, denoted by ν, *is the number of oscillations completed in a unit time.* Therefore, by definition, frequency is the reciprocal of time period, i.e.

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}} \tag{12.12}$$

Thus

$$\omega = 2\pi\nu = \frac{2\pi}{T} \tag{12.13}$$

The quantity ω is called the angular frequency which is 2π times the frequency. The unit for T in the MKS system is s (second), ν is measured in cps (cycles per second) or Hz (hertz) and ω in radians/second. The period (or frequency) is determined from the elastic and inertial properties of the system, the force constant K and the mass m.

Phase

The argument $(\omega t + \phi)$ of the sine function is called the *phase* of the motion. The constant ϕ is called the *initial phase* (i.e. phase at $t = 0$) or the phase constant. The knowledge of the phase constant ϕ enables us to find out how far from the mean position the oscillator was at time $t = 0$. For example, if

$$\phi = 0 \quad \text{then} \quad x(t) = A \sin \omega t$$

which means that the displacement was zero at time $t = 0$. When

$$\phi = \frac{\pi}{2},\ x(t) = A \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$

i.e. the displacement was maximum ($= A$) at time $t = 0$. Thus, *phase constant ϕ is a measure of how far the oscillator is away from its mean position at time $t = 0$*, or alternatively, *how much time elapsed before the oscillator last passed the mean position.* Amplitude A and phase constant ϕ are determined by the way the system is started at time $t = 0$. In Secs 12.6 and 12.10 the meaning of ω and ϕ will be made more clear.

Displacement-Time Curve of SHM

Knowing the three characteristics A, T and ϕ of an SHM we can plot $x(t)$ in Eq. (12.5) versus t. This plot gives the displacement-time curve of the SHM. In Fig. 12.3, displacement curves of some harmonic and nonhormonic oscillations are shown.

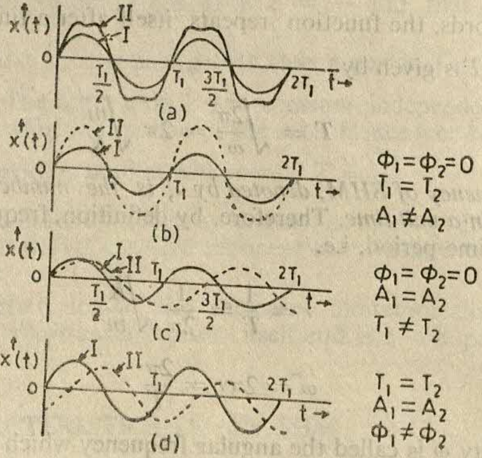


Fig. 12.3 Displacement-time curves

In Fig. 12.3a displacement curves I and II both represent periodic motions. Motion I is a harmonic oscillation but motion II is non-harmonic, as it does not correspond to Eq. (12.5) for harmonic oscillation which should give a sine curve. In Fig. 12.3b two harmonic oscillations I and II have equal phases and time periods but different amplitudes. In Fig. 12.3c two harmonic oscillations have equal phases and amplitudes but different time periods and in Fig. 12.3d two harmonic oscillations I and II have equal amplitudes and time periods but different phases.

12.6 RELATION BETWEEN LINEAR SHM AND UNIFORM CIRCULAR MOTION (THE REFERENCE CIRCLE)

The expression Fig. (12.5) for the displacement in SHM obtained in the last section has a very simple interpretation in terms of a relation between a hypothetical SHM along a line and uniform motion in a circle. This relation also gives a simple geometric meaning to the quantities ω and ϕ.

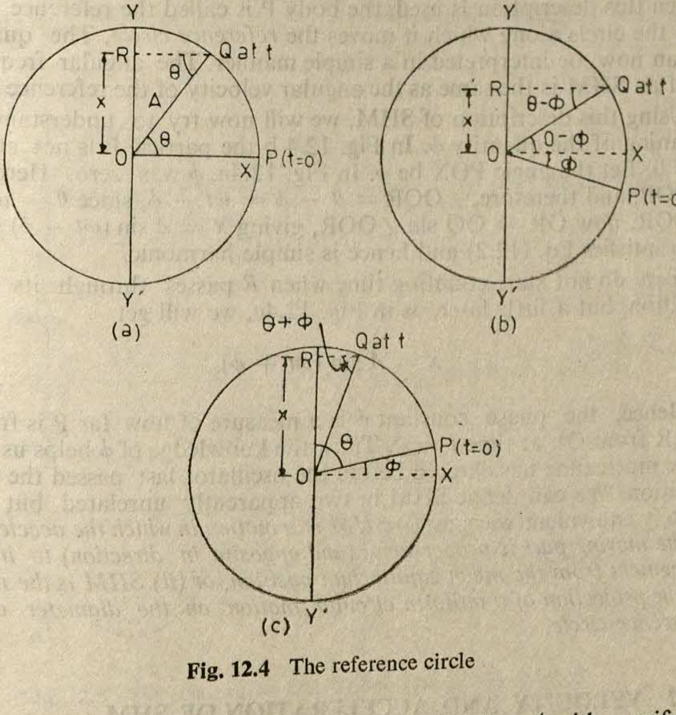


Fig. 12.4 The reference circle

Consider a particle P moving on a circle of radius A with a uniform velocity (Fig. 12.4). Let T be the time period of this circular motion and let Q be the position of the particle at an instant of time t (Fig.

12.4a). At time $t = 0$ the particle is at P. R is the foot of the perpendicular drawn from Q on the diameter YY′ of the circle. Let us study the motion of R as the particle P moves around the circle. At time $t = 0$ the point R is at O ($x = 0$) which we call the mean position At time $t = \frac{T}{4}$, R will be at Y, at $t = \frac{T}{2}$, R reaches O, at $t = \frac{3T}{4}$, R will reach Y′, and at $t = T$, R has come back to O, having completed one full oscillation. Thus, as P moves once round the circle, the foot of the perpendicular (or the projection of the circular motion on the diameter of the circle) moves along a straight line from O to Y, back to O, from O to Y′ and back to O again. During the time t when the particle moves from P to Q along the circle, the radius OP sweeps an angle θ. The angular velocity of the circular motion is ω = angle swept/time taken $= \frac{\theta}{t} = \frac{2\pi}{T}$ (since, when $t = T$, $\theta = 2\pi$) or $\theta = \omega t$. Now, in Δ OQR, OR = OQ sin $\angle$OQR or, $x = A \sin \omega t$, which agrees with Eq. (12.5) of the last section with $\phi = 0$. Therefore, the motion of R is simple harmonic. Thus, *SHM can be described as the projection of a uniform circular motion on the diameter of the circle.* When this description is used, the body P is called the reference body and the circle along which it moves the *reference circle*. The quantity ω can now be interpreted in a simple manner. The angular frequency ω of an SHM is the same as the angular velocity of the reference body.

Using this description of SHM, we will now try to understand the meaning of the quantity ϕ. In Fig. 12.4 b the particle P is not at X at $t = 0$. Let the angle POX be ϕ. In Fig. 12.4a, ϕ was zero. Here $\theta = \angle$QOP, and therefore, $\angle$OQR $= \theta - \phi = \omega t - \phi$, since $\theta = \omega t$. In $\angle$OQR, now OR = OQ sin $\angle$OQR, giving $x = A \sin (\omega t - \phi)$ which also satisfies Eq. (12.2) and hence is simple harmonic.

If we do not start counting time when R passes through its mean position, but a little later, as in Fig. 12.4c, we will get

$$x = A \sin (\omega t + \phi).$$

Hence, the phase constant ϕ is a measure of how far P is from X (or R from O), at time $t = 0$. Thus, the knowledge of ϕ helps us know how much time has elapsed before the oscillator last passed the mean position. We can define SHM in two apparently unrelated but completely equivalent ways as: (i) *SHM is a motion in which the acceleration of the moving part is proportional* (*and opposite in direction*) *to its displacement from the mean equilibrium position*, *or* (*ii*) *SHM is the motion of the projection of a uniform circular motion on the diameter of the reference circle.*

12.7 VELOCITY AND ACCELERATION OF SHM

It is instructive to learn how velocity and acceleration in an SHM vary with time. We know that displacement $x(t)$ is given by

$$x(t) = A \sin(\omega t + \phi)$$

The velocity V and acceleration a are given by

$$V = \frac{dx}{dt} = A\,\omega \cos(\omega t + \phi) = \pm A\omega \left(1 - \frac{x^2}{A^2}\right)^{1/2} \quad (12.14)$$

since $$\cos^2(\omega t + \phi) = 1 - \sin^2(\omega t + \phi)$$

and $$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= -A\,\omega^2 \sin(\omega t + \phi) = -\omega^2 x \quad (12.15)$$

We notice that when the displacement is maximum ($+A$ or $-A$) the velocity $V = 0$, because now the oscillator has to return and the velocity must change its direction. But when x is maximum ($+A$ or $-A$) the acceleration is also maximum ($-\omega^2 A$ and $+\omega^2 A$ respectively) and is directed opposite to the displacement. When $x = 0$, i.e. when $\sin(\omega t + \phi) = 0$, velocity V is maximum ($A\omega$ or $-A\omega$) and acceleration is zero. In Fig. 12.5 we have plotted separately the x versus t and the a versus t curves for an SHM, taking, for simplicity, $\phi = 0$.

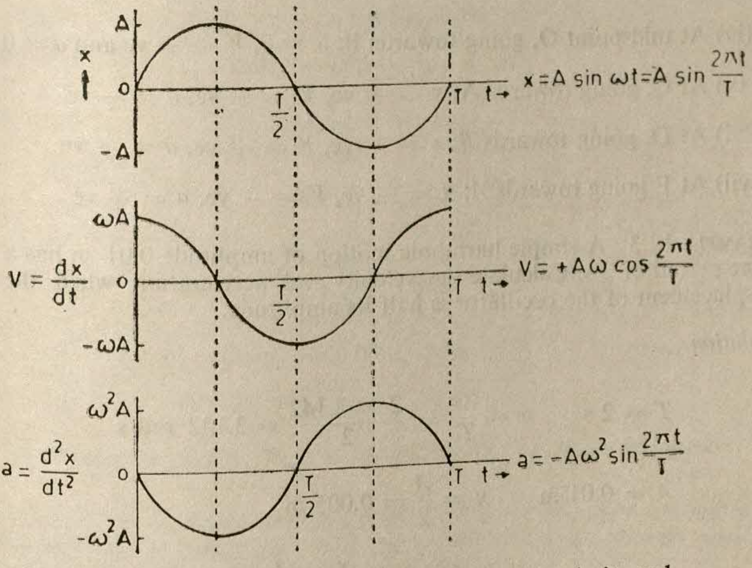


Fig. 12.5 The relation between displacement, velocity and acceleration in SHM.

Example 12.2 A particle is in linear simple harmonic motion between two extreme points A and B, 10 cm apart. (a) What is the amplitude of the motion? (b) If the direction from A to B is taken as the positive direction, what are the signs of displacement, velocity and acceleration of the particle when it is (i) at A, (ii) at B, (iii) at the midpoint of AB going towards A, (iv) at the midpoint of AB going towards B, (v) at 2.5 cm away from B going towards A, (vi) at 3 cm from A going towards B and (vii) at 4 cm from A going towards A?

Solution (a) Referring to Fig. 12.6 the amplitude of the motion = 5 cm. The mean position is O.

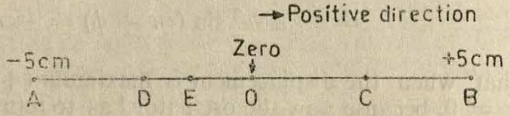


Fig. 12.6

(b) (i) At A; $x = -$ ve, $V = 0$ and $a = +$ ve

(ii) At B; $x = +$ ve, $V = 0$ and $a = -$ ve

(iii) At midpoint O, which is the mean position, going towards A;

$x = 0$, $V = -$ ve and $a = 0$

(iv) At mid-point O, going towards B; $x = 0$, $V = +$ ve and $a = 0$

(v) At C, going towards A; $x = +$ ve, $V = -$ ve, $a = -$ ve

(vi) At D, going towards *B*; $x = -$ ve, $V = +$ ve, $a = +$ ve

(vii) At E going towards *A*; $x = -$ ve, $V = -$ ve, $a = +$ ve

Example 12.3 A simple harmonic motion of amplitude 0.01 m has a time period of 2 s. Calculate the velocity and acceleration when the displacement of the oscillator is half its amplitude.

Solution

$$T = 2\text{ s} \qquad \omega = \frac{2\pi}{T} = \frac{2 \times 3.142}{2} = 3.142 \text{ rad/s}$$

$$A = 0.01\text{ m} \qquad x = \frac{A}{2} = 0.005\text{ m}$$

$$\therefore \qquad \sin(\omega t + \phi) = \frac{x}{A} = \frac{1}{2}$$

$$\text{Velocity } V = A\omega \cos(\omega t + \phi) = A\omega \{1 - \sin^2(\omega t + \phi)\}^{1/2}$$
$$= 0.01 \times 3.142 \times (1 - \tfrac{1}{4})^{1/2} = 0.0272 \text{ ms}^{-1}$$

$$\text{Acceleration } a = -A\omega^2 \sin(\sin(\omega t + \phi) = -0.01 \times (3.142)^2 \times \tfrac{1}{2}$$
$$= -0.0493 \text{ ms}^{-2}$$

EXAMPLE 12.4 A boy oscillates harmonically with amplitude 0.05 m. At a certain instant of time its displacement is 0.01 m and acceleration 1.0 m s^{-2}. Calculate (a) the period of oscillation; (b) its velocity at this instant; and (c) the maximum velocity.

Solution (a) $x = 0.01$ m $\quad A = 0.05$ m.

Acceleration $\frac{d^2x}{dt^2} = -1.0 \text{ ms}^{-2}$. (in SHM acceleration is directed opposite to displacement.)

$$\text{Now} \qquad \omega^2 = -\frac{d^2x}{dt^2}\bigg| x = \frac{1.0}{0.01} = 100 \text{ or } \omega = 10 \text{ rad/s}$$

$$\therefore \qquad T = \frac{2\pi}{\omega} = \frac{2 \times 3.142}{10} = 0.628 \text{ s}$$

$$\text{(b)} \qquad x = A \sin(\omega t + \phi)$$

$$\text{or} \qquad 0.01 = 0.05 \sin(\omega t + \phi)$$

$$\text{or} \qquad \sin(\omega t + \phi) = \frac{1}{5}$$

$$\therefore \qquad \cos(\omega t + \phi) = \sqrt{\frac{24}{25}}$$

$$\text{Velocity } \quad V = \frac{dx}{dt} = A\omega(\cos\omega t + \phi)$$
$$= 0.05 \times 10 \times \sqrt{\frac{24}{25}}$$
$$= 0.49 \text{ ms}^{-1}$$

(c) Maximum velocity $V_{\text{max}} = +A\omega = 0.05 \times 10 = 0.5 \text{ ms}^{-1}$

EXAMPLE 12.5 A body oscillates with SHM of amplitude 4 cm and a frequency of 5 Hz. At time $t = 0$, the body is at its equilibrium position ($x = 0$). Obtain the equation of the SHM.

Solution Let the equation of the SHM be

$$x = A \sin(\omega t + \phi) \tag{i}$$

where amplitude $A = 4$ cm and frequency $\nu = \frac{2\pi}{\omega} = 5$ Hz. Hence the angular frequency is $\omega = 2\pi\nu = 2\pi \times 5 = 10\pi$ rad s^{-1}.

Now at $t = 0$, $x = 0$. Substituting in Eq. (i) we have

$$0 = A \sin \phi \quad \text{giving } \phi = 0. \tag{ii}$$

Using (ii) in (i) we have

$$x = A \sin (\omega t + 0)$$

$$= A \sin \omega t$$

or

$$x = 4 \sin 10\pi t$$

This is the equation of the SHM in which x is in centimetres and t in seconds

Note: We could equally well consider a cosine function to describe the SHM as

$$x = A \cos (\omega t + \delta)$$

Setting $x = 0$ at $t = 0$ in this equation we have

$$0 = A \cos \delta \text{ giving } \delta = \frac{\pi}{2} \text{ so that}$$

$$x = A \cos \left(\omega t + \frac{\pi}{2} \right)$$

$$= A \sin \omega t$$

or

$$x = 4 \sin 10 \pi t$$

which is the same as the equation obtained above with a sine function.

EXAMPLE 12.6 The displacement x (in centimetres) of an oscillating particle varies with time t (in seconds) according to the equation

$$x = 2 \cos \left(0.5\pi t + \frac{\pi}{3} \right)$$

Find

(a) amplitude of oscillation
(b) the time period of oscillation
(c) the maximum velocity of the particle
(d) the maximum acceleration of the particle.

Solution The displacement of the particle is given by

$$x = 2 \cos \left(0.5\pi t + \frac{\pi}{3} \right) \text{ cm}$$

To find the amplitude and time period of the oscillation, we compare this equation with

$$x = A\cos(\omega t + \delta)$$

(a) Amplitude $A = 2$ cm

(b) Angular frequency $\omega = 0.5\pi$ rad s^{-1}

or
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.5\pi} = 4 \text{ s}$$

(c) Maximum velocity $V_{\text{max}} = |A\omega|$

$$= 2 \times 0.5\pi = \pi \text{ cm s}^{-1}$$
$$= 3.14 \text{ cm s}^{-1}$$

(d) Maximum acceleration $a_{\text{max}} = |-\omega^2 A|$

$$= \omega^2 A$$
$$= (0.5\pi)^2 \times 2$$
$$= \frac{\pi^2}{2} \text{ cm s}^{-2} = 4.935 \text{ cm s}^{-2}$$

Example 12.7 A particle executes SHM of amplitude 25 cm and time period 3 s. What is the minimum time required for the particle to move between two points located at 12.5 cm on either side of the mean position?

Solution

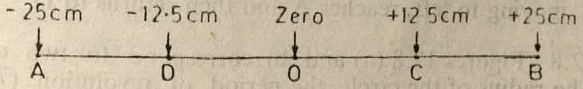


Fig. 12.7

Let A and B be the two extreme positions of the particle with O as the mean position. Displacements to the right of O are taken as positive while those to the left of O are taken as negative (Fig. 12.7).

Let the displacement of the particle in SHM be given by

$$x(t) = A\sin(\omega t + \phi) \qquad \text{(i)}$$

where $A = 25$ cm and $\omega = \frac{2\pi}{T} = \frac{2\pi}{3}$ rad s^{-1}

Let us suppose that at time $t = 0$, the particle is at extreme position B.

Setting $x = A$ at $t = 0$ in Eq. (i) we have

$$A = A\sin\phi$$

giving $\phi = \pi/2$.

Putting $\phi = \pi/2$ in Eq. (i), we get

$$x(t) = A \cos \omega t \qquad \text{(ii)}$$

where $A = 25$ cm.

Now let us say that the particle reaches point C at $t = t_1$ and point D at $t = t_2$. At C, the displacement $x(t_1) = +\ 12.5$ cm and D, it is $x(t_2) = -\ 12.5$ cm (see Fig. 12.7). So from (ii) we have

$$+\ 12.5 = 25 \cos \omega t_1$$

and $$-\ 12.5 = 25 \cos \omega t_2$$

or $$\cos \omega t_1 = +\ 0.5 \quad \text{or} \quad \omega t_1 = \pi/3$$

and $$\cos \omega t_2 = -\ 0.5 \quad \text{or} \quad \omega t_2 = \frac{2\pi}{3}$$

Hence $$\omega(t_2 - t_1) = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

$\therefore$ $$t_2 - t_1 = \frac{\pi}{3\omega} = \frac{T}{6} \qquad \left(\because \quad \omega = \frac{2\pi}{T}\right)$$

or $$(t_2 - t_1)_{\min} = \frac{3}{6} = 0.5 \text{ s}$$

Notice that $\cos \omega t_2 = -\ 0.5$ even for $t_2 = \frac{4\pi}{3}$. This value of t_2 does not correspond to the minimum time because this is the time at which the particle, moving to left, reaches A and then returns to D.

EXAMPLE 12.8 Figures 12.8 (a) and (b) correspond to two circular motions. The radius of the circle, the period of revolution (T), the

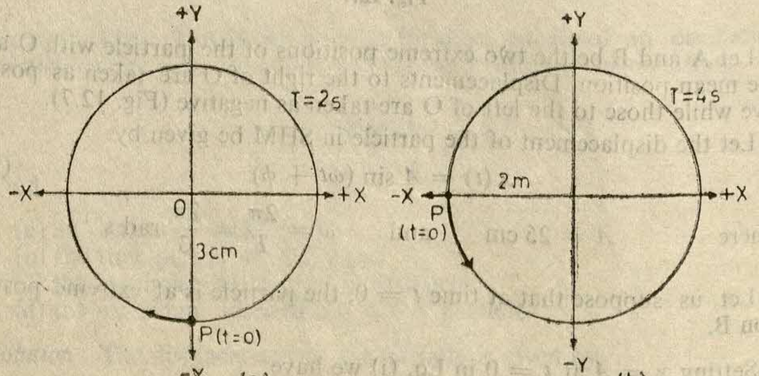


Fig. 12.8

initial position of particle P ($t = 0$) and the sense of revolution (clockwise or anticlockwise) are indicated on each diagram. Obtain the corresponding simple harmonic motions of the x-projections of the radius vector of the revolving particle P in each case.

Solution

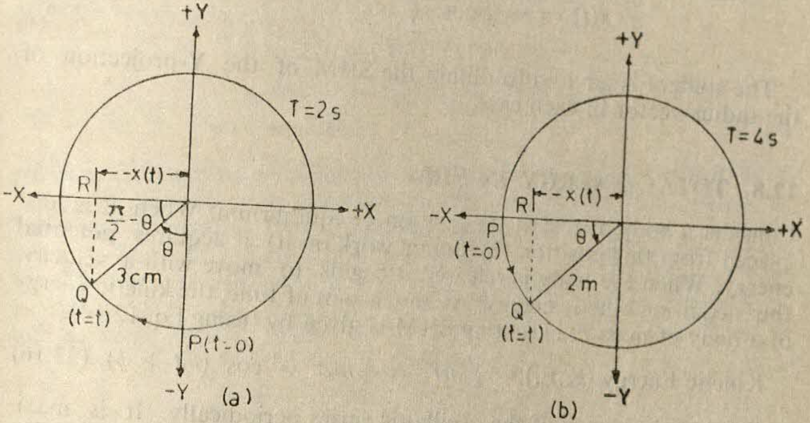


Fig. 12.9

(a) Let us say that the particle moves in the clockwise sense from P to Q in time t, the radius vector sweeping an angle $\theta = \omega t = \frac{2\pi t}{T} = \pi t$ radians ($\because$ $T = 2$ s) as shown in Fig. 12.9(a). R is the foot of the perpendicular drawn from Q on the X-axis and displacement OR $= - x(t)$, being to the left of origin O. In ΔOQR we have

$$\text{OR} = \text{OQ} \cos\left(\frac{\pi}{2} - \theta\right) = \text{OQ} \sin\theta$$

or

$$- x(t) = 3 \sin \pi t$$

$\because$

$$\text{radius OQ} = 3 \text{ cm and } \omega = \pi t$$

Here $x(t)$ is measured in centimetres. Hence the SHM is given by

$$x(t) = - 3 \sin \pi t$$

(b) In this case the particle moves in the anticlockwise sense from P to Q in time t, the radius vector sweeping an angle $\theta = \omega t = \frac{2\pi t}{T} = \frac{\pi t}{2}$ radians ($\because$ $T = 4$ s) as shown in Fig. 12.9 (b). R is the foot of the perpendicular drawn from Q on the X-axis and the displacement OR $= - x(t)$. In OQR we have

$$\text{OR} = \text{OQ} \cos \omega t$$

or $$-x(t) = 2\cos\frac{\pi t}{2} \left(\because \quad OQ = 2 \text{ m and } \omega = \frac{\pi t}{2}\right)$$

Here $x(t)$ is measured in metres. Hence the SHM is given by

$$x(t) = -2\cos\frac{\pi t}{2}$$

The student is advised to obtain the SHM of the Y-projection of the radius vector in each case.

12.8 TOTAL ENERGY IN SHM

Consider a body at rest at its position of equilibrium. When it is displaced from this position (by doing work on it) it acquires potential energy. When the body is released it begins to move with a velocity, thus acquiring kinetic energy. At any instant of time, the kinetic energy of a body of mass m executing SHM is given by (using Eq. 12.14).

$$\text{Kinetic Energy (K.E.)} = \tfrac{1}{2}\, mV^2 = \tfrac{1}{2}\, mA^2\, \omega^2 \cos^2(\omega t + \phi) \quad (22.16)$$

The kinetic energy of the oscillator varies periodically. It is maximum ($= \frac{1}{2}\, mA^2\omega^2$) when the velocity is maximum ($= \pm\, \omega A$) and displacement is zero, when the displacement is maximum ($x = \pm\, A$), velocity $V = 0$ and K.E. $= 0$. At this extreme position, the energy is, all potential, for intermediate positions (x lying between 0 and $\pm\, A$) the energy is partly kinetic and partly potential.

A closer look at the equation of the displacement [Eq. (12.5)]

$$x = A\sin(\omega t + \phi)$$

in SHM, reveals that the total energy of the oscillator must remain constant, because the maximum displacement ($= \pm\, A$) is regained every half-cycle. If no energy is dissipated (we have neglected dissipative or non-conservative force like friction) then all the potential energy becomes kinetic and vice versa.

Let us now compute potential energy at any instant of time t. Let x be the displacement at time t. The potential energy is given by the amount of work required to move the body from $x = 0$ to x, by applying a force. This force must be just enough to oppose the restoring force $F = -Kx$. In other words, the force to be applied must be $+\, Kx$

Work required to give an infinitesimal displacement dx

$$= \text{force} \times dx$$
$$= Kx\, dx$$

Total work done to displace the body from 0 to x

$$= \int_0^x Kx dx = \tfrac{1}{2}\, Kx^2$$

Thus, Potential energy (PE) $= \frac{1}{2} KA^2 \sin^2 (\omega t + \phi)$ (12.17)

$$= \frac{1}{2} m\omega^2 A^2 \sin^2 (\omega t + \phi)$$

where we have used Eqs (12.3) and (12.5)

Equations (12.16) and (12.17) give the instantaneous (i.e. at any instant of time t) values of kinetic and potential energy. The total energy E in SHM is, therefore, given by

$$\begin{aligned} E &= \text{KE} + \text{PE} \\ &= \tfrac{1}{2} m\omega^2 A^2 \{\cos^2 (\omega t + \phi) + \sin^2 (\omega t + \phi)\} \\ &= \tfrac{1}{2} m\omega^2 A^2 \qquad (12.18) \end{aligned}$$

which is constant as we would expect. It is obvious that the maximum values of kinetic and potential energy are equal (both equal to $\frac{1}{2} m\omega^2 A^2$) indicating that the energy exchange is complete. It is interesting to learn (Fig. 12.10) how the kinetic and potential energy of the harmonic oscillator vary with time. In plotting the curves in Fig. 12.10, we have, for simplicity, set $\phi = 0$, i.e. the particle passes through its mean position at time $t = 0$.

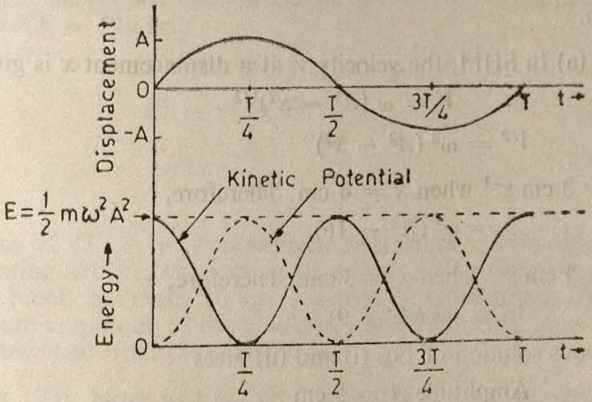


Fig. 12.10 Energy exchange in SHM

Note The energy of the oscillator may decrease not only due to friction in the system, but also due to radiation. The oscillating body imparts periodic motion to the particles of the medium in which it oscillates thus producing waves. For example, a tuning fork or a string produces sound waves in the medium which results in a decrease in energy.

It is instructive (and sometimes more convenient) to obtain the equation of SHM (Eq. 12.2) using energy considerations. The total energy of the oscillator is given by

$$E = \text{KE} + \text{PE}$$

$$E = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2} Kx^2 \qquad (12.19)$$

Since E is constant, $\frac{dE}{dt} = 0$. Differentiating Eq. (12.19) wrt time and setting

$$\frac{dE}{dt} = 0$$

we get

$$m \cdot \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + Kx \cdot \frac{dx}{dt} = 0$$

or
$$\frac{d^2x}{dt^2} = -\frac{K}{m}x$$

which is the Eq. (12.2) we have obtained earlier.

Example 12.9 A body executing linear SHM has a velocity of 3 cm s^{-1} when its displacement is 4 cm and a velocity of 4 cm s^{-1} when its displacement is 3 cm.

(a) Find the amplitude and period of the oscillation.

(b) If the mass of the body is 50 g, calculate the total energy of oscillation.

Solution (a) In SHM, the velocity V at a displacement x is given by

$$V = \omega (A^2 - x^2)^{1/2}$$

or
$$V^2 = \omega^2 (A^2 - x^2)$$

Now $V = 3$ cm s^{-1} when $x = 4$ cm. Therefore,

$$9 = \omega^2 (A^2 - 16) \qquad \text{(i)}$$

Also $V = 4$ cm s^{-1} when $x = 3$ cm. Therefore,

$$16 = \omega^2 (A^2 - 9) \qquad \text{(ii)}$$

Simultaneous solution of Eqs (i) and (ii) gives

Amplitude $A = 5$ cm

and angular frequency $\omega = 1$ rad s^{-1}

Hence Time period $T = \frac{2\pi}{\omega} = 2\pi$ s

$$\simeq 6.28 \text{ s}$$

(b)
$$m = 50 \text{ g} = 50 \times 10^{-3} \text{ kg}$$
$$A = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$$
$$\omega = 1 \text{ rad s}^{-1}$$
$$\text{Total energy} = \tfrac{1}{2} mA^2\omega^2$$
$$= \tfrac{1}{2} \times (50 \times 10^{-3}) \times (5 \times 10^{-2})^2 (1)^2$$
$$= 6.25 \times 10^{-5} \text{ J}$$

EXAMPLE 12.10 A particle is executing linear SHM of amplitude A. (a) What fraction of the total energy is kinetic when the displacement is half the amplitude? (b) At what displacement is the energy half kinetic and half potential?

Solution

$$\text{Kinetic energy (KE)} = \tfrac{1}{2}\, m\omega^2 (A^2 - x^2)$$
$$\text{Potential energy (PE)} = \tfrac{1}{2}\, m\omega^2 x^2$$
$$\text{Total energy } (E) = \tfrac{1}{2}\, m\omega^2 A^2$$

(a) When $x = A/2$,

$$\text{KE} = \tfrac{1}{2}\, m\,\omega^2 \left(A^2 - \frac{A^2}{4}\right) = \frac{3}{8}\, m\omega^2 A^2$$
$$E = \tfrac{1}{2}\, m\omega^2 A^2$$
$$\therefore \qquad \frac{\text{KE}}{E} = \frac{3}{4}$$

(b) The energy will be half kinetic and half potential at a value of x when KE = PE, i.e.

$$\tfrac{1}{2}\, m\,\omega^2 (A^2 - x^2) = \tfrac{1}{2}\, m\omega^2 x^2$$

or
$$A^2 - x^2 = x^2$$

or
$$x = \frac{A}{\sqrt{2}}$$

EXAMPLE 12.11 A horizontal platform (such as a piston in a machine) is executing SHM in the vertical direction with a frequency of 0.50 Hz. A block of mass 10 kg is placed on the platform. What is the maximum amplitude of the platform's SHM ensuring that the block is not detached from the platform?

Solution The block will not be detached from the platform, if the amplitude of the platform's SHM is such that, the maximum acceleration equals the acceleration due to gravity, i.e.

$$\omega^2 A_{\max} = g$$

or
$$A_{\max} = \frac{g}{\omega^2}$$
$$= \frac{g}{4\,\pi^2\,\nu^2}$$
$$= \frac{9.8}{4 \times (3.142)^2 \times (0.5)^2}$$
$$= 0.99 \text{ m}$$

Notice that $A_{\max}$ is independent of the mass of the block.

12.9 SOME EXAMPLES OF SHM

We will now analyse a few physical systems that oscillate with SHM. We will obtain the expression for the time period of oscillation in each case. It may be remarked that the variable x in SHM need not always be linear. It could represent angular displacement as in the case of a simple pendulum or charge as in the case of an electrical circuit involving a capacitance and an inductance. In torsional oscillations x is the twist, and in thermal oscillations the heat.

Mass-and-Spring System

When a force is applied to a spring to compress or stretch it, the resulting compression or elongation does not bear a simple relationship with the force applied. Figure 12.11 illustrates the force-displacement relationship for a spring. This relationship is, in general, not linear. Only for small displacements is the relationship linear (portion AB of the curve). The elastic force produced in the spring is given by $F = -kx$, where x is the change in the length of the spring when a force F is applied on it. The constant k is called the spring constant which is defined as the force required to cause a unit extension or compression in the spring. The unit of k in the MKS system is Nm^{-1}.

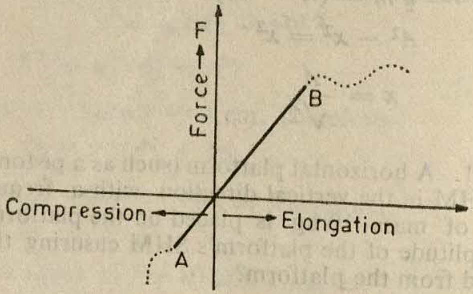


Fig. 12.11

Let us now analyse two simple systems of a mass oscillating on a spring.

Horizontal Oscillations Consider a massless spring of constant k, one end of which is fixed rigidly on a well while the other end is attached to a body of mass m, which is free to move on a frictionless horizontal surface (Fig. 12.12). Figure 12.12a is the position of static equilibrium, the spring being relaxed and no force acting on it. When the body is pulled to the right (Fig. 12.12b) through a distance x, the force exerted by the spring on the body is directed to the left and is given by $F = -kx$. This is the restoring force. Since the restoring force is proportional to the displacement (true only for small displacements) and is opposite in sign to the displacement, the resulting motion is simple harmonic. The body begins to move with a linear acceleration $a = \frac{d^2x}{dt^2}$ which, from Newton's law of motion, is given by

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$= -\omega^2 x$$

where $\omega = \sqrt{\frac{k}{m}}$

Also $\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

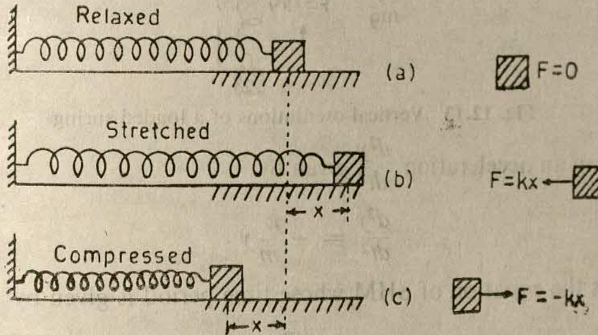


Fig. 12.12 Horizontai oscillations of a mass-spring system

Thus the expression for the time period of this oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{12.20}$$

Frequency ν of oscillation is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{12.21}$$

Vertical Oscillations Let us now consider the vertical oscillations of a loaded spring. Figure 12.13 illustrates the equilibrium position and force on the spring for two extreme positions of mass m.

In this case, the equilibrium state of the loaded spring is the state when the spring is stretched to a distance d by the force mg where g is the acceleration due to gravity. No force is acting on the body since

$$mg = kd \tag{12.22}$$

When the body is pulled through a distance y from the equilibrium position and released it starts oscillating with SHM because the restoring force is given by

$$F = -ky$$

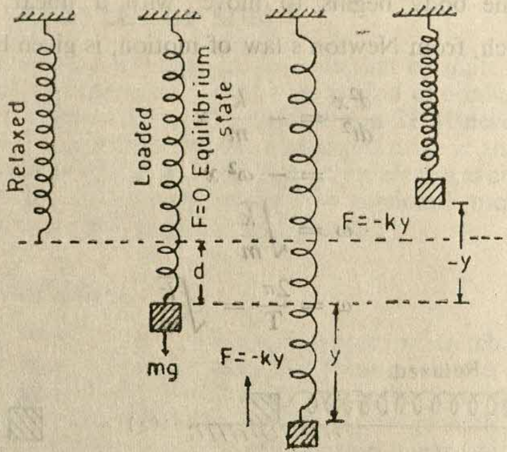


Fig. 12.13 Vertical oscillations of a loaded spring

imparting an acceleration $\frac{d^2y}{dt^2}$ given by

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

which is the equation of SHM whose time period is given by

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{12.23}$$

Equation (12.22) determines the constant k of the spring if m, g and d are known. The time period T given by (12.23) is the same as for horizontal oscillation. It depends only on mass m and spring constant k and is independent of gravity.

The Simple Pendulum

A simple pendulum (Fig. 12.14) is an idealized system consisting of a massless inextensible string, fixed rigidly at one end, having a point mass at the other. When the mass is displaced from the equilibrium position O and released, it oscillates. To determine the period of this oscillation we must compute the restoring force. Suppose that the mass is at P at any instant of time during oscillation and let α be the angle subtended by the string with the vertical. The force acting vertically downwards at P is mg. This force has a component $mg\cos\alpha$ which acts along the string and is balan-

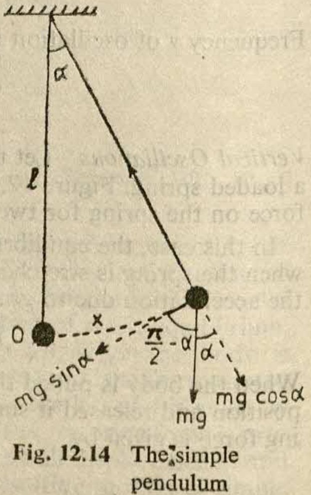


Fig. 12.14 The simple pendulum

ced by the tension on the string. The tangential component is $mg \sin \alpha$ and is directed opposite to increasing α. Thus, the restoring force is given by

$$\begin{aligned} F &= -\, mg \sin \alpha \\ &= -\, mg \left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} \cdots \right) \\ &= -\, mg\, \alpha \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120} - \cdots \right) \end{aligned}$$

where α is in radians.

The restoring force is not proportional to α, the angular displacement, and hence the oscillation is not simple harmonic. However, if α is so small that α^2 and higher order terms are negligible, we have

$$F = -\, mg\alpha$$

(The error in replacing $\sin \alpha \simeq \alpha$ is nearly 1 per cent for $\alpha = 0.25$ radian or 14°. For smaller angles, the error is even less.)

Now since x is the displacement OP along the arc, we have

$$F = -\, \frac{mg\, x}{l}$$

The acceleration of the bob, therefore, is

$$\frac{d^2x}{dt^2} = \frac{F}{m} = -\, \frac{g}{l}\, x$$

which represents an SHM of angular frequency

$$\omega = \sqrt{\frac{g}{l}}$$

and period

$$T = 2\pi \sqrt{\frac{l}{g}} \qquad (12.24)$$

Notice that the time period of a simple pendulum depends on its length l and the acceleration due to gravity at that place but is independent of the mass m of the bob.

Example 12.12 A helical spring stretches by 0.05 m when a mass of 0.5 kg is hung from it. A body of mass 1.0 kg is attached to one of its ends, the other end being rigidly fixed to the wall. The body is pulled 0.01 m along a horizontal frictionless surface and released. Determine (a) the force constant of the spring, (b) the frequency of oscillation, (c) the amplitude of oscillation, (d) the velocity and acceleration when the body is 0.005 m from its equilibrium position, (e) the maximum velocity and acceleration, and (f) the total energy of the oscillator. Assume the spring to have negligible mass and take $g = 10 \text{ m s}^{-2}$.

Solution (a) Force acting on the spring $= mg = 0.5 \times 10 = 5$ N This force extends the spring by 0.05 m.

$$\therefore \qquad k = \frac{5}{0.05} = 100 \text{ Nm}^{-1}$$

(b) Frequency $\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2 \times 3.14}\sqrt{\dfrac{100}{1.0}} = 1.59$ Hz

(c) The spring is pulled a distance 0.01 m and then released. Thus, initial displacement when energy is only potential must be = 0.01 m which is the amplitude A of oscillation.

(d) $$\text{Displacement } x = A \sin(\omega t + \phi)$$

$$\text{Velocity } V = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$$

$$\text{Acceleration } a = \frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t + \phi)$$

when $x = 0.005$ m, we have $0.005 = 0.01 \sin(\omega t + \phi)$

or $$\sin(\omega t + \phi) = \tfrac{1}{2}$$

$$\therefore \qquad \cos(\omega t + \phi) = \frac{\sqrt{3}}{2}$$

Now $$\omega = 2\pi\nu = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1.0}} = 10 \text{ rad/s}$$

$$V = 0.01 \times 10 \times \frac{\sqrt{3}}{3} = 0.087 \text{ m s}^{-1}$$

and $$a = -0.01 \times 10 \times 10 \times \tfrac{1}{2} = -0.5 \text{ m s}^{-2}$$

(e) $$V_{\max} = A\omega = 0.01 \times 10 = 0.1 \text{ m s}^{-1}$$

The velocity of the body is maximum when it passes through the equilibrium position $x = 0$ and this happens twice in each period. When it is released it travels to the left and has a maximum velocity of -0.1 m s^{-1}. After half a cycle it passes through the $x = 0$ position again, this time moving to the right with a maximum velocity $= +0.1 \text{ m s}^{-1}$

$$a_{\max} = -A\omega^2 = -0.01 \times 10 \times 10 = -1.0 \text{ m s}^{-2}$$

when x is maximum ($= \pm A$) the acceleration is maximum $= +1.0 \text{ m s}^{-2}$, since acceleration and displacement are in directions opposite to each other in SHM.

(f) The total energy of the oscillator is

$$E = \tfrac{1}{2}\, m\omega^2 A^2$$
$$= \tfrac{1}{2} \times 1.0 \times (0.01)^2 \times 10^2 = 0.005 \text{ J}$$

EXAMPLE 12.13 A spring with no mass attached to it hangs from a rigid support. A mass m is now hung on the lower end of the spring. The mass is supported on a platform so that the spring remains relaxed. The supporting platform is then suddenly removed and the mass begins to oscillate. The lowest position of the mass during the oscillation is 5 cm below the place where it was resting on the platform. (a). What is the frequency of oscillation? (b) What is the velocity when the mass is 2.5 cm below its original resting place? Take $g = 10 \text{ m s}^{-2}$ and neglect the mass of the spring.

Solution (a) It is clear from Fig. 12.15 that the separation between the two extreme positions of the oscillating mass is 5 cm. Therefore, the equilibrium position O is 2.5 cm below the supporting platform. In other words, force mg produces an extension $y = 2.5$ cm in the spring. If k is the force constant of the spring we have

$$mg = ky$$

or $$\frac{k}{m} = \frac{g}{y} = \frac{1000 \text{ cm s}^{-2}}{2.5 \text{ cm}} = 400 \text{ s}^{-2}$$

The angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{400} = 20 \text{ rad s}^{-1}$$

Hence frequency $$\nu = \frac{\omega}{2\pi} = \frac{20}{2 \times 3.142}$$
$$= 3.18 \text{ Hz}$$

(b) Amplitude of oscillation is

$$A = 2.5 \text{ cm}$$

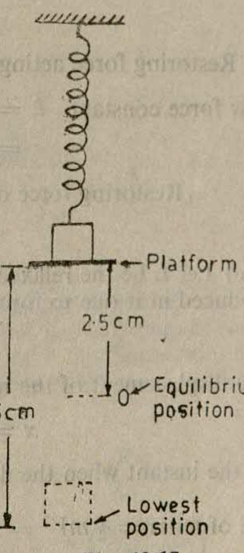


Fig. 12.15

When the mass is 2.5 cm below the platform, it is passing through the equilibrium position O and hence has the maximum velocity given by

$$V_{\text{max}} = \omega A$$
$$= 20 \times 2.5 = 50 \text{ cm s}^{-1}$$

EXAMPLE 12.14 If the mass $m = 5$ kg in Example 12.13, (a) what is the net force on the suspended mass at its lowest position? (b) What is the elastic restoring force on the mass due to the spring at its equilibrium position? (c) Show that the sum of the KE of the oscillating mass, its gravitation PE and the PE of the spring remains constant during oscillation. Take $g = 10 \text{ m s}^{-2}$ and neglect the mass of the spring.

Solution Refer to Fig. 12.15 again.

(a) At the lowest position of the mass, two forces act on it, both in the upward direction—(i) the reaction mg of the support and (ii) the force due to the acceleration of the mass at the lowest position (a_{max}).

Now $a_{\text{max}} = A\omega^2 = 2.5 \times (20)^2 = 1000 \text{ cm s}^{-2} = 10 \text{ m s}^{-2}$

$$\therefore \quad \text{Net force} = mg + ma^2_{\text{max}}$$

$$= m\,(g + a^2_{\text{max}})$$

$$= 5\,(10 + 10)$$

$$= 100 \text{ N}$$

(b) At the equilibrium position, the extension in the spring

$$= 2.5 \text{ cm} = 2.5 \times 10^{-2} \text{ m.}$$

$\therefore$ Restoring force acting upwards = force constant × extension

Now force constant $k = m\omega^2$

$$= 5 \times (20)^2 = 2000 \text{ Nm}^{-1}$$

$\therefore$ Restoring force due to spring $= 2000 \times 2.5 \times 10^{-2}$

$$= 50 \text{ N}$$

(c) Let L be the relaxed length of the spring and y the extension produced in it due to force mg so that

$$ky = mg \qquad \text{(i)}$$

The displacement of the mass during oscillation is given by

$$x = A \sin (\omega t + \phi) \qquad \text{(ii)}$$

At the instant when the displacement is x

$$\text{KE of mass} = \tfrac{1}{2} mV^2 = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2} mA^2\omega^2 \cos^2 (\omega t + \phi) \qquad \text{(iii)}$$

$$\text{PE of spring} = \tfrac{1}{2} k\,(y + x)^2 = \tfrac{1}{2} k\,(y^2 + 2yx + x^2)$$

$$= \tfrac{1}{2} ky^2 + kyx + \tfrac{1}{2} kx^2$$

Using (i) and (ii) and $\omega = \sqrt{\dfrac{k}{m}}$ we have

$$\text{PE of spring} = \tfrac{1}{2} ky^2 + mgx + \tfrac{1}{2} m\omega^2 A^2 \sin^2 (\omega t + \phi) \qquad \text{(iv)}$$

Taking gravitational PE at the mean position to be zero,

$$\text{Gravitational PE at } x = -\,mg\,x \qquad \text{(v)}$$

Adding (iii), (iv) and (v) we get

Total energy of mass-spring system

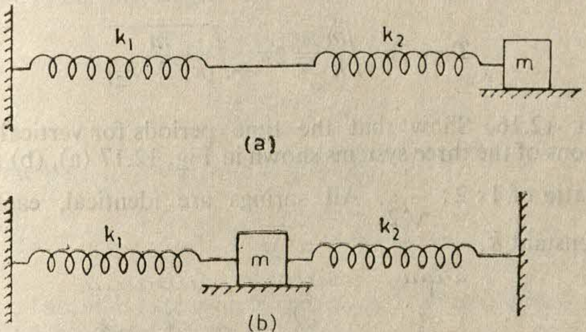
$$= \tfrac{1}{2}\, mA^2\omega^2 \cos^2(\omega t + \phi) + \tfrac{1}{2}\, ky^2 + mg\, x$$

$$+ \tfrac{1}{2}\, mA^2\omega^2 \sin^2(\omega t + \phi) - mg\, x$$

$$= \tfrac{1}{2}\, mA^2\omega^2 + \tfrac{1}{2}\, ky^2$$

which is independent of time t and hence remains constant.

EXAMPLE 12.15 Two springs of force constants k_1 and k_2 are connected to a mass m placed on a horizontal frictionless surface as shown in Figs 12.16 (a) and (b). Obtain the expression for the time period of horizontal oscillation in each case.



(a)

(b)

Fig. 12.16

Solution (a) Let the mass be displaced through a distance x, say to the right. Let x_1 and x_2 be the extensions caused in the springs of force constants k_1 and k_2 respectively, so that $x = x_1 + x_2$. Since the restoring force F exerted on the mass by each spring is the same, we have

$$F = -k_1 x_1 = -k_2 x_2$$

giving $$x_1 = -\frac{F}{k_1} \quad \text{and} \quad x_2 = -\frac{F}{k_2}$$

$$\therefore \qquad x = x_1 + x_2 = -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -F\left(\frac{k_1 + k_2}{k_1 k_2}\right)$$

Thus $$F = -\frac{k_1 k_2}{(k_1 + k_2)}\, x$$

or $$F = -kx$$

where $k = \dfrac{k_1 k_2}{(k_1 + k_2)}$ is the effective force constant of the combination.

Hence the time period of oscillation is

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) In this case, if the mass is displaced through a distance x, say to the right, the spring k_1 is extended by x and spring k_2 is compressed by x; so that the restoring force exerted by each spring on mass m is in the same direction, tending to bring it to its original equilibrium position. If F_1 and F_2 are the restoring forces due to k_1 and k_2 respectively, we have

$$F_1 = -k_1 x \quad \text{and} \quad F_2 = -k_2 x$$

$\therefore$ Total restoring force $F = -k_1 x - k_2 x = -(k_1 + k_2) x$

$$F = -kx$$

where $k = (k_1 + k_2)$ is the effective force constant of the combination. Hence

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{(k_1 + k_2)}}$$

EXAMPLE 12.16 Show that the time periods for vertical harmonic oscillations of the three systems shown in Fig. 12.17 (a), (b) and (c) are in the ratio of $1 : 2 : \frac{1}{\sqrt{2}}$. All springs are identical, each having a force constant k.

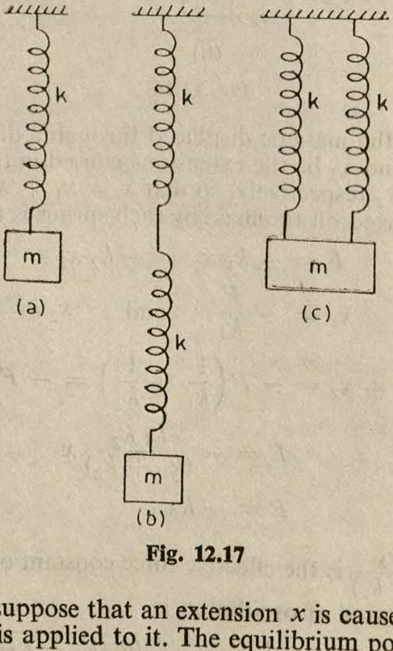


Fig. 12.17

Solution Let us suppose that an extension x is caused in the spring when a force mg is applied to it. The equilibrium position in case (a) is given by

$$F = mg = kx \tag{i}$$

The time period in this case is given by

$$T_a = 2\pi \sqrt{\frac{\text{mass}}{\text{spring constant}}} = 2\pi \sqrt{\frac{m}{k}} \tag{ii}$$

(b) In this case, the length of the spring is doubled. Hence a given force mg will double the extension. Let x' be the extension produced and k_{eff} be the force constant of the combination. Thus

$$F = mg = k_{\text{eff}}\, x'$$

$$= 2k_{\text{eff}}\, x \qquad (\because\ x' = 2x) \tag{iii}$$

Comparing (i) and (iii) we get

$$F = kx = 2k_{\text{eff}}\, x$$

or

$$k_{\text{eff}} = \frac{k}{2}$$

Thus the time period in this case is given by

$$T_b = 2\pi \sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi \sqrt{\frac{2m}{k}} = \sqrt{2}\ T_a \tag{iv}$$

(c) In this case, the extension x'' produced in each spring by a force mg is half that produced in case (a), i.e.

$$x'' = \frac{x}{2}$$

If k_{eff} is the force constant of the combination in this case, we have

$$F = mg = k_{\text{eff}}\, x'' = \frac{k_{\text{eff}}}{2}\, x \tag{vi}$$

Comparing (v) with (i) we have

$$k_{\text{eff}} = 2k$$

Hence,

$$T_c = 2\pi \sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi \sqrt{\frac{m}{2k}} = \frac{T_a}{\sqrt{2}} \tag{vi}$$

From (ii), (iv) and (vi) we have

$$T_a : T_b : T_c = 1 : 2 : \frac{1}{\sqrt{2}}$$

Example 12.17 A tray of mass $M = 12$ kg is supported on two identical springs as shown in Fig. 12.18. When the tray is depressed a little and released, it executes an SHM of period 1.5 s. (a) Find the force constant of each spring. (b) When a block of mass m is placed on the tray, the period of the SHM becomes 3.0 s. Find the value of m.

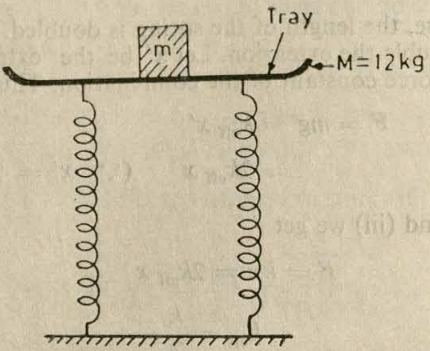


Fig. 12.18

Solution (a) As explained in Ex. 12.16, the time period of oscillation of M is given by

$$T = 2\pi\sqrt{\frac{M}{2k}}$$

where k is the force constant of each spring. Therefore

$$k = \frac{2\pi^2 M}{T^2} = \frac{2 \times (3.142)^2 \times 12}{(1.5)^2}$$

$$= 105.3 \text{ N m}^{-1}$$

(b) The new period T_1 is now given by

$$T_1 = 2\pi\sqrt{\frac{M+m}{2k}}$$

so that $$m + M = \frac{T_1^2 k}{2\pi^2}$$

or $$m + 12 = \frac{(3)^2 \times 105.3}{2 \times (3.142)^2} = 48$$

or $$m = 36 \text{ kg}$$

EXAMPLE 12.18 A vertical U-tube of uniform cross-sectional area A contains a liquid of density ρ. The liquid column is disturbed by gently blowing into the tube. Show that the resulting oscillation of the liquid is simple harmonic and find the period of oscillation. Neglect viscous effects.

Solution Let the liquid level in the left arm be depressed by, say x. The liquid level in the right arm is raised by an equal amount x. When left, the liquid levels will oscillate in each arm about their respective equilibrium positions A, A′ (see Fig. 12.19). If viscous effects are neglected, the liquid will never come to rest.

Since the column in the right arm is higher by $2x$ than the column in the left arm, the mass of this column of liquid is $m = 2A\rho x$. The gravitational force mg of this column provides the restoring force F. Thus

$$F = -mg = -2A\rho gx = -kx$$

where $k = 2A\rho g$.

If L is the total length of the liquid in the U-tube, the total mass M of the oscillating liquid is

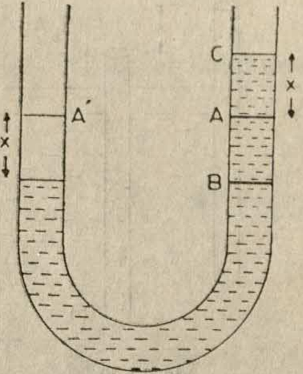


Fig. 12.19

$$M = \rho AL$$

Hence the acceleration of the liquid column is given by

$$\text{Acceleration } a = \frac{\text{force}}{\text{mass}}$$

$$= \frac{F}{m} = -\frac{kx}{M}$$

$$= -\frac{2A\rho g}{\rho AL}x$$

$$= -\frac{2g}{L}x$$

or $$a = -\omega^2 x \qquad \text{with } \omega = \frac{\sqrt{2g}}{L}$$

Hence the motion is simple harmonic. The time period of the motion is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{2g}}$$

Note: The period of oscillation is independent of the density ρ of the liquid and the cross-sectional area A of the U-tube.

EXAMPLE 12.19 A cylindrical piece of cork of height h and density ρ_c floats in a liquid of density ρ_l. The cork is depressed slightly and released. Show that the vertical motion of the cork is simple harmonic and find its period. Neglect viscous effects.

Solution Let A be the cross-sectional area of the cork and M its mass. Fig. 12.20 (a) shows the static equilibrium; the weight of the cork being balanced by the weight of the liquid it displaces.

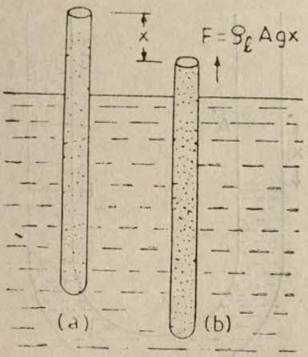


Fig. 12.20

If the cork is depressed through a distance x, as shown in Fig. 12.20 (b), the buoyant force on it increases by $\rho_l Agx$, because $\rho_l Ax$ is the mass of the liquid displaced by dipping; g being the acceleration due to gravity. If viscous effects are neglected, the restoring force on the cork is given by

$$F = -\rho_l Agx = -Kx$$

where $K = \rho_l Ag$. Since $F \propto -x$, the motion of the cork is simple harmonic. The time period of the motion is

$$T = 2\pi\sqrt{\frac{M}{K}}$$

where M is the mass of the cork $= Ah\rho_c$. Hence

$$T = 2\pi\sqrt{\frac{Ah\rho_c}{\rho_l Ag}} = 2\pi\sqrt{\frac{h\rho_c}{g\rho_l}}$$

EXAMPLE 12.20 An air chamber of volume V has a neck of cross-sectional area a into which a light ball m can move without friction. The diameter of the ball is equal to that of the neck of the chamber. Show that when the ball is pressed down a little distance and released, the ball executes SHM. Show that the period of the SHM is given by

$$T = 2\pi\sqrt{\frac{mV}{Ba^2}}$$

where B is the bulk modulus of air.

Solution Let P be the pressure of air in the chamber. When the ball is pressed down a distance x, the volume of air decreases from V to say $V - \Delta V$. Hence the pressure increases from P to $P + \Delta P$. The change in volume (see Fig. 12.21) is

$$\Delta V = ax$$

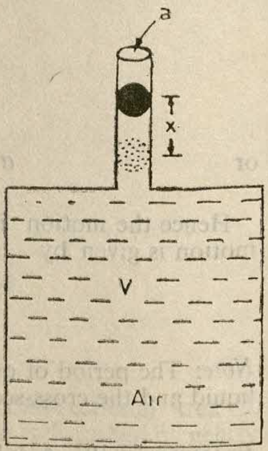


Fig. 12.21

The excess pressure ΔP is related to the bulk modulus B

as $$\Delta P = -B\frac{\Delta V}{V}$$

Now Restoring force on ball = excess pressure × cross-sectional area

or $$F = -\frac{Ba}{V}\Delta V$$

or $$F = -\frac{Ba^2}{V}x \qquad (\because \ \Delta V = ax)$$

or $$F = -Kx$$

where $K = Ba^2/V$

i.e. $F \propto -x$. Hence the motion of the ball is simple harmonic. If m is the mass of the ball, the time period of the SHM is

$$T = 2\pi\sqrt{\frac{m}{K}}$$

or $$T = 2\pi\sqrt{\frac{mV}{Ba^2}}$$

EXAMPLE 12.21 A simple pendulum is displaced from its equilibrium position O to a position P until the height of P above O is 0.05 m (see Fig. 12.22). It is then released. (a) Calulate the velocity of the bob when it passes the mean position O. Take $g = 10 \text{ m s}^{-2}$ and neglect friction. (b) What is the tension in the supporting string at this instant? The mass of the bob is 50 g and the time period is 2 s.

Solution (a) The potential energy of the pendulum at P

= work done (against gravity) in raising the bob of mass m to a vertical height h

$= mgh$

When the bob is released, it oscillates. The energy when it passes through O is entirely kinetic and is equal to $\frac{1}{2}mV^2$ where V is the velocity at point O.

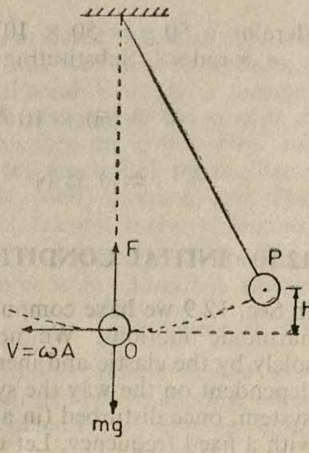


Fig. 12.22

Since friction is neglected, the entire potential energy at P is converted into kinetic energy at O, giving

$$\tfrac{1}{2}\, mV^2 = mgh$$

or
$$V = \sqrt{2gh}$$
$$= \sqrt{2 \times 10 \times 0.05} = 1 \text{ m s}^{-1}$$

(b)
$$\text{Time period } T = 2 \text{ s}$$

$\therefore$
$$\text{Angular frequency } \omega = \frac{2\pi}{T} = \pi \text{ rad s}^{-1}$$

At position O, the tension F in the supporting string is given by

$$(F - mg) = \text{centripetal force} = \frac{mV^2}{l}$$

where l = length of the string. We know that

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

Thus
$$l = \frac{g}{\omega^2}$$

Hence
$$F = mg + \frac{mV^2\omega^2}{g}$$
$$= m\left(g + \frac{V^2\omega^2}{g}\right)$$

Here $m = 50 \text{ g} = 50 \times 10^{-3}$ kg, $g = 10 \text{ m s}^{-2}$, $V = 1 \text{ m s}^{-1}$ and $\omega = \pi \text{ rad s}^{-1}$. Substituting these values we get

$$F = 50 \times 10^{-3} \times \left\{10 + \frac{(1)^2\,(\pi)^2}{10}\right\}$$
$$\simeq 0.55 \text{ N}$$

12.10 INITIAL CONDITIONS

In Sec. 12.9 we have computed the frequency (or period) of a few harmonic motions. We noticed that the frequency was determined solely by the elastic and inertial properties of the system and was not dependent on the way the system was set into oscillation initially. The system, once disturbed (in any arbitrary way) and released, oscillates with a fixed frequency. Let us now learn what determines the other two constants, namely, the amplitude A and phase constant ϕ of an oscillator. These constants are determined by how the system is set into oscillation. Consider the example of a simple pendulum. The pendulum can be set into oscillation in a variety of ways. We can displace it, say to the right, and let go, so that it has a finite initial displacement and no velocity. We can give it a kick when it is at its equilibrium position, in which case, it has a finite initial velocity but no initial displacement. We can also start with some initial displacement and then push when we let it go, in which case, it has finite initial displacement and finite initial velocity. The way the system is set into oscillation gives the *initial conditions.*

Suppose at time $t = 0$ (i.e. when the system is released) we give an initial displacement x_0 and an initial velocity V_0; x_0 and V_0 are both specified. This is the most general way in which a motion can be started. x_0 and V_0 will determine the constant A and ϕ. Using Eqs (12.5) and (12.14), and setting $t = 0$ we have

$$x_0 = A \sin \phi$$

and

$$V_0 = A\omega \cos \phi$$

$$\therefore \qquad \tan \phi = \frac{\omega x_0}{V_0}$$

and

$$A = \left(x_0^2 + \frac{V_0^2}{\omega^2}\right)^{1/2}$$

If at time $t = 0$, $x = x_0$ and $V_0 = 0$, then $A = x_0$, and $\phi = \frac{\pi}{2}$. If, on the other hand, $x = 0$ and $V = V_0$ at $t = 0$ then $A = \frac{V_0}{\omega}$ and $\phi = 0$.

12.11 DAMPED OSCILLATIONS

We have learnt that the total energy of a harmonic oscillator remains constant. Once started the oscillations continue for ever with a constant amplitude (which is determined from initial conditions) and a constant frequency (which is determined from force constant K and mass m of oscillator). Simple harmonic motions of this type are called *free and undamped*. In actual practice, however, the energy of the oscillator gradually decreases with time and it eventually comes to rest. This happens because, in actual physical systems, friction is always present.

We know that the elastic restoring force is displacement-dependent ($F = - kx$). The damping or frictional force is an additional restoring force which is velocity-dependent $\left(F_f = - b\,\frac{dx}{dt}\right)$, where b is a positive constant and is a measure of the friction present in the system. The equation of motion of an oscillator of mass m moving under a linear restoring force $- Kx$ and also subjected to a damping force $- b\,\frac{dx}{dt}$ will be

$$m\,\frac{d^2x}{dt^2} = - Kx - b\,\frac{dx}{dt}$$

To find out how the displacement x of the damped oscillator changes with time t, we need to solve this equation. The solution of this equation is outside the scope of this book. Here we simply quote the solution which is valid for low damping,

$$x(t) = A \exp(-bt/2m) \cos(\alpha t + \delta) \tag{12.25}$$

where $\alpha = \left(\frac{K}{m} - \frac{b^2}{4m^2}\right)^{1/2}$ is the angular frequency of the damped oscillator. We know that $\omega = K/m$. Therefore

$$\alpha = \left(\omega^2 - \frac{b^2}{4m^2}\right)^{1/2} \tag{12.26}$$

Thus ω is the angular frequency in the absence of damping; ω is called the *natural* frequency of the oscillator.

Equation (12.25) shows that the motion of the damped oscillator is periodic (its period is $2\pi/\alpha$) but not simple harmonic because its amplitude $A \exp(-bt/2m)$ decreases exponentially with time. The energy of the oscillation, therefore, decreases with time, the loss of energy due to friction appearing as heat in the medium (see Fig. 12.23).

Equation (12.26) tells us that $\alpha < \omega$, i.e. the frequency of the oscillator decreases due to damping. Thus the *friction has three effects*: (*i*) *it changes the simple harmonic motion to periodic motion*, (*ii*) *it decreases the amplitude of the oscillation and* (*iii*) *it reduces the frequency of oscillation.*

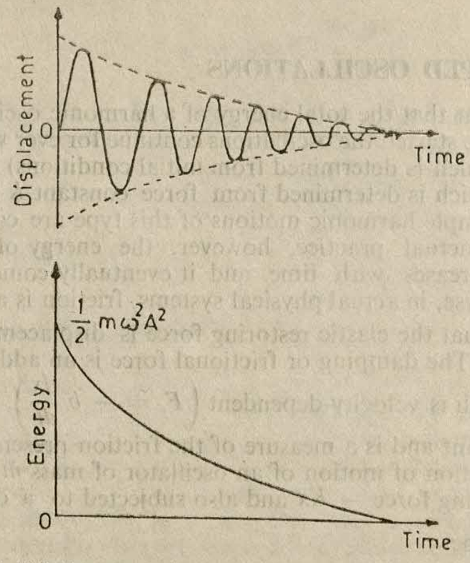


Fig. 12.23 Displacement-time and energy-time curves for damped oscillation

12.12 FORCED OSCILLATIONS AND RESONANCE

Since the free oscillations of a damped oscillator will eventually die out, it is necessary to supply energy to keep the system in oscillation. A problem of great practical importance is that of a damped harmonic oscillator driven by an externally applied harmonic force of the type $F(t) = F_0 \sin \sigma t$, where F_0 is the magnitude of the force applied and σ its angular frequency. The oscillator is then said to execute *forced* (or maintained) oscillations. Very often a system is set into oscillation by linking (or coupling) it in some way with another oscillating system (which we will call the driver). For example, in a resonance tube, the air column vibrates because it is linked (by sound waves) to the vibrations of a tuning fork. The diaphragm of a loud-speaker vibrates because it is linked (by current oscillations) to the output circuit of an amplifier. The oscillator picks up energy from the driver and oscillates. Initially, the oscillator likes to oscillate at its own frequency $\left(\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}\right)$ but the driver would like it to oscillate at its own frequency $\left(\nu' = \frac{\sigma}{2\pi}\right)$ and after some time it succeeds. At this time, the oscillator has forgotten its time-history and begins to oscillate at a constant amplitude at the frequency of the driver. The oscillations are then called *steady-state forced oscillations*. In the steady state the rate at which power is lost through friction equals the rate at which it is fed to the oscillator by the driver. When the frequeney of the driver is varied, the amplitude of the forced oscillator also changes. In Fig. 12.24 we have plotted the amplitude of the steady-state forced oscillation as the frequency of the driver is varied from zero to a large value (see Supplement at the end of the Chapter).

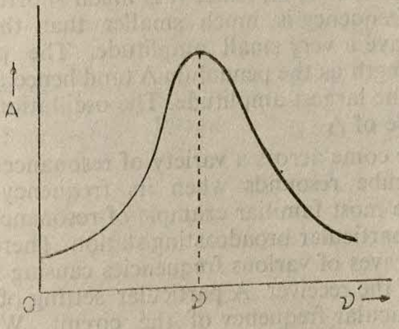


Fig. 12.24 Displacement of a forced oscillator as frequency ν' of the driver is varied. ν is the frequency of free-damped oscillations

We notice that the displacement (and hence energy) of forced oscillations is very small if $\nu' \ll \nu$ and $\nu' \gg \nu$. But when $\nu' \simeq \nu$ the amplitude of the forced oscillations becomes very large. In other words, when the frequency of the driving force is equal to the frequency of free-damped oscillations, the oscillator responds most favourably to the driving force and extracts maximum energy from it. The case $\nu' = \nu$ is called *resonance* and the oscillations are then called resonant oscillations.

Resonance can be best illustrated by a very simple experiment. From the same elastic string PQ, (Fig. 12.25) suspended four pendulums A, B, C and D.

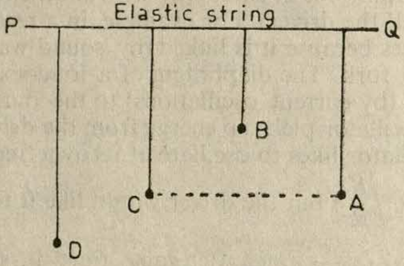


Fig. 12.25 Resonance in pendulum motion

The pendulum A (the driver) is set into oscillation. The pendulums B, C and D will also start oscillating because they are coupled or linked to A by the elastic string. Initially, the motion of B, C and D will be erratic. After some time (when the steady state is reached) the pendulums B, C and D will all start oscillating with the frequency of the pendulum A. The oscillations of pendulums B, C and D are forced oscillations. We will notice that the pendulum B, whose frequency is much larger than that of A, (since it is much shorter) and the pendulum D, whose frequency is much smaller than that of A, (since it is much longer) have a very small amplitude. The pendulum C which has the same length as the pendulum A (and hence the same frequency) oscillates with the largest amplitude. The oscillations of C are in resonance with those of A.

In physics we come across a variety of resonances. The air-column in a resonance tube resounds when its frequency agrees with that of the fork. The most familiar example of resonance is when we tune our radio to a particular broadcasting station. There are many stations sending radio waves of various frequencies causing forced oscillations in the circuit of the receiver. A particular setting of the tuner corresponds to a particular frequency of the circuit. When this frequency equals that of the waves from a particular broadcasting station, the power absorption is maximum and hence we hear only that station. A sodium chloride crystal, which consists of positively and negatively charged ions, can absorb energy if subjected to an oscillating electric field. When the frequency of the relative oscillations of ions matches the frequency of the electric field, the crystal absorbs maximum energy from the field.

Some resonances can cause disaster. A column of army men marching over a bridge can set in a forced oscillation of the bridge. If the frequency of their footsteps happens to match one of the natural frequencies of the bridge, (which is determined by the dimensions and elastic properties of the material of the structure) resonance will occur and the bridge will oscillate with a destructively large amplitude. This is the reason why soldiers break step when crossing a bridge.

SUPPLEMENT

Mathematical Treatment of Forced Oscillations

For the sake of simplicity we consider the case in which damping is absent. An undamped oscillator of natural angular frequency $\omega = \sqrt{K/m}$ is subjected to a harmonic force $F(t) = F_0 \sin \sigma t$ where σ is the angular frequency of the applied force. The equation of motion of the forced oscillator is

$$m \frac{d^2x}{dt^2} = -Kx + F_0 \sin \sigma t$$

or

$$\frac{d^2x}{dt^2} + \omega^2 x = f_0 \sin \sigma t \tag{S.1}$$

where $\omega = \sqrt{K/m}$ and $f_0 = F_0/m$; m being the mass of the oscillator. To determine the response of the oscillator to the externally applied periodic force $F(t)$, we need to solve Eq. (S.1). Before we attempt to solve this equation, let us perform the experiment illustrated in Fig. 12.26.

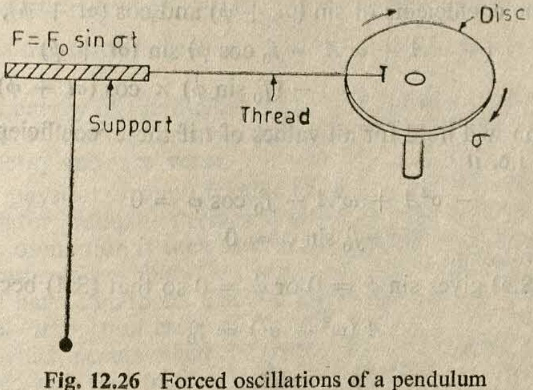


Fig. 12.26 Forced oscillations of a pendulum

Consider a simple pendulum hanging from a support which can execute horizontal oscillations by attaching it to a rotating disc (having a motor not shown in the diagram) by means of a thread or wire.

Let us suppose that at time $t = 0$ the pendulum is at its rest position when the support starts oscillating. The pendulum begins to oscillate under the influence of the periodic impulses imparted to it by the oscillating support. We will notice that for some time, the motion of the pendulum is quite erratic. But after some, the pendulum is seen to execute SHM of a constant amplitude at the same frequency as that of the support, i.e. the frequency of the applied force which is ν' $(=\sigma/2\pi)$. Notice that the pendulum does not oscillate at its own natural frequency ν $(= \omega/2\pi)$ given by $\nu = \frac{1}{2\pi}\sqrt{g/l}$. This is the steady state of the forced oscillator. We will analyse the steady-state behaviour of the forced oscillator as it is important from the point of view of practical application.

The displacement of the forced oscillator in the steady state must, therefore, be given by

$$x(t) = A \sin(\sigma t + \phi) \tag{S.2}$$

where σ is the angular frequency (= that of the applied force) of the oscillator and ϕ its phase constant. Differentiating Eq. (S.2) twice with respect to t we get

$$\frac{d^2x}{dt^2} = -\sigma^2 A \sin(\sigma t + \phi) \tag{S.3}$$

Using Eqs (S.2) and (S.3) in Eq. (S.1) we get

$$\begin{aligned} -\sigma^2 A \sin(\sigma t + \phi) + \omega^2 A \sin(\sigma t + \phi) &= f_0 \sin \sigma t \\ &= f_0 \{\sin(\sigma t + \phi - \phi)\} \\ &= f_0 \{\sin(\sigma t + \phi)\cos\phi + \cos(\sigma t + \phi)\sin\phi\} \end{aligned}$$

collecting the coefficients of $\sin(\sigma t + \phi)$ and $\cos(\sigma t + \phi)$, we have

$$\begin{aligned} (-\sigma^2 A + \omega^2 A - f_0\cos\phi)\sin(\sigma t + \phi) \\ -(f_0 \sin\phi) \times \cos(\sigma t + \phi) = 0 \end{aligned}$$

This equation will hold for all values of t if these coefficients separately vanish, i.e. if

$$-\sigma^2 A + \omega^2 A - f_0\cos\phi = 0 \tag{S.4}$$

and

$$f_0 \sin\phi = 0 \tag{S.5}$$

Condition (S.5) gives $\sin\phi = 0$ or $\phi = 0$ so that (S.4) becomes

$$A(\omega^2 - \sigma^2) = f_0$$

or

$$A = \frac{f_0}{\omega^2 - \sigma^2}$$

or

$$A = \frac{F_0/m}{4\pi^2(\nu^2 - \nu'^2)} \tag{S.6}$$

where $\omega = 2\pi\nu$, $\sigma = 2\pi\nu'$ aud $f_0 = F_0/m$.

Equation (S.6) gives the amplitude of the forced oscillator in the steady state. Notice that when $\nu' \simeq \nu$; $A \to \infty$, i.e. if the frequency of the applied force equals that of the natural (free) oscillations, the amplitude of the oscillator becomes infinity (i.e. very large). This is the case of resonance. If damping is included, the expression for A is modified and the amplitude of the forced damped oscillator is not infinite at resonance, as depicted earlier in Fig. 12.24; it will be large but finite at resonance.

SUMMARY

The most general definition of SHM is the motion in which the restoring force (and hence the acceleration) is proportional (and opposite in direction) to the displacement from the equilibrium position. Equivalently, it is the motion of the projection of a uniform circular motion on the diameter of the reference circle.

The three characteristics of an SHM are the amplitude A, period T (or frequency ν) and phase constant (or initial phase) ϕ. Amplitude is the magnitude of the maximum displacement from the equilibrium position. The period of an SHM is the smallest time interval during which the motion repeats itself. Frequency ν, which is the inverse of period, is the number of full oscillations completed in one second. The phase constant is a measure of the time that elapses before the oscillator last passed its equilibrium position. The amplitude A and the phase constant ϕ are determined from the initial conditions which tell us how the oscillation is started at time $t = 0$. The period T is determined from the elastic (i.e. force constant) and inertial (i.e. the mass) properties of the system.

The displacement of the oscillator in SHM is given by $x = A \sin (\omega t + \phi)$, where $\omega \left(= \frac{2\pi}{T} = 2\pi\nu \right)$ is the angular frequency of the motion. T is measured in seconds, ν in Hertz and ω in radian per second.

The total energy of the oscillator in SHM is given by $E = \frac{1}{2} m\omega^2 A^2$ and is constant. During the motion, kinetic energy is transformed into potential energy and vice versa.

In actual physical systems, friction is always present and the energy of the oscillator gradually decreases with time and it eventually comes to rest. The oscillation is then said to be damped. A problem of great practical importance is that of a damped oscillator driven by an externally applied harmonic force. The oscillations are then said to be forced. An extremely important case of forced oscillations is the resonant oscillation which occurs when the frequency of the driving force happens to agree with that of free-damped oscillations. At resonance, the damped oscillator responds most favourably to the driving force and extracts maximum energy from it.

EXERCISES

1. Define SHM. Give two examples of SHM not mentioned in the text.
2. Name the three physical parameters that characterize an SHM. Give the meaning of each.
3. 'All simple harmonic motions are periodic, but all periodic motions are not necessarily simple harmonic'. Comment.
4. What are the dimensions of period, frequency, angular frequency and force constant? Give the MKS unit for each.
5. The motion of a particle is given by $x = A \cos (\omega t + \phi)$. Show that the motion is harmonic. Write down the expressions for velocity and acceleration of the perticle. Is the acceleration zero at any point in its path?
6. If the mass of the particle in Question 5 is m, obtain the expression for its kinetic energy. Show that the kinetic energy is zero when $x = \pm A$.
7. A particle executes an SHM of amplitude A. At what displacement is the energy half-kinetic and half potential?
8. Show that the equation $x = A \sin \left(\frac{2\pi t}{T} + \phi\right)$ of a particle executing SHM implies that the energy of the particle does not change with time.
9. Under which conditions is the motion of a simple pendulum harmonic? Deduce the expression for its period and show that it is independent of the mass of the pendulum.
10. Obtain the expression for the frequency of vertical oscillations of a massless spring of force constant K loaded with mass m and hence show that the frequency is independent of the length of the spring. Is the frequency also independent of acceleration due to gravity?
11. Distinguish between free, damped, forced and resonant vibrations. Give an example of each.
12. Establish the equation of motion of a forced· undamped oscillator and solve it for the steady state.
13. What are periodic, harmonic and non-periodic functions. Give one example of each.
14. State the Fourier theorem about periodic functions.

PROBLEMS

1. A particle executes SHM of amplitude 5 cm and a period of 2 s. Find the speed of the particle at a point where its acceleration is half the maximum value.
2. A body moves with an SHM of frequency 0.5 Hz and amplitude 4 cm. Starting at a time when the displacement is 4 cm, find the values of displacement, velocity and acceleration 1.25 s later.
3. A particle oscillates with SHM of amplitude 5 cm and period 6 s. How long will it take to move from one end of its path on one side of the equilibrium position to a position 2.5 cm on the other side of the equilibrium position? What is the magnitude of its velocity at this point?

4. The equation of a simple harmonic motion is given by

$$x = 3 \sin 5\pi t + 4 \cos 5\pi t$$

where displacement x is in centimetres and time t in seconds. Find the amplitude, period and phase constant of the motion.

5. A particle in a linear SHM has a maximum velocity of 20 cm s^{-1} and a maximum acceleration of 80 cm s^{-2}. Determine the amplitude and angular frequency of oscillation.

6. The amplitude of the SHM of a particle is 5 cm. When its displacement is 3 cm, its velocity has a magnitude of 0.5 m s^{-1}. Find the period of the motion. If the mass of the particle is 10 g. What is its total energy?

7. A load of mass 0.5 kg hangs from a spring of force constant 10 N m^{-1}. The mass is pulled down 0.05 m from its equilibrium position and then released. (a) What is the distance between two most widely separated positions of the mass? (b) How long does it take to traverse this distance?

8. A heavy object, placed on a shock absorber such as a rubber pad, compresses it by 1 cm. If the object is given a vertical tap, it will oscillate. Find the frequency of oscillation. Neglect damping.

   [*Hint*: Assume that the pad acts like an elastic spring].

9. A pole of cross-sectional area A and mass M is floating vertically in a liquid of density ρ. The centre of gravity of the pole is below the liquid surface. If the pole is pressed down a little and released, show that the resulting motion is simple harmonic. Obtain an expression for the period of the motion.

10. A uniform spring has a certain mass suspended from it. The spring is cut into two equal halves and the same mass is suspended from one of the halves. Will the frequency of vertical oscillations be the same as before? If not, how is the frequency in the second case related to that in the first?

11. A uniform spring of force constant k is cut into two pieces whose lengths are in the ratio of 1 : 2. What is the force constant of each piece in terms of k?

12. A pendulum clock shows accurate time. If the length of its pendulum is increased by 0.1%, what will be the error in time per day?

13. By what percentage should the length of the pendulum of a clock be changed so that it keeps correct time when moved from a place where $g = 9.80$ m s^{-2} to a place where $g = 9.82$ m s^{-2}?

14. A person normally weighing 60 kg stands on a platform which is oscillating up and down harmonically with a period of 1.0 s and an amplitude of 10 cm. If a weighing machine on the platform gives the person's weight against time, what will be the maximum and minimum readings shown by it. Take $g = 9.8$ m s^{-2}.

15. A platform executes SHM in a vertical direction with an amplitude of 5 cm and an angular frequency of 20 rad s^{-1}. A block is placed on the platfrom when it is at the lowest point of its path. (a) At what point will the block leave the platform? (b) How far will the block rise above the highest point reached by the platform? Take $g = 10$ m s^{-2}.

16. Suppose a tunnel is dug through the earth (assumed to be a homogeneous sphere) from one side to the other along a diameter. A body dropped into the tunnel takes time t_1 to reach the centre of the earth. If the acceleration due to gravity had remained unchanged, the body would have taken time t_2 to traverse this distance. What is the ratio $t_1 : t_2$? Neglect friction offered to the motion.

    [*Hint*: In the first case, the motion of the body is simple harmonic with a period $T = 2\pi \sqrt{\frac{R}{g}}$, where R is the radius of the earth and g is the acceleration due to gravity at its surface].

17. A body moves with SHM about a point $x = 0$. At time $t = 0$ it has a displacement of 5 cm and zero velocity. If the frequency of the motion is 0.5 Hz, find the magnitudes of the displacement, velocity and acceleration at $t = 1.25$ s.

18. The displacement of an oscillating particle is given by

    $$x = 3 \sin \pi t + 4 \cos \pi t$$

    where x is in centimetres and t in seconds. Show that the motion of the particle is simple harmonic and find its amplitude.

19. A horizontal platform with an object placed on it executes SHM of amplitude 2.5 cm in the vertical direction. What must be the least period of these oscillations so that the object is not detached from the platform? Take $g = 10 \text{ m s}^{-2}$.

20. A small spherical steel ball is placed a little way away from the centre of a large concave mirror whose radius of curvature $R = 2.5$ m. When the ball is released, it begins to oscillate about the centre. Show that the motion of the ball is simple harmonic and find its period. Neglect friction and take $g = 10 \text{ m s}^{-2}$.

21. Two masses m_1 and m_2 are suspended together by a spring of force constant k. When the masses are in equilibrium, mass m_1 is removed without disturbing the system. Find the angular frequency and amplitude of oscillation of m_2. Neglect the mass of the spring.

22. A mass m is suspended at the end of a massless wire of length L and cross-sectional area A. If Y is the Young's modulus of the material of the wire, obtain the expression for the period of the SHM along the vertical line.

23. Which of the following functions of time represents (a) simple harmonic, (b) periodic but not simple harmonic and (c) non-period motion? Find the period T in each case of periodic motion; ω is a positive constant.

    (i) $\cos \omega t - \sin \omega t$

    (ii) $\sin^3 \omega t$

    (iii) $3 \sin \left(\frac{\pi}{3} - 2 \omega t\right)$

    (iv) $\cos \omega t + \cos 3 \omega t - \cos 5 \omega t$

    (v) $\omega t (1 + \omega t)$

    (vi) $e^{+\omega}$

13

Waves

13.1 INTRODUCTION

There are essentially two ways of transporting energy from the place where it is produced to that where it is intended to be utilized. The first involves the actual transport of matter. For example, a bullet fired from a gun carries its kinetic energy with it which can be used at another location. The second method by which energy can be transported is much more useful and important; it involves what we call a wave process. The wave carries energy but there is no transfer of matter. When a drummer beats a drum its sound is heard at distant points. The sound carries energy as it can move the diaphragm of the ear. When a stone is dropped in the still water in a pond, water waves move steadily along the water until they reach the shore. If there is a small floating object, like a piece of cork, it will move up and down near its own location, which indicates that the molecules of water do not move along with the wave. When a bulb is turned on, the room is flooded with light. Light waves also carry energy. It is possible to transmit an electric signal (or a message) from one place to another. Although these various processes of transport of energy are different, yet they have a common feature which we will call *wave motion.*

The key word in wave motion is *disturbance* or perturbation. The reason why a variety of wave motions are possible is that there are various ways of disturbing the physical state of a body. In the case of water waves, the disturbance is the change in the position of water molecules relative to the equilibrium state (flat surface) brought about by dropping a stone. For sound waves, the changes of pressure can be regarded as a disturbance. For this disturbance to travel from one point of a medium to another, the particles of the medium must be coupled to one another by some force so that the disturbance created at one point can be handed down to its neighbours.

It is essential, at this stage, to clarify the meaning of the word 'disturbance'. The scope of its meaning need not be limited, in a narrow sense, to mean the actual physical displacement of particles of a medium. In fact, a material medium is not even necessary if some physical property of space can exist in vacuum. We know that electric and magnetic fields can exist in vacuum. Disturbance, in this case, could be a perturbation (or change) in these fields. This disturbanc etravels in vacuum, needing no material medium for its propagation. Light waves are an example of such a wave.

In this chapter we will study only those waves which require a material medium for their propagation. We will now make an extremely important assumption, necessary for the existence of wave motion. We will ignore the molecular structure of matter and assume the medium to be continuous. The medium contains a large number of particles (e.g. one litre of air has 2.7×10^{22} molecules). If these are distributed within a limited space, the average distance between them becomes exceedingly small. As an approximation one can imagine the number of particles to be approaching infinity and intermolecular separation tending to zero. Such a system is said to be continuous. Implicit in this description is the existence of waves.

13.2 WAVES IN A CONTINUOUS MEDIUM

Waves on a Stretched String

Suppose you are holding one end of a long rope, the other end of which is fixed to a peg on the wall. (Fig. 13.1). Move your hand suddenly upwards and bring it back to its original position. A kink (or pulse) is created in the string near your hand and you will see it move down the length of the rope. The particles of the string simply move up and down as the pulse passes by. The pulse obviously moves with a finite velocity. Let the pulse labelled 1 be created at time $t = 0$. It travels a distance x in time t when a second pulse labelled 2 is created

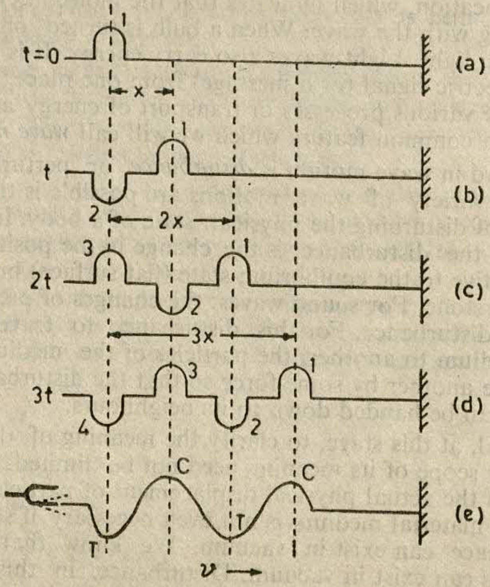


Fig. 13.1 Propagation of pulses on a string

by moving the hand, say down-wards, and bringing it back to its original position. Whatever be its shape or size, the second pulse will travel the same distance x in time t. The velocity of each pulse is $v = \frac{x}{t}$. At time $2t$ another pulse labelled 3, of the same shape as pulse 1, is produced and so on. The sequence till time $3t$ is depicted in Figs 13.1 a, b, c and d.

On the basis of these observations, the following two conclusions about pulse propagation can be drawn: (a) the pulse travels on the string carrying energy with it, but the string does not bodily move with it and no matter is transported; and (b) the velocity of the pulse is independent of the shape or size of the pulse. The pulse velocity v depends only on the medium.

We will now enter into the domain of waves which are of immediate concern to us. You will notice that a succession of pulses were created in the string at regular intervals of time and this was easily accomplished by jerking our hand up or down at regular intervals. Now suppose we create pulses on the string so that there is no time interval between them. We do not wait between two pulses. This can be easily accomplished by tying the free end of the string to a prong of tuning fork and setting it into harmonic oscillation. The result is depicted in Fig. 13.1e. Such a wave created by the simple harmonic oscillation of any particle of the string is called a *harmonic wave*. We will be dealing only with harmonic waves. Such waves are travelling waves, moving down the length of the string. They eventually reach the fixed end and are reflected there. We will discuss reflection later. For the moment we assume that no reflection has taken place. In Sec. 13.4 we will analyse such a wave in greater detail.

Waves on a Spring (Compressional Wave)

Take a long spring one end of which is attached to a piston which can move right or left, the other end being fixed to the wall. Figure 13.2 a depicts the state of affairs at time $t = 0$. The piston is now made to oscillate with an SHM of period, say, T. When the piston moves to the right, it compresses the coils of the spring immediately next to it. As time passes this compression labelled C travels down the length of the spring but the coils only vibrate back and forth. When the piston moves to the left, it extends the coils and this so-called rarefaction also travels along the spring following the compression. We say that a compressional wave travels along the spring. Figures 13.2b, c, d and e, depict the state of affairs at $t = \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}$ and T respectively.

13.3 TYPES OF WAVES

There are two types of wave motions: (i) transverse and (ii) longitudinal.

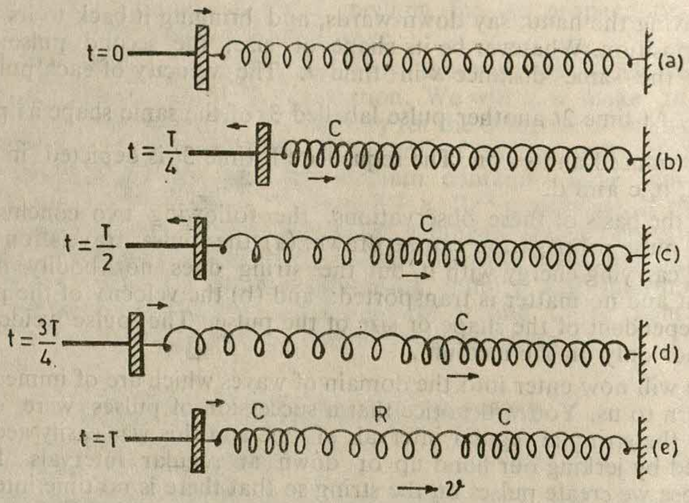


Fig. 13.2 Compressional wave in a spring

Transverse Waves

In transverse waves the particles of the medium vibrate at right angles to the direction in which the wave propagates. Waves on a string depicted in Fig. 13.1 are transverse. Water waves are also transverse. Electromagnetic waves (which include light waves) are transverse waves. Here, however, the disturbance that travels is not a result of the vibration of material particles but oscillation of electric and magnetic fields which occurs at right angles to the direction in which the wave travels.

Longitudinal Waves

In longitudinal waves the particles of the medium vibrate along the direction of wave propagation. The wave in the spring shown in Fig. 13.2 is longitudinal. The most familiar example of a longitudinal wave is a sound wave. When we speak, the particles of air next to our mouth get displaced producing a compression. Thus a compressional wave (analogous to the wave on a spring) travels in air. Sound waves can travel in liquids, solids and gases.

13.4 CHARACTERISTICS OF A HARMONIC WAVE

In order to understand the propagation of a harmonic wave in a medium let us again consider the example of a transverse wave on a stretched string. We have seen in Sec. 13.2 how such a wave can be produced. We will now examine it in greater detail. To fix our ideas we shall consider the motion of the first nine particles of the string (see Fig. 13.3). The particle labelled 1 is made to oscillate with SHM (say by attaching it to a prong of a tuning fork) of period T and amplitude A. For the sake of simplicity. we assume that at $t = 0$ the particle 1 is at its mean position (Fig. 13.3a) so that its displacement is given by

$$y(t) = A \sin \omega t = A \sin \frac{2\pi t}{T} \tag{13.1}$$

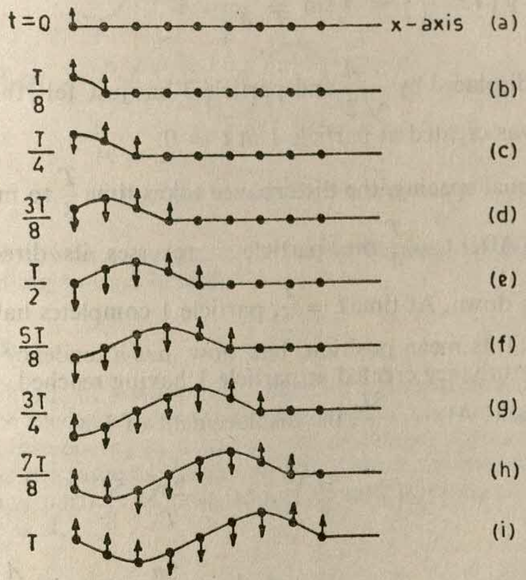


Fig. 13.3 State of particle oscillations and wave propagation

This disturbance created at particle 1 (located at $x = 0$) will be transmitted to particles 2, 3, 4,. . . and so on. The particles 1 to 9 are identical and equispaced. Let the interparticle spacing be such that the disturbance takes time $\frac{T}{8}$ to travel from one particle to the next. The state of motion of the particles at time

$$t = 0, \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2}, \frac{5T}{8}, \frac{3T}{4}, \frac{7T}{8} \text{ and } T$$

are depicted in Fig. 13.3. The sequence of Fig. 13.3 may be understood by recalling the reference circle in Ch. 12. The arrows indicate the direction in which the particle will move the next moment. The spacing between the particles of the string has been exaggerated for convenience. At time $t = 0$, particle 1 executing SHM given by Eq. (13.1) is at its mean position and will move up the next moment. At time $t = \frac{T}{8}$ (Fig. 13.3b), particle 1 is displaced by

$$y\left(t = \frac{T}{8}\right) = A \sin \frac{2\pi}{T} \cdot \frac{T}{8} = A \sin \frac{\pi}{4} = \frac{A}{\sqrt{2}}$$

and particle 2 has just felt the disturbance and will have the tendency to move up as particle 1 had at $t = 0$. At $t = \frac{T}{4}$ (Fig. 13.3c) particle 1 is displaced by

$$y\left(t = \frac{T}{4}\right) = A \sin \frac{2\pi}{T} \frac{T}{4} = A$$

Particle 2 is displaced by $\frac{A}{\sqrt{2}}$ and particle 3 has just felt the disturbance which was created at particle 1 at $t = 0$.

Because of equal spacing, the disturbance takes time $\frac{T}{8}$ to move to the next particle. After $t = \frac{T}{4}$ the particle 1 reverses its direction and starts moving down. At time $t = \frac{T}{2}$, particle 1 completes half its cycle and is again at its mean position but now has a tendency to move down, the disturbance created at particle 1 having reached as far as the fifth particle. At $t = \frac{5T}{8}$, the displacement of 1 is

$$y\left(t = \frac{5T}{8}\right) = A \sin \frac{2\pi}{T} \cdot \frac{5T}{8}$$

$$= A \sin \frac{5\pi}{4} = - \sin \frac{A}{4}$$

$$= - \frac{A}{\sqrt{2}}$$

At time $t = T$ when particle 1 completes one full vibration and passes through its mean position upward, the disturbance reaches particle 9 which has the tendency to move up. Figure 13.4 shows what the string will look like at $t = 2T$, and $t = 3T$ as particle 1 completes two and three vibrations respectively. We notice that as time passes new crests and troughs labelled C and T respectively are created giving the impression that something is travelling down the length of the string. The disturbance (or wave) travels along the string and the particles execute SHM with the same period and amplitude. But the state of motion (or phase) of particle oscillation is different for different particles. This is the crucial point. We notice in Fig. 13.3i that the two nearest particles in the same state of motion (i.e. having the same phase) are 1 and 9. They are both at their mean positions and are moving up. The phase of particle 5 is opposite to that of 1 or 9 as it is moving down.

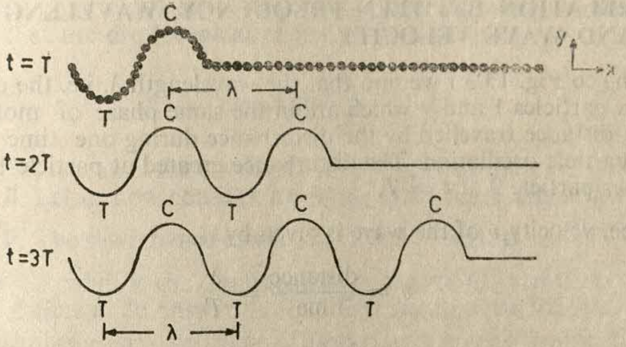


Fig. 13.4 Travelling harmonic wave

We are now in a position to define some characteristics of a harmonic travelling wave. These are:

Amplitude

The amplitude of a wave is the maximum displacement of the particles of the medium from their mean position. Assuming that the wave does not lose energy as it travels, the maximum displacement of any particle is the same as the amplitude of the harmonic disturbance created at any particle of the medium. In our example, the amplitude of the wave is A.

Period

The time period of a wave is the period of harmonic oscillations of particles of the medium which is the same for all particles. In our example, T is the period of the wave. The frequency ν of the wave is inverse of period T $\left(\text{i.e. } \nu = \frac{1}{T}\right)$.

Wave Velocity

Wave velocity v is the distance travelled by the wave in one second.

Wavelength

The wavelength (denoted by λ) is defined as the distance measured along the direction of propagation of wave between two nearest points which are in the same phase of vibration. In Fig. 13.3 i the particles 1 and 9 are in the same phase. The distance between them is the wavelength of the wave. (See also Fig. 13.4).

Another very important characteristic of a wave, namely, its phase, will be dealt with separately in Sec. 13.9.

13.5 RELATION BETWEEN FREQUENCY, WAVELENGTH AND WAVE VELOCITY

Referring to Fig. 13.3 i we find that the wavelength λ, i.e. the distance between particles 1 and 9 which are in the same phase of motion, is just the distance travelled by the disturbance during one time period (T) of particle oscillation. The disturbance created at particle 1 at $t = 0$ reaches particle 9 at $t = T$.

Hence, velocity v of the wave is given by

$$v = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T} \tag{13.2}$$

Since $$\nu = \frac{1}{T}$$

we have

$$v = \nu\,\lambda \tag{13.3}$$

This very important relation between wavelength, frequency and wave velocity holds for transverse as well as longitudinal waves. We could impart an SHM to particle 1 in Fig. 13.3 along the x-axis. This disturbance will travel from one particle to the next and at time $t = T$ particle 9 will be disturbed and its phase will be the same as that of particle 1. We will then get a longitudinal wave travelling along the x-axis. Since the wave velocity v is determined only by the elastic and inertial property of the medium, it is a constant for a given medium. Frequency ν is characterized by the source which produces a disturbance, e.g. a tuning fork in the example of Sec. 13.4. Hence, if ν is changed, wavelength λ changes so as to preserve product $\nu\lambda$ which is equal to v.

13.6 MATHEMATICAL DESCRIPTION OF HARMONIC WAVE: THE WAVE EQUATION

We have learnt that wave motion (transvesse or longitudinal) is a result of particle oscillation. The state of motion (or phase) is, in general, different for different particles. The particles in the crest are in a different state of motion from those in the trough. A similar statement can be made for compression and rarefaction in the case of a longitudinal wave. What we observe (i.e. waves) is really the state of motion of various particles of the medium. In formulating such a wave motion in mathematical terms we must be able to relate the state of motion between any two particles.

Consider a wave travelling along the x-axis with velocity v. This wave was created by giving harmonic oscillations of period T and amplitude A to a particle P located, say, at $x = 0$. We assume, for simplicity, that the phase constant ϕ of this harmonic oscillation is zero so that the displacement of particle P is given by

$$y\,(0, t) = A \sin \frac{2\pi t}{T} \tag{13.4}$$

where $y\,(0, t)$ stands for displacement at time t of the particle located at $x = 0$. Let us now consider a particle Q on the x-axis at a distance x from P. The disturbance created at P will reach Q in $\frac{x}{v}$ seconds. We now ask what is the displacement $y\,(x, t)$ of Q located at x at any instant of time t. To answer this question we argue as follows.

The displacement of particle Q located at x is equal to the displacement the particle P, located at $x = 0$, had $\frac{x}{v}$ seconds earlier.

In other words, the displacement of particle Q at an instant of time t

= displacement of particle P at an instant of time

$$t' = t - \frac{x}{v}$$

$$= A \sin \frac{2\pi t'}{T}, \text{ since displacement of P is given by Eq. (13.4)}$$

for all t

$$= A \sin \left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right\}$$

Since the LHS of the above equality is $y\,(x, t)$ we get,

$$y\,(x, t) = A \sin \left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right\} \tag{13.5}$$

This is the equation for a harmonic wave travelling along the positive x-direction. In deriving this equation we have assumed that (i) the amplitude A of particle oscillations does not change in the course of propagation of the wave and (ii) the medium is homogeneous, i.e. the wave velocity v does not change from place to place.

For a wave travelling along the negative x-direction, the equation is

$$y\,(x, t) = A \sin \left\{\frac{2\pi}{T}\left(t + \frac{x}{v}\right)\right\} \tag{13.6}$$

We may use Eq. (13.5) to check our definition of wavelength λ. By definition, the state of motion of two particles at a distance λ apart must be the same. Setting $x = \lambda$ in Eq. (13.5) we have

$$\begin{aligned} y(\lambda, t) &= A \sin\left\{\frac{2\pi}{T}\left(t - \frac{\lambda}{v}\right)\right\} \\ &= A \sin\left\{\frac{2\pi t}{T} - 2\pi\right\} \quad \left(\because \ v = \frac{\lambda}{T}\right) \\ &= A \sin\frac{2\pi t}{T} \\ &= y(0, t) \end{aligned}$$

In other words, the function $y(x, t)$ repeats itself after a distance λ. Wavelength λ is also called the *spatial periodicity* of the wave. The wave is thus *doubly periodic*. It has *temporal* (in relation to time) *periodicity* T and *spatial* (in relation to space) *periodicity* λ. Equation (13.2) gives the relation between these two very important characteristics of wave-motion. Using Eq. (13.2) and substituting for $T = \frac{\lambda}{v}$, Eq. (13.5) can be rewritten, in an alternative form, as

$$y(x, t) = A \sin\left\{\frac{2\pi v}{\lambda}\left(t - \frac{x}{v}\right)\right\}$$

or

$$y(x, t) = A \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\} \tag{13.7}$$

We can rewrite Eq. (13.5) in yet another more compact form by defining the two quantities, $k = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi}{T}$. In terms of k and ω we can recast Eq. (13.5) as

$$\begin{aligned} y(x, t) &= A \sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{vT}\right) \\ &= A \sin\left(\omega t - \frac{2\pi x}{\lambda}\right) \quad (\because \ vT = \lambda) \end{aligned}$$

or

$$y(x, t) = A \sin(\omega t - kx) \tag{13.8}$$

The quantity k is called the *wave number* and ω the *angular frequency* of the wave.

13.7 THE WAVE FUNCTION

In the preceding sections we have described an important class of wave motions called *harmonic waves*. Harmonic waves are those waves which travel in a medium as a result of harmonic oscillations of the particles of the medium. They are, therefore, described by sine (or cosine) function [see Eq. (13.7)]. Figure 13.4 shows the shape of a string when a harmonic wave travels in it. Notice that the shape of the string is described by a sine function. The shape of the disturbed string is called its *wave profile.*

Harmonic waves are not the only kinds of waves. For example in Figs 13.1 (a), (b), (c) and (d) we have seen how pulses travel in a string. These pulses (also called wave pulses) travel with the same speed as that of a harmonic wave shown in Fig. 13.1 (e). Pulses are also described by a displacement $y(x, t)$ which is a function of both x and t. Such a function is called a *wave function.* Equation (13.7) shows the explicit form of the wave function for a harmonic wave travelling in the $+x$ direction.

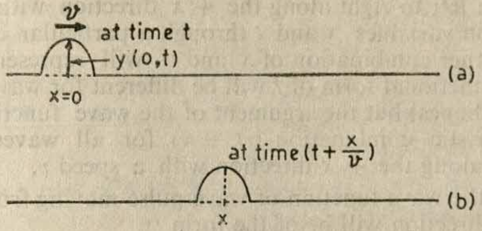


Fig. 13.5 Wave function of a wave pulse

We will now obtain the general form of the wave function for any kind of a wave. We will consider the example of a pulse travelling in a long string with a speed, say, v. Consider a pulse located around $x = 0$ at time t as shown in Fig. 13.5a. Figure 13.5b shows the pulse when it arrives at a point x on the string. The time taken by the pulse to travel a distance x is x/v. Let $y(0, t)$ stand for the particle displacement in the pulse at $x = 0$ at time t. Let us say that

$$y(0, t) = f(t)$$

where $f(t)$ is an unspecified function of t. It is clear that

Displacement at $x = 0$ at time t = displacement at x at a later time $t + \frac{x}{v}$

or Displacement at x at time t = displacement at $x = 0$ at an earlier time $t^* = t - \frac{x}{v}$

$$= f(t^*)$$

$$= f\left(t - \frac{x}{v}\right)$$

If the symbol $y\,(x, t)$ stands for the displacement at x at time t, we have

$$y\,(x, t) = f\left(t - \frac{x}{v}\right)$$

Since v is a constant, we may write

or $$y\,(x, t) = f\,(vt - x) \tag{13.9}$$

The wave function $f\,(vt - x)$ is a function of both variables x and t. It must be clearly understood that the wave function for a pulse moving from left to right along the $+\,x$ direction with a speed v depends only on variables x and t through a particular combination $(x - vt)$. No other combination of x and t will represent a travelling pulse. The functional form of f will be different for waves of different profiles (or shapes) but the argument of the wave function must have the characteristic combination $(vt - x)$ for all waves moving from left to right along the $+\,x$ direction with a speed v.

Similarly the wave function of wave pulse moving from right to left in the $-\,x$ direction will be of the form

$$y\,(x, t) = g\,(vt + x) \tag{13.10}$$

For a harmonic wave, we have obtained the explicit functional forms of wave functions $f\,(vt - x)$ and $g\,(vt + x)$. These forms are given in Eqs (13.5) and (13.6) respectively. Notice the characteristic combinations $(vt - x)$ and $(vt + x)$.

The general form of the wave function may be written as

$$y\,(x, t) = f\,(bt - ax) \tag{13.11}$$

which represents a wave travelling at a speed $v = b/a$. If b/a is positive, the function (13.11) will represent a wave travelling in the $+\,x$ direction. But if b/a is negative, it represents a wave travelling in the $-\,x$ direction. For a harmonic wave, the wave function is [see Eq. (13.7)].

$$y\,(x, t) = A \sin\,(bt - ax) \tag{13.12}$$

where $b = 2\pi v/\lambda = 2\pi\nu$ and $a = 2\pi/\lambda$ are constants of the wave. The parameters A, b and a completely describe a harmonic wave.

Thus we find that the wave function $f\,(bt - ax)$ describes all kinds of waves. We will, however, restrict ourselves only to harmonic waves described by the wave function in Eq. (13.12). With this discussion on general wave functions, we will now return to the special case of harmonic wave functions.

13.8 GRAPHICAL REPRESENTATION OF A HARMONIC WAVE

Equation (13.5) of a harmonic wave contains three variables—particle displacement y, particle location x and time t. The graph of y when

x and t are both changing is, therefore, a three-dimensional curve. To draw a graph on a two-dimensional plane, one of the two independent variables (x and t) is kept constant and the other is varied. Let us study how y varies with x at various values of time t.

1. $t = 0$: Equation (13.5) becomes

$$y(x, 0) = -A \sin \frac{2\pi x}{vT} = -A \sin \frac{2\pi x}{\lambda}$$

$x \rightarrow$	0	$\frac{\lambda}{4}$	$\frac{\lambda}{2}$	$\frac{3\lambda}{4}$	λ	$\frac{5\lambda}{5}$	$\frac{3\lambda}{2}$	$\frac{7\lambda}{4}$	2λ
$y(x, 0) \rightarrow$	0	−A	0	+A	0	−A	0	+A	0

Figure 13.6a is a plot of $y(x, 0)$ against x.

2. $t = \frac{T}{4}$: Putting $t = \frac{T}{4}$ in Eq. (13.5) we get

$$y\left(x, \frac{T}{4}\right) = A \sin\left(\frac{\pi}{2} - \frac{2\pi x}{\lambda}\right) = A \cos \frac{2\pi x}{\lambda}$$

$x \rightarrow$	0	$\frac{\lambda}{4}$	$\frac{\lambda}{2}$	$\frac{3\lambda}{4}$	λ	$\frac{5\lambda}{4}$	$\frac{3\lambda}{2}$	$\frac{7\lambda}{4}$	2λ
$y\left(x, \frac{T}{4}\right) \rightarrow$	+A	0	−A	0	+A	0	—A	0	+A

Curve (b) in Fig. 13.6 represents $y\left(x, \frac{T}{4}\right)$ against x.

3. At $t = \frac{T}{2}$ Eq. (13.5) reduces to

$$y\left(x, \frac{T}{2}\right) = A \sin \frac{2\pi x}{\lambda}$$

which is depicted in Fig. 13.6c.

4. At $t = \frac{3T}{4}$ we get

$$y\left(x, \frac{3T}{4}\right) = -A \cos \frac{2\pi x}{\lambda}$$

which is plotted in Fig. 13.6d.

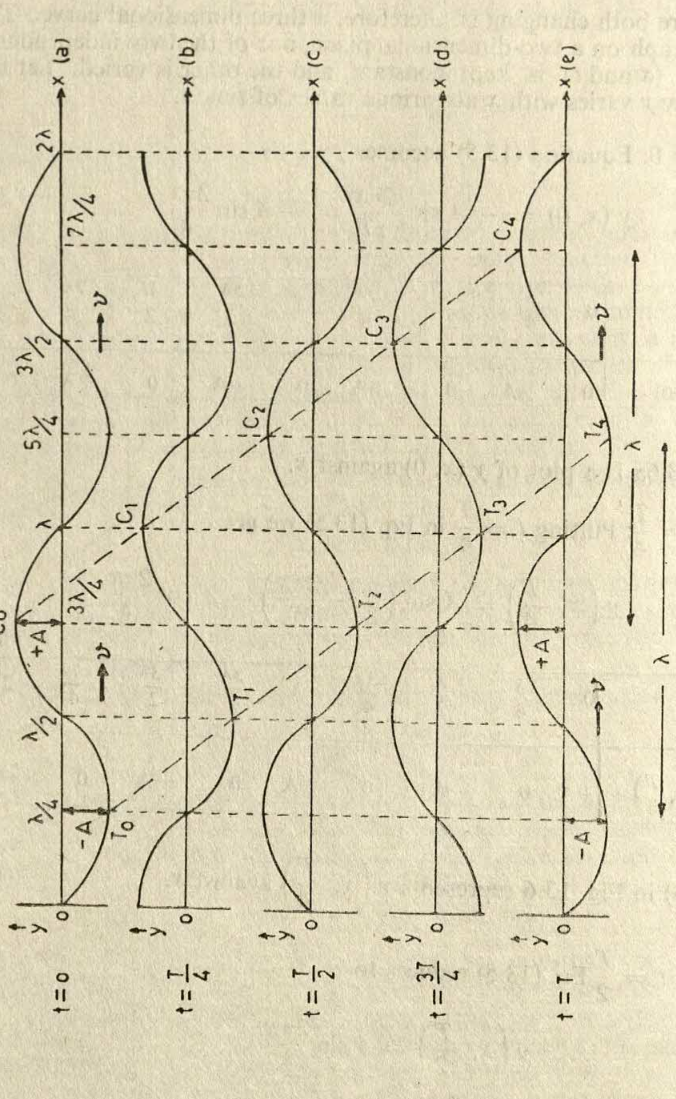


Fig. 13.6 Harmonic travelling wave

5. At time $t = T$ we recover the graph in Fig. 13.6a since

$$y(x, T) = A \sin\left(2\pi - \frac{2\pi x}{\lambda}\right)$$

$$= -A \sin \frac{2\pi x}{\lambda}$$

Figure 13.6 is a graphical representation of a harmonic wave of amplitude A, period T, and wavelength λ, being created somewhere at the left of the point $x = 0$ and travelling to the right with a velocity v. For simplicity, we have plotted y against x, for x lying between 0 and 2λ. Waves, of course, extend beyond $x = 2\lambda$. Figures 13.6b, c, d and e show what the waves look like at successive instants of time $t = \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}$ and T. One can imagine these curves to be instant photographs of the wave-train taken at five successive instants of time. The figure clearly shows that the whole pattern travels to the right with a velocity $v = \frac{\lambda}{T}$. The wave troughs at these five time-values have been marked as T_0, T_1, T_2, T_3 and T_4. The corresponding wave crests are C_0, C_1, C_2, C_3 and C_4. One can clearly visualize how crests and troughs move as time passes. We have seen that a transverse wave can be readily represented graphically since the displacements are perpendicular to the direction of propagation of the wave. A graph such as in Fig. 13.6 is the picture of the actual wave. The situation for a longitudinal wave is quite different. For a longitudinal wave in a gas there is a crowding of particles in a compression where their density is high, and in a rarefaction it is low. Figure 13.7a is the picture of a longitudinal wave. This is not a very convenient representation. Instead, we represent a longitudinal wave by a graph which shows how a certain property of the medium, like particle density, differs at various parts of the wave train. In Fig. 13.7b the straight horizontal line represents normal density. All points above this line represent

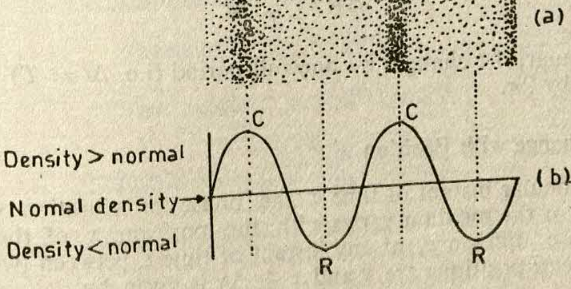


Fig. 13.7 Pictorial and graphical representation of a longitudinal wave

densities larger than normal and those below represent less than normal densities. Another property of the medium, namely, particle displacement, can be similarly represented graphically. We must however, realize that the graph is not in any sense a picture of a wave.

13.9 PHASE AND PHASE DIFFERENCE

In Sec. 13.6 we obtained Eq. (13.5) for a harmonic wave, with the simplifying assumption, that at time $t = 0$ the displacement of the particle at $x = 0$ was zero, i.e. its phase constant $\phi = 0$. If, however, the particle at $x = 0$ was not passing through its mean position at $t = 0$, Eq. (13.4) would be replaced by $y\,(0, t) = A \sin\left(\frac{2\pi t}{T} + \phi\right)$ where ϕ is the phase constant (i.e. phase at $t = 0$) of the particle motion. Equation (13.5) would then read

$$y\,(x, t) = A \sin\left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right) + \phi\right\}$$

The argument of the sine function is called phase Φ in a harmonic wave. Thus, phase Φ at space point x and time t in a harmonic wave is given by

$$\Phi = \frac{2\pi}{T}\left(t - \frac{x}{v}\right) + \phi \qquad (13.13)$$

It is evident from this equation that the phase changes with time t as well as space-point x.

Phase Change with Time

The phase of a given particle (i.e. x fixed) changes with time. As time changes from t to $t + \Delta t$ its phase changes from Φ to $\Phi + \Delta\Phi$ where $\Delta\Phi$, using Eq. 13.13 is given by

$$\Delta\Phi = \frac{2\pi}{T}\,\Delta t$$

In particular, if t changes by one full period (i.e. $\Delta t = T$) the phase changes by 2π.

Phase Change with Position

At a particular instant of time t the phase of vibration of various particles of the medium varies with the position x of the particles. The phase difference, at any instant of time t, between two particles whose mean positions are x and $x + \Delta x$ is given by

$$\Delta\Phi = -\,\frac{2\pi}{\lambda}\,\Delta x$$

The minus sign indicates that, for a wave travelling in the $+\ x$ direction, the particles at higher values of x lag behind in phase. This is because they receive the disturbance later than those behind them. From the equation it is evident that if $\Delta x = \lambda$ the phase change is 2π.

The phase difference (rather than phase) is a very important and useful concept in relation to wave motion. We will use it to discuss the interference of waves. In fact, one often defines wavelength λ and period T in terms of phase change. Wavelength λ of a wave is the distance between points (or particles) whose phases differ by 2π. Particles whose phases differ by π (or 3π or 5π, etc.) are said to be in opposite phase. Similarly, the period T is the time during which the phase of a given particle changes by 2π.

EXAMPLE 13.1 When a plane wave traverses a medium, the displacement of particles is given by

$$y\,(x, t) = 0.01 \sin 2\pi\,(2t - 0.01\,x)$$

where y and x are expressed in metres and t in seconds. Calculate (a) the amplitude, wavelength, velocity and frequency of the wave: (b) the phase difference between two positions of the same particle at a time interval of 0.25 s; (c) the phase difference, at a given instant of time, between two particles 50 m apart.

Solution (a) The given equation can be rewritten in the form

$$y\,(x, t) = 0.01 \sin \left\{\frac{2\pi}{100}\,(200\,t - x)\right\}$$

Comparing this with the wave equation,

$$y\,(x, t) = A \sin \left\{\frac{2\pi}{\lambda}\,(vt - x)\right\}$$

we have,

Amplitude $A = 0.01$ m

Wavelength $= 100$ m

Wave velocity $v = 200\ \text{m s}^{-1}$

and frequency $\nu = \dfrac{v}{\lambda} = \dfrac{200}{100} = 2$ Hz

(b) Phase change in a time interval of Δt is

$$\Delta\Phi = \frac{2\pi}{T}\cdot\Delta t = 2\pi\nu\Delta t$$

or phase difference $= 2\pi \times 2 \times 0.25 = \pi = 180°$. In other words, the particle phase is reversed in a time 0.25 s which is obvious since its period

$$T = \frac{1}{\nu} = 0.5 \text{ s}$$

(c) Phase difference for path difference of Δx is

$$\Delta\Phi = -\frac{2\pi}{\lambda} \cdot \Delta x = -\frac{2\pi}{100} \times 50 = -\pi = -180°$$

In other words, the particle located 50 m (which is half the wavelength) ahead of another particle lags in phase by 180°.

EXAMPLE 13.2 A transverse harmonic wave of amplitude 0.01 m is generated at one end ($x = 0$) of a long horizontal string by a tuning fork of frequency 500 Hz. At a given instant of time the displacement of the particle at $x = 0.1$ m is -0.005 m and that of the particle at $x = 0.2$ m is $+0.005$ m. Calculate the wavelenth and the wave velocity. Obtain the equation of the wave assuming that the wave travels along the $+x$ direction and that the end $x = 0$ is at the equilibrium position at $t = 0$.

Solution Since the wave is travelling along the $+x$ direction and the displacement of the end $x = 0$ is zero at time $t = 0$, the general equation of this wave is

$$y(x, t) = A \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\} \quad \text{(i)}$$

where $A = 0.01$ m.

When $x = 0.1$ m, $y = -0.005$ m

$$\therefore \quad -0.005 = 0.01 \sin\left\{\frac{2\pi}{\lambda}(vt - x_1)\right\}$$

where $x_1 = 0.1$ m.

or

$$\sin\left\{\frac{2\pi}{\lambda}(vt - x_1)\right\} = -\frac{1}{2}$$

$$\therefore \quad \text{Phase } \Phi_1 = \frac{2\pi}{\lambda}(vt - x_1) = \frac{7\pi}{6} \quad \text{(ii)}$$

When $x = 0.2$ m, $y = +0.005$. Therefore, we have

$$+0.005 + 0.01 \sin\left\{\frac{2\pi}{\lambda}(vt - x_2)\right\}$$

where $x_2 = 0.2$ m.

or $$\Phi_2 = \frac{2\pi}{\lambda}(vt - x_2) = \frac{\pi}{6} \qquad \text{(iii)}$$

From Eqs (ii) and (iii)

$$\Delta\Phi = \Phi_1 - \Phi_2 = \pi$$

Now $$\Delta\Phi = -\frac{2\pi}{\lambda}\Delta x$$

Thus, $$\pi = -\frac{2\pi}{\lambda}(x_1 - x_2) = -\frac{2\pi}{\lambda}(0.1 - 0.2)$$

or $$\lambda = 0.2 \text{ m}$$

Now,

Frequency ν of the wave = frequency of the tuning fork
= 500 Hz

Hence Wave velocity $v = \nu\lambda = 500 \times 0.2 = 100 \text{ m s}^{-1}$

Substituting for A. λ and v in Eq. (i) we get

$$y(x, t) = 0.01 \sin\{10\pi(100t - x)\}$$

This is the equation of the wave where y and x are in metres and t in seconds.

Example 13.3 The particle displacements in a travelling harmonic wave are given by

$$y(x, t) = 2.0 \cos 2\pi(10t - 0.008x + 0.35)$$

where x and y are in centimetres and t is in seconds. What is the phase difference between oscillatory motion at two points separated by a distance (a) 4 m, (b) 0.5 m. (c) $\frac{\lambda}{2}$ and (d) $\frac{3\lambda}{4}$?

Solution The phase difference $\Delta\Phi$ between particles separated by a distance Δx is given by

$$|\Delta\Phi| = \frac{2\pi}{\lambda}\cdot\Delta x$$

where λ is the wavelength. Now we know that

$$\frac{2\pi}{\lambda} = \text{coefficient of } x \text{ in the argument of the sine or cosine function}$$
$$= 2\pi \times 0.008$$

$$\therefore \qquad \lambda = \frac{1}{0.008} = 125 \text{ cm}$$

(∵ x is in centimetres)

$$= 1.25 \text{ m}$$

(a) For $x = 4$ m, $\Delta\Phi = \frac{2\pi}{1.25} \times 4 = 6.4\,\pi$ rad $= 0.4\pi$ rad, since a phase change of 2π, 4π, 6π, etc. implies no phase change.

(b) For $x = 0.5$ m, $\Delta\Phi = \frac{2\pi}{1.25} \times 0.5 = 0.8\pi$ rad

(c) For $x = \frac{\lambda}{2}$, $\Delta\Phi = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$ rad

(d) For $x = \frac{3\lambda}{4}$, $\Delta\Phi = \frac{2\pi}{\lambda} \cdot \frac{3\lambda}{4} = \frac{3\pi}{2}$ rad.

EXAMPLE 13.4 A travelling wave on a string is described by

$$y\,(x, t) = 7.5 \sin\left(0.005\, x + 12t + \frac{\pi}{4}\right)$$

where y and x are in centimetres and t is in seconds.

(a) Find the displacement and velocity of the particle of the string at a point $x = 1$ cm and $t = 1$ s. In this velocity equal to the wave velocity?

(b) Locate the points on the string which have the same transverse displacement and velocity as the $x = 1$ cm point has at $t = 2$ s, 5 s and 11 s.

Solution The particle displacements are given by

$$y\,(x, t) = 7.5 \sin\left(0.005\, x + 12t + \frac{\pi}{4}\right) \qquad \text{(i)}$$

Let us first find the wavelength and the wave velocity. Now

$$\frac{2\pi}{\lambda} = \text{coefficient of } x \text{ in the argument of sine function}$$

$$= 0.005$$

$$\therefore \qquad \lambda = \frac{2\pi}{0.005} = 1256.8 \text{ cm} = 12.57 \text{ m}$$

Also, $2\pi \frac{v}{\lambda}$ = coefficient of t in the argument of the sine function$=12$.

$$\therefore \qquad v = \frac{12\lambda}{2\pi} = \frac{12}{2\pi} \times \frac{2\pi}{0.005} = 2400 \text{ cm s}^{-1}$$

$$= 24 \text{ m s}^{-1}$$

Since the wave is travelling in the $-x$ direction at a speed of 24 m s^{-1}, the wave velocity is -24 m s^{-1}.

(a) Putting $x = 1$ cm and $t = 1$ s in Eq. (i), we have

$$\begin{aligned}
\text{Displacement at } x = 1 \text{ cm} \quad &\text{and} \quad t = 1 \text{ s} \\
&= 7.5 \sin\left(0.005 \times 1 + 12 \times 1 + \frac{\pi}{4}\right) \\
&= 7.5 \sin(12.005 + 0.785) \quad (\because \pi = 3.142) \\
&= 7.5 \sin(12.79 \text{ rad}) \\
&= 7.5 \sin(732.7^\circ) \quad (\because \pi \text{ rad} = 180^\circ) \\
&= 7.5 \sin(4\pi + 12.7^\circ) \\
&= 7.5 \sin(12.7^\circ) \\
&= 7.5 \times 0.2198 \\
&= 1.65 \text{ cm}
\end{aligned}$$

Particle velocity is obtained by differentiating Eq. (i) with respect to t. We get

$$V(x, t) = \frac{dy}{dt} = 7.5 \times 12 \times \cos\left(0.005 \times 12t + \frac{\pi}{4}\right) \qquad \text{(ii)}$$

Putting $x = 1$ cm and $t = 1$ s in Eq. (ii) we get, as before

$$\begin{aligned}
\text{Velocity at } x = 1 \text{ cm and } t = 1 \text{ s} &= 7.5 \times 12 \times \cos 12.7^\circ \\
&= 87.8 \text{ cm s}^{-1}
\end{aligned}$$

which is not equal to the wave velocity of -24 m s^{-1}.

(b) By definition of wavelength (λ), all points on the string which are $\pm \lambda, \pm 2\lambda, \ldots = \pm n\lambda$ away from $x = 1$ cm (here n is an integer and $\lambda = 12.67$ m) have the same displacement and velocity as those at $x = 1$ cm for all t values.

13.10 VELOCITIES IN WAVE MOTION

We have learnt that the velocity of a wave depends on the elastic and inertial properties of the medium in which it travels. In terms of these properties one can obtain the expression for wave velocity in the medium. This derivation involves the use of differential calculus and is outside the scope of the present book. We can, however, use the method of dimensional analysis to determine how the wave velocity depends on these two properties.

Velocity of a Transverse Wave on a Stretched String

Consider a transverse wave travelling on a string of mass m, per unit length stretched by a tension F. We can now make a guess as to the factors on which the wave velocity v should depend. The wave velocity must obviously depend on the elastic and inertial properties of the medium. Tension F is a measure of the elasticity of the string. When the tension is increased the elastic restoring force, on an element of the string which is displaced, increases. So, v must depend on F. The mass of the element of the string determines its inertia. As the velocity v does not depend on how long the string is, we guess that it should depend on m, the mass of a unit length the string. We may also except v to depend on the radius r of the string. The factors r and m together determine the density $\rho = \frac{m}{\pi r^2}$.

Thus, we assume that v is given by

$$v = KF^a m^b r^c$$

where K is a dimensionless constant and a, b and c are unknown exponents. In terms of mass M, length L and time T, the dimensions of various quantities are

$F = \mathrm{MLT^{-2}}$ the dimensions of force

$m = \mathrm{ML^{-1}}$ the dimension of m

$r = \mathrm{L}$

$v = \mathrm{LT^{-1}}$

Substituting these dimensions in the equation, we get

$$(\mathrm{LT^{-1}}) = (\mathrm{MLT^{-2}})^a\,(\mathrm{ML^{-1}})^b\,(\mathrm{L})^c$$

K being dimensionless

or
$$(\mathrm{LT^{-1}}) = (\mathrm{M}^{a+b}\,\mathrm{L}^{a-b+c}\,\mathrm{T}^{-2a})$$

Equating the exponents we get

$$a + b = 0$$

$$a - b + c = 1$$

$$-2a = -1$$

which yields

$$a = \tfrac{1}{2}, \qquad b = -\tfrac{1}{2} \quad \text{and} \quad c = 0$$

Hence

$$v = K\sqrt{\frac{F}{m}}$$

Starting from a reasonable guess as to the factors likely to be significant, we have obtained an expression for wave velocity without using any physical law. We cannot however, evaluate the constant K, which can be done either by a detailed mathematical analysis of the problem or by doing an experiment. It turns out that $K = 1$. The velocity of a transverse wave travelling on a string, of mass m per unit length, stretched by a tension F is given by

$$v = \sqrt{\frac{F}{m}}$$

The force in a string is called tension and is usually denoted by a symbol T (not to be confused with time period) and not F. Thus

$$v = \sqrt{\frac{T}{m}} \qquad (13.14)$$

If T is in newton and m in kg m^{-1}, v will be in m s^{-1}.

EXAMPLE 13.5 Transverse waves are generated in two uniform steel wires A and B of diameters 10^{-3} m and 0.5×10^{-3} m respectively, by attaching their free end to a vibrating source of frequency 500 Hz. Find the ratio of the wavelengths if they are stretched with the same tension.

Solution The density of a wire of mass M, length L and diameter d is given by

$$\rho = \frac{4M}{\pi d^2 L} = \frac{4m}{\pi d^2}$$

now

$$v_A = \sqrt{\frac{T}{m_A}}$$

and

$$v_B = \sqrt{\frac{T}{m_B}}$$

$$\frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}} = \frac{d_B}{d_A}$$

but $v_A = \nu\lambda_A$ and $v_B = \nu\lambda_B$, ν being the frequency of the source.

Hence

$$\frac{\lambda_A}{\lambda_B} = \frac{v_A}{v_B} = \frac{d_B}{d_A} = \frac{0.5 \times 10^{-3}}{10^{-3}} = 0.5$$

Velocity of Sound (longitudinal) Waves in a Medium

Let us try to guess what factors are likely to influence sound velocity in a medium. We can visualize the molecules of the medium (solid, liquid or gas) as small balls coupled to each other by small springs. The springs represent intermolecular forces. When a ball is pushed to the right, a compression wave will travel in the direction in which the ball oscillates. The wave velocity will obviously depend on the mass of the balls. The heavier the balls, the greater will be their inertia and the less the acceleration produced by a given force (Newton's second law) and hence the longer will be the time required for the disturbance to travel. Thus, v should depend on the density of the medium and we expect the wave velocity to be less for a denser medium. Secondly, if the springs are stiffer (greater intermolecular attraction), the larger will be the restoring force and we expect a greater wave velocity. The modulus of elasticity E is a measure of the 'springiness' of a medium. Hence, we expect v to depend on E. Thus we write

$$v = KE^a \rho^b$$

$$\text{Dimensions of } E = \frac{\text{stress}}{\text{strain}} = \frac{\text{force}}{\text{area}}$$

$$= \frac{\mathrm{MLT^{-2}}}{\mathrm{L^2}} = (\mathrm{ML^{-1}\,T^{-2}})$$

$$\text{Dimension of } \rho = (\mathrm{ML^{-3}})$$

Therefore, we have,

$$(\mathrm{LT^{-1}}) = (\mathrm{MT^{-1}\,T^{-2}})^a\,(\mathrm{ML^{-3}})^b$$

$$= (\mathrm{M}^{a+b}\,\mathrm{L}^{-a-3b}\,\mathrm{L}^{-2a})$$

$$\therefore \quad a + b = 0$$

$$-a - 3b = 1$$

$$-2a = -1$$

which give $a = \frac{1}{2} \quad b = -\frac{1}{2}$

Hence $v = K\sqrt{\dfrac{E}{\rho}}$

The dimensionless constant K, as before, can be determined either by an elaborate theory or by an experiment which gives $K = 1$.

$$\therefore \quad v = \sqrt{\frac{E}{\rho}} \tag{13.15}$$

For a solid rod in which the wave travels along the rod, the relevant modulus of elasticity is Young's modulus Y. Thus for a solid,

$$v = \sqrt{\frac{Y}{\rho}} \tag{13.16}$$

For a gas or a liquid where the wave propagates in all directions, the relevant modulus of elasticity is the bulk modulus B,

$$v = \sqrt{\frac{B}{\rho}} \tag{13.17}$$

If the modulus of elasticity is expressed in Nm^{-2} and density in kg m^{-3}, the wave velocity v will be in m s^{-1}.

EXAMPLE 13.6 Calculate the velocity of sound in (a) water and (b) steel. Given density of steel = 7800 kg m^{-3}, Young's modulus of steel = 20×10^{10} Nm^{-2} and bulk modulus of water = 0.20×10^{10} Nm^{-2}.

Solution (a) $E\,(\text{water}) = 0.20 \times 10^{10}\ \text{Nm}^{-2}$

$$\rho\,(\text{water}) = 1000\ \text{kg m}^{-3}$$

$$v\,(\text{water}) = \sqrt{\frac{0.20 \times 10^{10}}{1000}} = 1414\ \text{m s}^{-1}$$

(b) $Y\,(\text{steel}) = 20 \times 10^{10}\ \text{Nm}^{-2}$

$$\rho\,(\text{steel}) = 7800\ \text{kg m}^{-3}$$

$$v\,(\text{steel}) = \sqrt{\frac{20 \times 10^{10}}{9800}} = 5060\ \text{m s}^{-1}$$

The experimental value at 0 °C is $5.10 \times 10^{3}\ \text{m s}^{-1}$

The following table gives measured values of the velocity of sound in various substances at 0 °C.

Table 13.1 Velocity of sound in various substances at 0 °C (metres per second)

Substance	*Velocity*	*Substance*	*Velocity*
Air	332	steel	4975
Hydrogen	1268	lead	1420
Carbon dioxide	259	glass	4860
Fresh water	1435	wood	4000
Sea water	1454	rubber	5000
Mercury	1484	—	—

13.11 NEWTON'S FORMULA AND LAPLACE'S CORRECTION

Newton gave the first theoretical expression for the velocity of sound wave in air. He assumed that when a sound wave travels through a gaseous medium, the temperature variations in the regions of compression and rarefaction are negligible. The conditions are, therefore, isothermal and Boyle's law can be applied. Consider a volume V of a gas at pressure P. As a result of sound having travelled in it, let the change in pressure be ΔP and the corresponding change in volume by ΔV. From Boyle's law, we have

$$PV = \text{constant (isothermal change)}$$

Differentiating partially we have

$$P.\,\Delta V + P.\,\Delta V = 0$$

or
$$P = + \frac{\Delta P}{-\,\Delta V/V} = \frac{\text{excess pressure}}{\text{volume strain}}$$
$$= B_i \text{ (by definition)}$$

where B_i is the volume elasticity (or bulk modulus) of the gas. when isothermal conditions hold. Remember when ΔP is positive, ΔV will be negative (compression) and vice versa. Hence, B_i is always positive. Therefore, under isothermal conditions

$$v = \sqrt{\frac{P}{\rho}} \qquad (13.18)$$

This is Newton' formula for the velocity of sound in a gas, For air at standard temperature and pressure (STP), the density $\rho = 1.29 \text{ kg m}^{-3}$ and pressure $P = 0.76$ m of mercury $= 0.76 \times (13.6 \times 10^3) \times 9.8$ Nm^{-2}. Substituting for P and ρ in formula (13.18) the velocity of sound in air at STP is equal to

$$v = \left(\frac{0.76 \times 13.6 \times 10^3 \times 9.8}{1.29}\right)^{1/2}$$
$$= 280 \text{ m s}^{-1}$$

The experimental value, from various experiments, for the velocity of sound at STP is 332 m s^{-1} which is considerably higher (by about 16%) than the above theoretical value obtained from Newton's formula. Newton was unable to give a satisfactory explanation for this discrepancy. The first satisfactory explanation was given by a French scientist, Laplace, in 1816.

Laplace's Correction

As a sound wave travels in a gas, the region of compression is heated and that of rarefaction cooled. The thermal conductivity of a gas is small and these thermal changes occur so rapidly that the heat developed in compression and cooling produced in rarefaction is not transferred out or in to achieve thermalization during the short time-scale. The time-scale is the time required by sound to travel from compression to rarefaction. The wave 'sees' regions of unequal temperature. Laplace pointed out that the thermal changes are adiabatic which obey

$$PV^{\gamma} = \text{constant}$$

where $\gamma = \frac{C_p}{C_v}$, C_p and C_v being specific heats of gas at constant pressure and at constant volume respectively. Differentiating partially we get,

$$P \times \gamma V^{\gamma-1}\,\Delta V + \Delta P V^{\gamma} = 0$$

or

$$\gamma P = -\frac{\Delta P V^{\gamma}}{V^{\gamma-1}\,\Delta V} = \frac{\Delta P}{-\Delta V/V}$$

$$= B_a$$

where B_a is the volume elasticity, the subscript a refers to the adiabatic change.

Thus

$$v = \sqrt{\frac{\gamma P}{\rho}} \qquad (13.19)$$

Taking $\gamma = 1.4$ for air, velocity of sound in air at STP is

$$v = \left(\frac{1.4 \times 0.76 \times 13.6 \times 10^3 \times 9.8}{1.29}\right)^{1/2} = 331 \text{ m s}^{-1}$$

which is in close agreement with the experimental value, thereby establishing the correctness of Laplace's explanation.

13.12 EFFECT OF CHANGE OF PRESSURE, TEMPERATURE AND HUMIDITY ON VELOCITY OF SOUND IN A GAS

Change of Pressure

The pressure P and volume V of a gas of mass m at a given temperature are related as

$$PV = \text{constant}$$

or

$$\frac{Pm}{\rho} = \text{constant, where } \rho \text{ is the density of the gas}$$

or

$$\frac{P}{\rho} = \text{constant}$$

Hence the velocity of sound in a gas is independent of its pressure.

Change of Temperature

Let v_t and v_0 be the velocity of sound in gas at t °C or 0 °C respectively. From Eq. 13.19 we have

$$v_t = \sqrt{\frac{\gamma P}{\rho}}$$

and $$v_0 = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

or $$\frac{v_t}{v_0} = \sqrt{\frac{P}{P_0} \cdot \frac{\rho_0}{\rho}} \qquad (13.20)$$

From the gas equation,

$$\frac{P_0 V_0}{T} = \frac{PV}{T}$$

where $T = (273 + t)$ °C and $T_0 = 273$ °C are absolute temperatures at t °C and 0 °C.

or $$\frac{P_0 m}{\rho_0 T_0} = \frac{Pm}{\rho T}$$

or $$\frac{P}{P_0} \cdot \frac{\rho_0}{\rho} = \frac{T}{T_0}$$

Substituting in Eq. (13.20) we have

$$\frac{v_t}{v_0} = \sqrt{\frac{T}{T_0}} = \left(\frac{273 + t}{273}\right)^{1/2} = \left(1 + \frac{t}{273}\right)^{1/2} \qquad (13.21)$$

Thus, the velocity of sound in a gas is proportional to the square root of its absolute temperature. For small changes in temperature $\frac{t}{273} \ll 1$ and we can use the binomial expansion to write,

$v_t = v_0 \left(1 + \frac{1}{2} \cdot \frac{t}{273}\right)$, terms of order $\left(\frac{t}{273}\right)^2$ and higher are neglected.

$v_t = v_0 + 0.61t$, since for air $v_0 = 332$ m s^{-1}. Thus, the velocity of sound in air increases by about 0.61 m s^{-1} for each degree Celsius rise in temperature.

Change of Humidity

The presence of water vapour in a gas lowers its density. Consequently, sound travels faster in humid gas than in dry gas under the same conditions. An increase in humidity increases the speed of sound.

EXAMPLE 13.7 A string of mass 2.50 kg stretches to a length 20.0 m when subjected to a tension of 200 N. A transverse pulse is created at one end of the string. How long does it take to reach the other end of the string?

Solution

$$\text{Mass per unit length } (m) = \frac{2.5 \text{ kg}}{20 \text{ m}} = 0.125 \text{ kg m}^{-1}$$

$$\text{Tension } (T) = 200 \text{ N}$$

$$\therefore \qquad \text{Speed of the pulse } (v) = \sqrt{\frac{T}{m}}$$

$$= \sqrt{\frac{200}{0.125}}$$

$$= 40 \text{ m s}^{-1}$$

∴ Time taken by the pulse to travel the entire length (20 m) of the string $= \dfrac{20 \text{ m}}{40 \text{ m s}^{-1}} = 0.5$ s.

EXAMPLE 13.8 A person drops a stone from the top of a tower 340 m high which splashes into the water of a pond near the base of the tower. If the speed of sound in air is 340 m s^{-1}, find the time taken between the dropping of the stone and the hearing of the splash by him. Take $g = 9.8$ m s^{-2}.

Solution The time taken is $t = t_1 + t_2$, where t_1 is the time taken by the stone to fall through a distance h, the height of the tower and t_2 is the time taken by the sound to travel a distance $= h$. These times are given by

$$h = \tfrac{1}{2} g t_1^2$$

and

$$h = v t_2$$

where $v = 340$ m s^{-1} is the speed of sound. Therefore

$$t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 340}{9.8}} = 8.33 \text{ s}$$

and

$$t_2 = \frac{h}{v} = \frac{340 \text{ m}}{340 \text{ m s}^{-1}} = 1.0 \text{ s}$$

$$\therefore \qquad t = t_1 + t_2 = 9.33 \text{ s}$$

EXAMPLE 13.9 Compare the velocities of sound in hydrogen (H_2) and carbon dioxide (CO_2). The ratio (γ) of specific heats of H_2 and CO_2 are respectively 1.4 and 1.3.

Solution

$$v_1 = \sqrt{\frac{\gamma_1 P}{\rho_1}} \qquad \text{and} \qquad v_2 = \sqrt{\frac{\gamma_2 P}{\rho_2}}$$

$$\therefore \qquad \frac{v_1}{v_2} = \sqrt{\frac{\gamma_1 \rho_2}{\gamma_2 \rho_1}}$$

Since density of a gas is proportional to its molecular weight,

$$\frac{\rho_2}{\rho_1} = \frac{44.01}{2.016} = 21.83$$

$$\therefore \qquad \frac{v_1}{v_2} = \left(\frac{1.4}{1.3} \times 21.83\right)^{1/2} = 4.85$$

Velocity of sound in hydrogen is 4.85 times that in carbon dioxide.

Example 13.10 The velocity of sound in air at 15 °C is 340 m s^{-1}. Calculate the velocity at (a) 0 °C and (b) 30 °C. What is the velocity at 15 °C if the barometric pressure changes from 76 to 75 cm Hg?

Solution

$$\frac{v_t}{v_0} = \left(\frac{273 + t}{273}\right)^{1/2}$$

(a) When $t = 15$ °C, $v_t = 340$ m s^{-1}

$$v_0 = 340 \times \left(\frac{273}{273 + 15}\right)^{1/2} = 340 \times \sqrt{\frac{273}{288}} = 331 \text{ m s}^{-1}$$

(b) When $t = 30$ °C

$$v_{30} = v_0\left(\frac{273 + 30}{273}\right)^{1/2} = 331 \times \sqrt{\frac{303}{273}} = 349 \text{ m s}^{-1}$$

The velocity of sound is not affected by any change in pressure. At 15 °C, it remains 340 m s^{-1} when the atmospheric pressure changes from 76 to 75 cm Hg.

Example 13.11 In a laboratory experiment (room temperature being 15 °C) the wavelength of a note of sound of frequency 500 Hz is found to be 0.68 m. If the density of air at STP is 1.29 kg m^{-3}, calculate the ratio of the specific heats of air.

Solution

Velocity of sound (v_t) at 15 °C $= \nu\lambda = 500 \times 0.68 = 340$ m s^{-1}

Velocity (v_0) at 0 °C $= 340 \times \sqrt{\dfrac{273}{273 + 15}} = 331$ m s^{-1}

At STP

$$\rho_0 = 1.29 \text{ kg m}^{-3}$$

and $\qquad P_0 = 0.76 \times 13.6 \times 10^3 \times 9.8 \text{ N m}^{-2}$

Since $$v_0 = \sqrt{\frac{\gamma P_0}{\rho_0}}, \quad \gamma = \frac{C_p}{C_v}$$

$$\gamma = v_0^2 \frac{\rho_0}{P_0} = \frac{331 \times 331 \times 1.29}{0.76 \times 13.6 \times 10^3 \times 9.8} = 1.39$$

13.13 BOUNDARY EFFECTS

So far we have discussed waves that travel unobstructed in a medium. But the medium must end somewhere. For example, water waves in a pond will eventually reach the boundary of the pond. Similarly the waves on a string will reach the ends of the string. If a wave travelling in one medium meets the boundary of another medium, it is partly reflected back into the first medium and partly transmitted (or refracted) into the second medium. We shall first discuss reflection of waves.

Reflection of Waves

It is instructive to consider what happens when a pulse travelling on a string encounters a discontinuity, which we will take to be a rigid wall where the end of the string is fixed as shown in Fig. 13.8. When the pulse travelling from the left reaches the point of attachment, the string exerts an upward force on the wall. By Newton's third law of motion, the wall, in turn, merely by remaining stationary, exert a downward force on the string. The result is that the string is accelerated in the downward direction, thus creating a reflected pulse, with negative displacement, that travels in the reverse direction. Therefore, a crest is reflected as a trough.

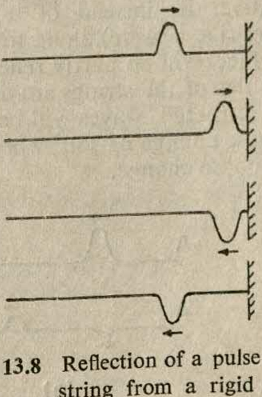


Fig. 13.8 Reflection of a pulse in a string from a rigid wall shown in four stages at equal intervals of time

If, instead of a pulse, we have a harmonic wave incident on the wall, a reflected harmonic wave will travel in the opposite direction in which the particle displacements are reversed. Thus, if the particle displacements in the incident wave (which travels in + x direction) are given by

$$y_i(x, t) = A \sin \left\{ \frac{2\pi}{\lambda} (vt - x) \right\}, \tag{13.22}$$

the particle displacements in the reflected wave (which travels in the − x direction) will be given by

$$y_r(x, t) = -A \sin \left\{ \frac{2\pi}{\lambda} (vt + x) \right\} \tag{13.23}$$

Since the reflected wave travels in the same medium as the incident wave, its speed, wavelength and frequency will be the same as those of the incident wave. Since

$$-A\sin\left\{\frac{2\pi}{\lambda}(vt+x)\right\} = +A\sin\left\{\frac{2\pi}{\lambda}(vt+x)\pm\pi\right\}$$

the reflected wave suffers a phase change of π radians with respect to the incident wave. The reflected wave obeys the usual laws of reflection. These conclusions hold not only for waves on a string but also for all types of waves including sound and light waves.

Refraction of Waves

We have discussed above the mechanism of reflection of a pulse in a stretched string at an infinitely massive wall. Let us see what happens when a positive pulse, started at the free end of a light string, meets a heavier string which is joined to the other end of the lighter string (Fig. 13.9a). Let us assume that both strings are stretched with the same tension. The linear density (mass per unit length) of string A is less than that of string B. When the pulse reaches the point of attachment of the strings, it is partly transmitted and partly reflected. The transmitted (or refracted) pulse is positive and the reflected pulse is negative. If, instead of a single pulse, we send a periodic series of pulses (i.e. a wave) along the string A, this wave on reaching the interface will be partly reflected and partly refracted. Since the linear densities of the strings are different, the velocities of the reflected and the refracted waves will be different. Also, the reflected wave suffers a phase change of 180° whereas the refracted wave does not undergo any phase change.

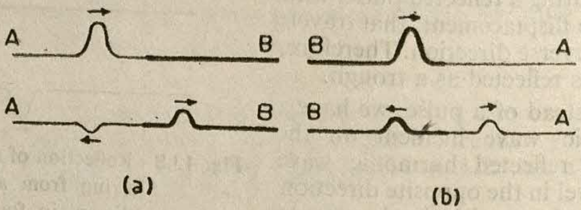


Fig. 13.9 (a) Reflection and refraction of a pulse when the second string is denser than the first. Note a phase change of 180° in the reflected wave (b) Reflected and refracted pulses when the second string is lighter than the first. (Note that the reflected wave does not undergo any phase change)

We can do the two-string experiment by sending the incident pulse in the heavier string as illustrated in Fig. 13.9b. The pulse is partly reflected and partly transmitted. The reflected wave is not inverted in this case, indicating that no phase change occurs when a pulse, travelling in a denser string, is reflected at the boundary of the lighter string. These conclusions about one-dimensional waves on strings hold for all types of waves.

Note The wave velocity and the wavelength of the refracted wave are different from those of the incident wave, their frequencies, however, are equal. The refracted wave obeys Snell's law of refraction. Since the frequencies of the incident and refracted waves are equal, we have

$$\nu = \frac{v_i}{\lambda_i} = \frac{v_r}{\lambda_r} \tag{13.24}$$

where subscripts *i* and *r* stand for the incident and refracted waves respectively.

EXAMPLE 13.12 A bat emits ultrasonic sound of frequency 102 kHz in air. If this sound meets a water surface, what is the wavelength of (a) the reflected sound and (b) the transmitted sound? Speed of sound in air = 340 m s^{-1} and in water = 1428 m s^{-1}.

Solution

$$\text{Frequency } (\nu) = 102 \text{ kHz} = 1.02 \times 10^5 \text{ Hz}$$

(a) The reflected wave travels in air. From Eq. (13.24) its wavelength is

$$\lambda_a = \frac{v_a}{\nu} = \frac{340}{1.02 \times 10^5} = 3.33 \times 10^{-3} \text{ m}$$

(b) The transmitted wave travels in water, its wavelength will be

$$\lambda_w = \frac{v_w}{\nu} = \frac{1428}{1.02 \times 10^5} = 1.4 \times 10^{-2} \text{ m}$$

13.14 SUPERPOSITION OF WAVES

We have so far discussed the propagation of a single wave in a medium. In practice, however, one often comes across the possibility of two or more waves travelling through a medium simultaneously. A boatman may receive the sound wave from the siren of a ship directly from it as well as the wave reflected by the seawater. A floating log may simultaneously receive water waves created by two or more boats. The transverse wave on a string will eventually get reflected at the fixed end and we have two waves travelling in opposite directions on the string. We use the term *interference* to describe the physical effects of superposition of two or more waves travelling through the same region of space.

The Superposition Principle

Consider two waves travelling in a medium. Let us focus our attention on the motion of one particle of the medium. When a wave reaches this particle, it suffers one displacement. When two waves simultaneously cross this particle, it suffers two displacements, one due to each wave. *The resultant displacement of the particle is given by the algebraic sum of the individual displacements given to it by the two waves.* This is the principle of superposition.

Superposition of Two Pulses

Let us apply the superposition principle to find out what happens when two pulses of identical shapes travelling in opposite directions on a string cross each other. Figure 13.10 shows the two pulses moving towards each other with a speed of 1 m s^{-1}. Figure 13.10 a to *f* show the wave form after every second. They cross each other between $t = 2$ s and $t = 3$ s as shown in Figs. 13.10 c and 13.10 d. These figures show that when the two pulses overlap (or superpose) the resultant wave profile is the algebraic sum of the displacements due to each pulse.

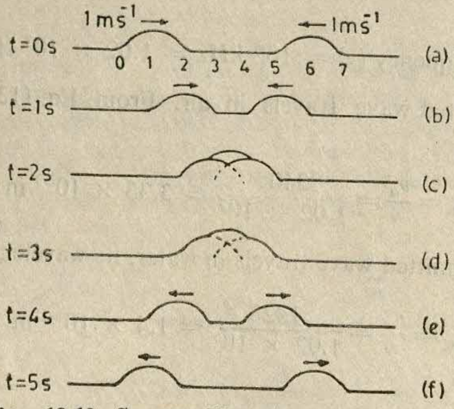


Fig. 13.10 Superposition of two identical pulses travelling in opposite directions

Figure 13.11 shows what happens when two equal and opposite pulses moving in opposite directions cross each other. In Fig. 14.11 c we see that there is an instant when the string appears undisturbed. In this situation the positive pulse overlaps the negative pulse and seem to cancel each other. This again shows that the resultant wave profile is the algebraic sum of individual waves.

From Figs 13.10 and 13.11 we notice that two pulses continue to retain their individual shapes afer crossing each other. However, at the instant they cross each other, the appearance of the wave profile is different from the shape of either individual pulse.

Figures 13.10 c and d are examples of *constructive* interference and Fig. 13.11 c is an example of *destructive interference.*

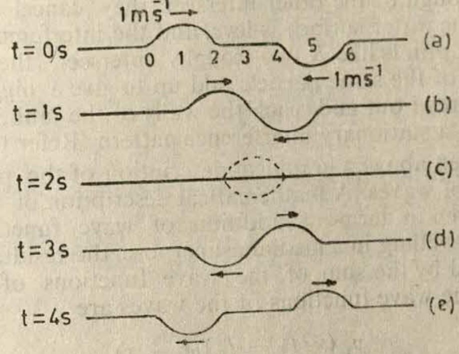


Fig. 13.11 Superposition of two equal and opposite pulses travelling in opposite directions

Superposition of Harmonic Waves

You must have observed the interference pattern produced when circular waves, resulting from dropping two stones into a pond of water, cross over each other. In the laboratory this interference can be demonstrated by attaching a stylus to each prong of a tuning fork and placing it over a big ripple tank. The fork is set into vibration and each stylus is brought into contact with water. Two sets of concentric circular waves start simultaneously from the point of contact of each stylus (Fig. 13.12). The wave trains travel in the form of crests and troughs on the surface of water. The solid circles represent crests and the dotted ones troughs.

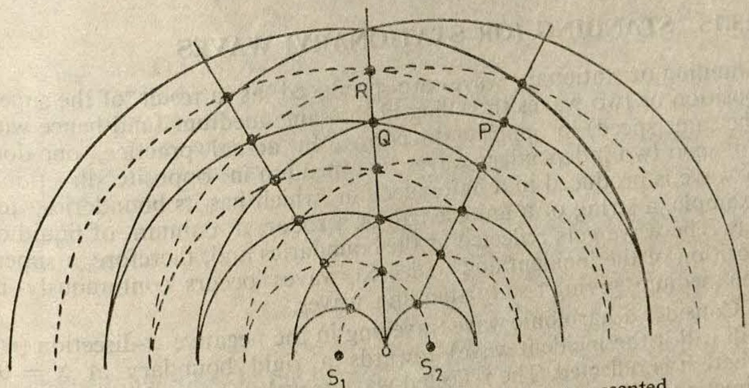


Fig. 13.12 Interference pattern, (Crests are represented by solid circles and troughs by dotted ones)

At points like Q where two crests intersect, a larger crest is formed, the displacements of the particle due to two waves adding up. This is called *constructive* interference. At points like P where a crest of one wave and a trough of the other intersect, they cancel each other. At these points the water surface is level and the interference is said to be *destructive*. At points like R two troughs intersect, the two negative displacements of the same particle add up to give a bigger trough. As these waves spread out and reach the walls of the tank, they get reflected and we get a stationary interference pattern (Refer to Sec. 13.15).

We have given above a graphical description of the phenomenon of superposition of waves. A mathematical description of the superposition can be given in terms of addition of wave functions. If two or more waves travelling in a medium superpose, the resultant wave function is obtained by the sum of the wave functions of the individual waves, i.e. if the wave functions of the waves are

$$\begin{aligned} y_1(x,t) &= f_1(vt-x) \\ y_2(x,t) &= f_2(vt-x) \\ \vdots\quad & \quad\ \vdots \\ y_n(x,t) &= f_n(vt-x) \end{aligned}$$

then the resultant wave is given by

$$y = y_1(x,t) + y_2(x,t) + \ldots + y_n(x,t)$$

or $$y = f_1(vt-x) + f_2(vt-x) + \ldots + f_n(vt-x) \quad (13.25)$$

This is the principle of superposition. This principle holds only if the amplitude of the disturbance is not too large. In the following two sections we will use the principle of superposition to discuss standing waves and beats.

13.15 STANDING (OR STATIONARY) WAVES

Standing or stationary waves are produced as a result of the superposition of two waves travelling in the same medium (and hence with the same speed) in opposite directions. In actual practice, one does not send two independent waves in a medium in opposite directions. A wave is produced in a finite medium which has its boundaries, for example, a string of a finite length or a rodor a column of liquid or gas. The wave gets reflected at the boundaries and, therefore, a superposition of the incident and reflected waves occurs continuously in the medium, giving rise to standing waves.

Consider a harmonic wave travelling in the negative x-direction (we will call it the incident wave) towards a rigid boundary at $x = 0$, where it is reflected. The reflected wave travels along the positive x-direction with a reversal of amplitude.

The particle displacements y_i and y_r due to the incident and reflected waves respectively are given by

$$y_i = A \sin \left\{\frac{2\pi}{\lambda}(vt + x)\right\} \tag{13.26}$$

and

$$y_r = -A \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\} \tag{13.27}$$

where A is the amplitude, λ is the wavelength and v is the velocity of each wave.

Using the principle of superposition, the resultant wave y is given by

$$\begin{aligned} y &= y_i + y_r \\ &= A\left[\sin\left\{\frac{2\pi}{\lambda}(vt + x)\right\} - \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}\right] \end{aligned}$$

Recalling that $\sin\alpha - \sin\beta = +2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$

we have

$$y = 2A \sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi\, vt}{\lambda}\right) \tag{13.28}$$

which may also be written as

$$y = 2A \sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{T}\right)$$

where $T = \frac{\lambda}{v}$ is the period of each wave.

Equation (13.28) represents a standing or stationary wave. It is obvious that this equation does not represent a travelling wave, since it no longer has the characteristic form involving the combination $(vt - x)$ in the argument of the trigonometric function. Remember that only functions, such as $f(vt - x)$ or $g(vt + x)$, represent a travelling wave.

Let us see what happens at some arbitrary point x_1. For this point the equation is

$$y = \left(2A \sin\frac{2\pi x_1}{\lambda}\right)\cos\frac{2\pi t}{T}$$

Since all quantities in the brackets are constant, we can write it as

$$y = R\cos\frac{2\pi t}{T} = R\cos\omega t \qquad \left(\because\ \omega = \frac{2\pi}{T}\right)$$

where $R = 2A \sin\left(\frac{2\pi x_1}{\lambda}\right)$ is a constant for that point. This is the equation of an SHM whose amplitude R depends on the value of x chosen. It may be worthwhile to remark here that in a travelling harmonic wave also the motion of each particle is simple harmonic but the amplitude of the motion is the same for all values of x. Thus *standing waves are really not waves*. They can at most be described as simple harmonic motions of the particles of a medium; the different particles having different amplitudes but all having the same period T. Such simple harmonic motions of the particles of a medium are called *normal modes*. We will consider two cases by way of illustration.

Normal Modes of a Stretched String Fixed at Both Ends

Consider a uniform string of length L stretched by a tension T along the x-axis, with its ends rigidly fixed to $x = 0$ and $x = L$. A transverse wave is produced in the string by plucking it as in a sitar or guitar. Let us say that this wave travels along the string in the negative x direction and gets reflected when it meets the fixed end at $x = 0$.

Equation (13.28) represents the effect of superposition of these two waves where y is the resultant displacement. At the ends of the string where it is rigidly fixed, there can be no motion at all. Thus the two boundary conditions which must be satisfied are:

$$y = 0 \quad \text{at} \quad x = 0$$

and

$$y = 0 \quad \text{at} \quad x = L$$

The first boundary condition that y must vanish at $x = 0$ is automatically satisfied by Eq. 13.28). To satisfy the second boundary condition we put $x = L$ in Eq. (13.28) and require that y vanish at $x = L$ for all values of time t, i.e.

$$2A \sin\left(\frac{2\pi L}{\lambda}\right) \cos\left(\frac{2\pi vt}{\lambda}\right) = 0$$

This requirement can be satisfied for all values of *t only* if

$$\sin\left(\frac{2\pi L}{\lambda}\right) = 0$$

or

$$\frac{2\pi L}{\lambda} = 0, \pi, \; 2\pi \ldots, \text{etc.}$$

or

$$\frac{2\pi L}{\lambda} = n\pi \;\; ; \;\; n = 0, 1, 2, 3, \ldots, \text{etc.}$$

For each value of n there is a corresponding value of λ. So we can write

$$\frac{2\pi L}{\lambda_n} = n\pi$$

or

$$\lambda_n = \frac{2L}{n} \tag{13.29}$$

The frequency of vibration of the string is then given by

$$\nu_n = \frac{v}{\lambda_n}$$

where v is the speed of the transverse waves on the string and is given by [see Eq. (13.14)]

$$v = \sqrt{\frac{T}{m}}$$

where m is the linear mass density (i.e. mass per unit length) of the string and T is the tension. Thus we have

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{m}} \tag{13.30}$$

Notice that the value of $n = 0$ is unphysical since then the wavelength $\lambda_0 \to \infty$ and the frequency $\nu_0 = 0$. We will now discuss the various *normal modes* of vibrations of the string. Notice that λ cannot take all values; it can take only discrete values given by Eq. (13.29) corresponding to $n = 1, 2, 3, \ldots$, etc. The fact that λ_n and hence the frequency ν_n can have only some permitted values is a consequence of the bonndary conditions which determine what frequencies will the normal modes have.

For a given value of n, the resultant particle displacements are obtained from Eq. (13.28) by using Eq. (13.29), which gives

$$y = 2A \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{2\pi t}{T_n}\right) \tag{13.31}$$

where $T_n \left(= \frac{v}{\lambda_n} = \frac{1}{\nu_n}\right)$ is the period of vibrations for the given value of n.

$n = 1$ (*Fundamental mode or first Harmonic*) Setting $n = 1$ in Eqs (13.29), (13.30) and (13.31) we have

$$\lambda_1 = 2L \tag{13.32}$$

$$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{m}} \tag{13.33}$$

and $$y = 2A \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{2\pi t}{T_1}\right) \tag{13.34}$$

or $$y = R \cos\left(\frac{2\pi t}{T_1}\right)$$

where $R = 2A \sin\left(\frac{\pi x}{L}\right)$ is the amplitude in this mode. Notice that the displacement y given by Eq. (13.34) is zero at $x = 0$ and $x = L$ for all values of t; as it should be because these points are fixed. There is no other value x between $x = 0$ and $x = L$ where y can vanish. Also notice that amplitude R is maximum at $x = \frac{L}{2}$. The points of the medium which are permanently at rest are called *nodes.* The points where the amplitude is maximum (+ve or −ve) are called *antinodes.*

Thus we find that for $n = 1$, there is a node at each fixed end and an antinode in the middle. The vibration given by Eq. (13.34) is depicted in Fig. 13.13a at $t = 0, T_1/4, T_1/2, 3T_1/4$ and T_1. The string vibrates in one segment with a frequency ν_1 given by Eq. (13.33). This is called the *fundamental mode* or the *first harmonic* of vibration of the string.

n = *2* (*Second Harmonic*) When $n = 2$, Eqs (13.29), (13.30) and 13.31) give

$$\lambda_2 = L \tag{13.35}$$

$$\nu_2 = \frac{1}{L}\sqrt{\frac{T}{m}} \tag{13.36}$$

and $$y = 2A \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi t}{T_2}\right) \tag{13.37}$$

where $T_2 = 1/\nu_2$ is the period of vibration.

Notice from Eq. (13.37) that $y = 0$ at all times for $x = 0, \frac{L}{2}$, and L and the amplitude of vibration is maximum (+ve or −ve) at $x = \frac{L}{4}$ and $\frac{3L}{4}$. Thus there are three nodes (at $x = 0, \frac{L}{2}$ and L) and two antinodes$\left(\text{at } x = \frac{L}{4} \text{ and } \frac{3L}{4}\right)$. Notice an additional node in the middle point $x = \frac{L}{2}$ which is permanently at rest; this point corresponds to destructive interference. The vibration of the string given by Eq. (13.37) is depicted in Fig. 13.13b at $t = 0, \frac{T_2}{4}, \frac{T_2}{2}, \frac{3T_2}{4}$ and T_2. The string vibrates in two segments with a frequency ν_2 given by Eq. (13.36).

n = 3 (*Third Harmonic*) Figure 13.13 c shows the normal mode of vibration of the string for $n = 3$. The string vibrates in three segments with a frequency $\nu_3 = \frac{3}{2L}\sqrt{\frac{T}{m}}$. This is called the third harmonic. The frequencies and shapes of higher harmonic can be similarly obtained.

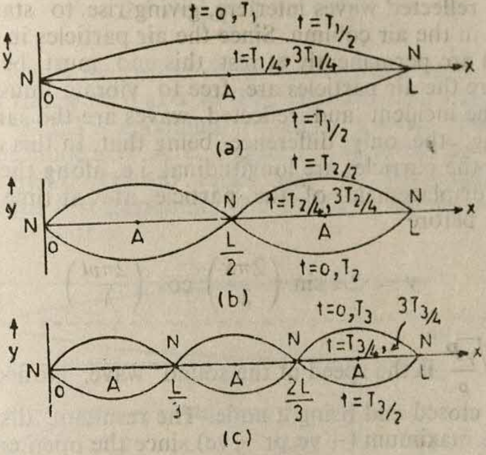


Fig. 13.13 First three normal modes of a string fixed at its ends

Thus, there are various normal modes of vibrations of a stretched string fixed at both ends. The frequencies $\nu_1, \nu_2, \nu_3, \ldots$, etc. of these modes are in the ratio of $1 : 2 : 3, \ldots$

In our analysis so far we have considered a superposition of only one pair of waves, namely, the incident wave travelling from right to left and the reflected wave travelling from left to right. We must bear in mind that the reflected wave will be again reflected at the end $x = L$ and will travel in the direction of the previous incident wave. There will be numerous such reflections and we must consider the superposition of numerous pairs of waves. Fortunately, it turns out that the standing wave patterns depicted in Fig. 13.13 are not materially modified. The reason is not hard to seek. Take, for example, the fundamental mode. For this mode the phase difference between the first incident wave and the second incident wave (i.e. the first incident wave after a reflection at $x = 0$ and another at $x = L$) is equal to $2\pi/\lambda_1$ times the extra path, i.e. $2L$ traversed. Thus, this phase difference $= \frac{2\pi}{\lambda_1} \cdot 2L = \frac{2\pi}{2L} \cdot 2L = 2\pi$. Hence, the second incident wave is exactly like the first. The same argument holds for the first and second reflected waves. This is true not only for the fundamental mode but also for higher harmonics.

Normal Modes of an Air Column in a Pipe

A detailed study of the vibrations of a bounded volume of air is essential for the understanding of wind instruments, such as the flute, pipe, etc. Consider a cylindrical pipe of length L lying along the x-axis, one of the ends ($x = 0$) being closed and the other ($x = L$) open. A sound wave is sent along the pipe. This is reflected at the closed end. The direct and the reflected waves interfere, giving rise to standing longitudinal waves in the air column. Since the air particles in contact with the closed end are permanently at rest, this end must be a node. The open end where the air particles are free to vibrate must be an antinode. Thus, the incident and reflected waves are the same as in the case of a string, the only difference being that, in this case, the displacements of the particles are longitudinal, i.e. along the x-direction. The resultant displacement of the particle at x at time t is given by Eq. (13.28) as before

$$y = 2A \sin\left(\frac{2\pi x}{\lambda}\right) \cos\left(\frac{2\pi v t}{\lambda}\right) \qquad (13.38)$$

where $v = \sqrt{\frac{\gamma P}{\rho}}$ is the speed of the sound wave. Notice that $y = 0$ at $x = 0$, the closed end being a node. The resultant displacement at $x = L$ will be maximum (+ve or −ve) since the open end is a free or loose boundary. From Eq. (13.38) we see that this condition is satisfied if

$$\sin\frac{2\pi L}{\lambda} = \pm 1$$

or

$$\frac{2\pi L}{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

or

$$\lambda_p = \frac{4L}{p} \qquad (13.39)$$

where $p = 1, 3, 5, \ldots$. The corresponding frequency is given by

$$\nu_p = \frac{v}{\lambda_p} = \frac{pv}{4L} \qquad (13.40)$$

Using Eq. (13.39) in Eq. (13.38) we have $\left(\because \frac{v}{\lambda} = \frac{1}{T}\right)$

$$y = 2A \sin\left(\frac{p\pi x}{2L}\right) \times \cos\left(\frac{2\pi t}{T_p}\right) \qquad (13.41)$$

The normal modes of vibration of the air column in a closed pipe are as follows.

p = 1 (*Fundamental mode or First Harmonic*) Putting $p = 1$ inE qs (13.39) to (13.41) we get

$$\lambda_1 = 4L$$

$$\nu_1 = \frac{v}{4L} \qquad (13.42)$$

$$y = 2A \sin\left(\frac{\pi x}{2L}\right) \cos\left(\frac{2\pi t}{T_1}\right) \qquad (13.43)$$

where $T_1 = 1/\nu_1$ is the period of the fundamental mode. Notice from this equation that $y = 0$ at $x = 0$ for all values of t and the amplitude $2A \sin(\pi x/2L)$ is maximum ($= 2A$) at $x = L$. There is a node at $x = 0$ and antinode at $x = L$. Figure 13.14a shows how the displacement y given by Eq. (13.43) varies with x. This is callad the fundamental mode (or first harmonic); the frequency of sound emitted by the pipe is $\nu_1 = v/4L$.

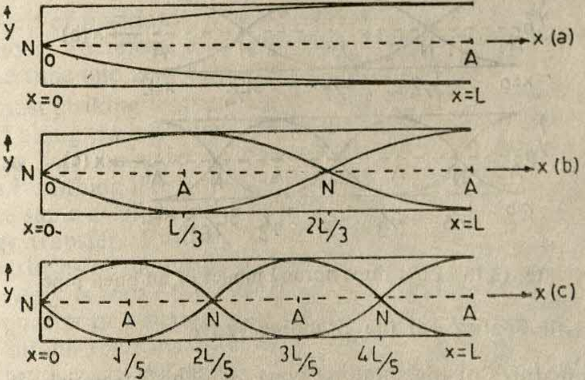


Fig. 13.14 First three normal modes of a closed pipe

p = 3 (*Third Harmonic*) Setting $p = 3$ in Eqs (13.40) and (13.41), the frequency and particle displacements are given by

$$\nu_3 = \frac{3v}{4L}$$

and
$$y = 2A \sin\left(\frac{3\pi x}{2L}\right) \cos\left(\frac{2\pi t}{T_3}\right)$$

From this equation we find that there are two nodes at $x = 0$ and $x = \frac{2L}{3}$ and two antinodes at $x = \frac{L}{3}$ and $x = L$. This mode is depicted in Fig. 13.14b. The frequency of sound emitted by the pipe in this mode is $\nu_3 = \frac{3v}{4L} = 3\nu_1$.

$n = 5$ (*Fifth harmonic*) The fifth harmonic is depicted in Fig. 13.14c. The frequency of sound emitted by the pipe in this mode is $\nu_5 = 5v/4L = 5\nu_1$ and so on for higher harmonics.

Thus in a closed pipe, the frequencies of the normal modes of vibration of the air column are in the ratio of 1 : 3 : 5, : . . . ., etc. Only odd harmonics are present.

The shapes and frequencies of the first three normal modes in an open pipe are shown in Fig. 13.15. The frequency of the fundamental mode is $\nu_1 = v/2L$. The frequencies of the higher harmonics are $2\nu_1$, $3\nu_1$, $4\nu_1$, ... Thus, in an open pipe, all harmonics (even as well as odd) are present.

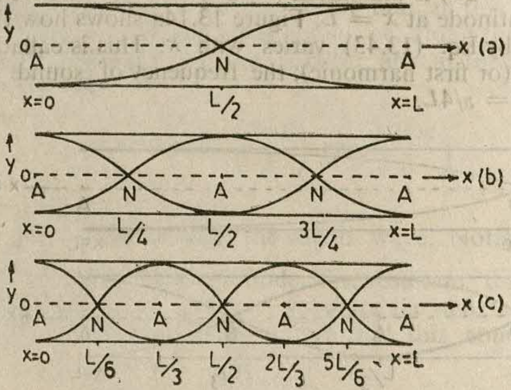


Fig. 13.15 First three normal modes of an open pipe

Characteristic Features of the Standing Wave

The main features of the standing wave, which distinguish it from a travelling wave are as follows:

1. In a travelling wave the amplitude of particle oscillation is the same at all positions of the medium. On the other hand, in a stationary (or standing) wave, the amplitudes of oscillations are different at different places. At nodes the amplitude is zero and at antinodes the amplitude is maximum, being equal to the sum of the amplitudes of the constituent waves. At other intermediate positions the amplitude varies between these two limits. The distance between two consecutive nodes or antinodes is half the wavelength of the constituent waves.

2. In a travelling wave the particle velocity is given by

$$\frac{dy}{dt} = A \cdot \frac{2\pi v}{\lambda} \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

The velocity varies from place to place and from time to time. The maximum $\left(= \pm \dfrac{2\pi A v}{\lambda} = \pm 2\pi A\nu\right)$ and minimum (= 0) values of velocity are the same for all particles but are reached at different times. On the other hand, in a standing wave the particle velocity is given by

$$\frac{dy}{dt} = -2A \sin\left(\frac{2\pi x}{\lambda}\right) \sin\left(\frac{2\pi vt}{\lambda}\right)$$

The velocity varies from place to place and from time to time. The maximum and minimum values of particle velocity are different for different positions but are reached at the same time. The velocity is maximum at the antinodes and zero at the nodes.

3. In a travelling wave, the phase of particle oscillations is $\frac{2\pi}{T}\left(t - \frac{x}{v}\right)$ which depends on both x and t. Therefore, different particles acquire their maximum displacement at different intervals of time. This interval is determined by the magnitude of the wave velocity. In a standing wave, the phase of particle oscillations is $\frac{2\pi t}{T}$, which is independent of position, i.e. at a given time the phase is the same at all positions. Therefore, all points acquire their maximum displacement at one time and zero displacement also at one time.

4. The most striking feature of a standing wave is that energy is not transported along the medium. The reason is that energy cannot conceivably flow past the nodes of the medium which are permanently at rest. There is nothing like a wavefront in a standing wave, since the phase is the same at all points, changing only with time. Hence, there is no energy transfer. Further, in a standing wave, twice in each cycle, all particles of the medium have zero displacement so that the potential energy is zero—the energy of the system is then entirely kinetic. A quarter of a period later, all particles of the medium simultaneously attain their maximum displacement and at this instant the kinetic energy is zero, the energy of the medium being entirely potential. At other instants the energy is partly potential and partly kinetic. Hence, in a standing wave, the energy periodically becomes entirely kinetic and entirely potential. In a travelling wave the average energy of the medium is half potential and half kinetic.

Example 13.13 Standing waves are produced by the superposition of two waves

$$y_1 = 0.05 \sin(3\pi t - 2x)$$

and

$$y_2 = 0.05 \sin(3\pi t + 2x)$$

where y and x are measured in metres and t in seconds. Find the amplitude of a particle at $x = 0.5$ m.

Solution The resultant displacement is given by

$$y = y_1 + y_2 = 0.05\{\sin(3\pi t - 2x) + \sin(3\pi t + 2x)\}$$

Using the trigonometric relation

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

we have $$y = 0.1 \cos 2x \cdot \sin 3\pi t$$

or $$y = R \sin 3\pi t$$

where R, the amplitude of standing waves, is given by $R = 0.1 \cos 2x$ with

$$x = 0.5 \text{ m}$$

$$\cos 2x = \cos(2 \times 0.5 \text{ rad})$$

$$= \cos(1 \text{ rad})$$

$$= \cos 57.3° = 0.54$$

$\therefore$ Amplitude R at $x = 0.5$ is $0.1 \times 0.54 = 0.054$ m

EXAMPLE 13.14 The transverse displacement of a string fixed at both ends is given by

$$y = 0.06 \sin\left(\frac{2\pi x}{3}\right) \cos(120\pi t) \qquad \text{(i)}$$

where x and y are in metres and t is in seconds. The length of the string is 1.5 m and its mass is 3.0×10^{-2} kg. Answer the following questions:

(a) Does function (i) represent a travelling or a stationary wave?
(b) What is the amplitude at a point $x = 0.5$ m?
(c) What is the velocity at point $x = 0.75$ m at time $t = 0.25$ s?
(d) Write down the equations of the component waves whose superposition gives the vibration given in Eq. (i) above. What are the wavelength, frequency and speed of propagation of each wave?
(e) Determine the tension in the string.
(f) Do all points on the string vibrate with the same (i) frequency, (ii) phase and (iii) amplitude?

Solution (a) The function (i) does not have the characteristic form $(vt + x)$ (in the argument of the trigonometric function) of a travelling wave. Hence the function (i) does not represent a travelling wave; it represents a stationary wave.

(b) Displacement is maximum when $\cos(120\pi t) = 1$. Hence the amplitude $= 0.06 \sin\left(\frac{2\pi x}{3}\right)$.

At $x = 0.5$ m, the amplitude $= 0.06 \sin\left(\frac{2\pi}{3} \times 0.5\right)$

$$= 0.06 \sin\left(\frac{\pi}{3}\right) = 0.052 \text{ m}$$

(c) The velocity at point x at time t is obtained by differentiating Eq. (i) with respect to t.

$$V = \frac{dy}{dt} = -(0.06 \times 120\pi) \sin\left(\frac{2\pi x}{3}\right) \sin(120\pi t)$$

Now at $t = 0.25$ s, $\sin(120\pi t) = \sin(30\pi) = 0$. Hence at $t = 0.25$s, the velocity is zero for all values of x including $x = 0.75$ m.

The stationary wave in the question may be written as

$$y = 2A \sin \alpha \cos \beta$$

where $A = 0.03$ m, $\alpha = \frac{2\pi x}{3}$ and $\beta = 120\pi t$.

Now $\quad 2A \sin \alpha \cos \beta = A \sin(\alpha + \beta) + A \sin(\alpha - \beta)$

$$= 0.03 \sin\left(\frac{2\pi x}{3} + 120\pi t\right) + 0.03 \sin\left(\frac{2\pi x}{3} - 120\pi t\right)$$

i.e. $\quad y = y_1 + y_2$

Hence the two component waves are

$$y_1 = 0.03 \sin\left(\frac{2\pi x}{3} + 120\pi t\right)$$

and

$$y_2 = 0.03 \sin\left(\frac{2\pi x}{3} - 120\pi t\right)$$

$$= -0.03 \sin\left(120\pi t - \frac{2\pi x}{3}\right)$$

Let λ be the wavelength, ν the frequency and v the speed of each wave. Then

$$\frac{2\pi}{\lambda} = \text{coefficient of } x \text{ in the argument of the sine function} = \frac{2\pi}{3}$$

or $\quad \lambda = 3$ m.

Also $\quad \omega = 2\pi\nu =$ coefficient of t in the argument of the sine function

$$= 120\pi$$

which gives $\nu = 60$ Hz.

Hence $\quad v = \nu\lambda = 60 \times 3 = 180 \text{ m s}^{-1}$

(e) Mass per unit length $(m) = \frac{3.0 \times 10^{-2}}{1.5} = 2.0 \times 10^{-2} \text{ kg m}^{-1}$

We know that $v = \sqrt{\frac{T}{m}}$, where T is the tension in the string.

$$\therefore \quad T = mv^2 = 2.0 \times 10^{-2} \times (180)^2 = 648 \text{ N}$$

(f) In a stationary wave on a string, all points on the string vibrate with the same frequency and the same phase, but the amplitude is different at different points (i.e. different values of x). The amplitude is zero at nodes and maximum at antinodes.

EXAMPLE 13.15 A wire of density 9 g cm^{-3} is stretched between two clamps 100 cm apart while being subjected to an extension of 0.05 cm. What is the lowest frequency of transverse vibrations of the wire, assuming that the Young's modulus of the material of the wire = 0.9 × 10^{11} Nm^{-2}.

Solution Young's modulus

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{T/A}{l/L} = \frac{TL}{Al}$$

where T = tension, L = original length = 100 cm, l = extension = 0.05 cm and A = area of cross-section of the wire.

Hence,

$$\frac{T}{A} = \frac{Yl}{L} = \frac{0.9 \times 10^{11} \times 0.05}{100} = 4.5 \times 10^{7}$$

$$\text{Now mass per unit length } m = \frac{\text{mass}}{\text{length}}$$

$$= \frac{\text{volume} \times \text{density}}{\text{length}}$$

$$= \frac{\text{area} \times \text{length} \times \text{density}}{\text{length}}$$

$$= \text{area} \times \text{density}$$

$$= A\rho$$

Hence the lowest frequency (i.e. frequency of the fundamental model) is given by

$$\nu = \frac{1}{2L}\sqrt{\frac{T}{m}} = \frac{1}{2L}\sqrt{\frac{T}{A\rho}}$$

$$= \frac{1}{2 \times 1.00}\sqrt{\frac{4.5 \times 10^{7}}{9000}}$$

$$= 35.3 \text{ Hz}$$

EXAMPLE 13.16 A wire having a linear density of 0.05 g cm^{-1} is stretched between two rigid supports with a tension of 4.5 × 10^{2} N. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the wire resonates is 490 Hz. Determine the length of the wire.

Solution Let 420 Hz be the pth harmonic, then 490 Hz the $(p+1)$th harmonic. Therefore

$$420 = \frac{p}{2L}\sqrt{\frac{T}{m}} \tag{i}$$

and

$$490 = \frac{p+1}{2L}\sqrt{\frac{T}{m}} \tag{ii}$$

Dividing (i) and (ii) we have

$$\frac{490}{420} = \frac{p+1}{p} \text{ giving } p = 6$$

Substituting this value of p in Eq. (i) we get ($m = 0.05 \text{ g cm}^{-1} = 0.05 \times 10^{-1} \text{ kg m}^{-1}$)

$$420 = \frac{6}{2L} \times \left(\frac{4.5 \times 10^2}{0.05 \times 10^{-1}}\right)^{1/2}$$

which gives

$$L = 2.14 \text{ m}$$

EXAMPLE 13.17 A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source? Will the same source be in resonance with the pipe if both ends are open? Speed of sound = 340 m s^{-1}.

Solution Let N be the frequency of the source and ν_p that of the pth harmonic of a closed pipe. The source will resonantly excite that harmonic mode of the pipe for which

$$N = \nu_p$$

for any value of $p = 1, 3, 5, \ldots$. Now for a closed pipe we know that

$$\nu_p = \frac{pv}{4L}$$

Therefore, for resonance,

$$N = \frac{pv}{4L}$$

Putting $N = 430$ Hz, $v = 340 \text{ m s}^{-1}$ and $L = 20 \text{ cm} = 0.2 \text{ m}$ we have

$$430 = \frac{p \times 340}{4 \times 0.2}$$

which gives $p = 1.01 \simeq 1$

Hence the source of frequency 430 Hz will resonantly excite the first harmonic (i.e. the fundamental mode).

in an open pipe, the condition of resonance with the same source will be

$$N = \nu_n = \frac{nv}{2L}, \quad \text{where } n = 1, 2, 3, \ldots$$

$$\therefore \qquad n = \frac{N \times 2L}{v}$$

$$= \frac{430 \times 2 \times 0.2}{340} = 0.5$$

which is not an integer. Hence the source will not be in resonance with any harmonic of the open pipe.

EXAMPLE 13.18 A half-metre long tube open at one end, with a moveable piston shows resonance with a tuning fork of frequency 512 Hz when the tube length is 16.0 cm and 49.0 cm. Calculate the speed of sound at the temperature of the experiment and determine the end-correction. What is this correction for?

Solution End-correction.

In our discussion we have taken the open end of a closed pipe to be an antinode. This is not strictly true. In fact, the particles of air just at the open end are not perfectly free because of the restriction imposed by the pipe. The true antinode is slightly away from the open end as shown in Fig. 13.16. The distance *e* is called the end-correction. The effective length of the pipe = $(L + e)$.

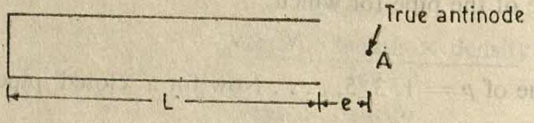


Fig. 13.16

The tuning fork is in resonance at two lengths of the pipe, viz. 16 cm and 49 cm. Since the second resonance length is about three times the first, it is clear that the fork is in resonance with the first harmonic (i.e. the fundamental mode) when $L_1 = 16$ cm and with the third harmonic when $L_2 = 49$ cm. Thus we have

$$N = \nu_1 \frac{v}{4\,(L_1 + e)} \tag{i}$$

Also

$$N = \nu_3 = \frac{3v}{4\,(L_2 + e)} \tag{ii}$$

where N is the frequency of the tuning fork. Dividing (ii) by (i) we get

$$3\,(16.0 + e) = 49.0 + e \qquad \text{or} \quad e = 0.5 \text{ cm}$$

Then the speed of sound as obtained from (i) is

$$\begin{aligned} v &= 4N(L_1 + e) \\ &= 4 \times 512 \times (16.0 + 0.5) \\ &= 33792 \text{ cms}^{-1} = 337.92 \text{ m s}^{-1} \\ &\qquad \simeq 338 \text{ m s}^{-1} \end{aligned}$$

13.16 BEATS

We will now apply the principle of superposition to another case of interference in which two harmonic waves of different frequencies superpose travelling in the same direction in the same medium. Consider two harmonic waves of frequencies ν_1 and ν_2 travelling in the $+x$ direction in a medium and reaching the point x of the medium simultaneously. The displacements of a particle located at x, due to the two waves, at any instant of time, will then be given by

$$y_1 = A_1 \sin\left\{\frac{2\pi}{\lambda_1}(vt - x)\right\}$$

and
$$y_2 = A_2 \sin\left\{\frac{2\pi}{\lambda_2}(vt - x)\right\}$$

Since the two waves travel in the same medium, their speeds are the same. Since their frequencies are different, their wavelengths will also be different. We will now make the following assumptions which simplify the analysis considerably without significantly altering the conclusions about beats. We assume that

(i) the amplitudes of the two interfering waves are equal, i.e. $A_1 = A_2 = A$.

(ii) the observation point is located at $x = 0$.

Under these assumptions, the above equations reduce to

($\because$ $v/\lambda_1 = \nu_1$ and $v/\lambda_2 = \nu_2$)

$$y_1 = A \sin(2\pi\nu_1 t) \tag{13.44}$$

and
$$y_2 = A \sin(2\pi\nu_2 t) \tag{13.45}$$

Using the principle of superposition, the resultant displacement is given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\{\sin(2\pi\nu_1 t) + \sin(2\pi\nu_2 t)\} \end{aligned}$$

or
$$y = 2A \cos\left\{2\pi\frac{(\nu_1 - \nu_2)}{2}t\right\} \sin\left\{2\pi\frac{(\nu_1 + \nu_2)}{2}t\right\} \tag{13.46}$$

Let us define

$$\nu_m = \tfrac{1}{2}(\nu_1 - \nu_2)$$

and
$$\nu_a = \tfrac{1}{2}(\nu_1 + \nu_2)$$

Quantity ν_m is called the 'modulation' frequency and ν_a the average frequency. In terms of ν_m and ν_a, Eq. (13.46) reads

$$y = 2A \cos (2\pi\nu_m t) \sin (2\pi\nu_a t)$$

or

$$y = R \sin (2\pi\nu_a t) \tag{13.47}$$

where

$$R = 2A \cos (2\pi\nu_m t) \tag{13.48}$$

The apparent resemblance of Eq. (13.47) with the equation of SHM is misleading. In fact the oscillation represented by Eq. (13.47) is not harmonic, since its 'amplitude' R given by Eq. (13.48) varies with time. If ν_1 is only slightly different from ν_2 so that ν_m $[= \frac{1}{2}(\nu_1 - \nu_2)]$ is very small compared to ν_a $[= \frac{1}{2}(\nu_1 + \nu_2)]$ then R will vary very slowly with time. Equation (13.47) will then represent a periodic rapid oscillation of frequency ν_a 'modulated' by a slowly varying oscillation of frequency ν_m.

The amplitude R of the resultant motion is maximum (+ ve or − ve) if [see Eq. (13.48)]

$$\cos 2\pi\nu_m t = \pm 1$$

or

$$2\pi\nu_m t = 0, \pi, 2\pi, \ldots$$

or

$$2\pi \frac{(\nu_1 - \nu_2)}{2} t = 0, \pi, 2\pi, \ldots$$

or

$$t = 0, \frac{1}{\nu_1 - \nu_1}, \frac{2}{\nu_1 - \nu_2}, \ldots$$

The time interval between two consecutive maxima is

$$t_b = \frac{1}{\nu_1 - \nu_2}$$

Hence the frequency of maxima, i.e. the number of times R becomes maximum per second is

$$\nu_b = \frac{1}{t_b} = \nu_1 - \nu_2$$

The amplitude R will be minimum (= zero) if

$$\cos (2\pi\nu_m t) = 0$$

or

$$2\pi\nu_m t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

or

$$t = \frac{1}{2(\nu_1 - \nu_2)}, \frac{3}{2(\nu_1 - \nu_2)}, \frac{5}{2(\nu_1 - \nu_2)}, \ldots$$

Hence the frequency of minima is also

$$\nu_b = \nu_1 - \nu_2$$

Beats are most easily observed in sound waves. Since the loudness (or intensity) of a wave is proportional to the square of its amplitude, the intensity of the resulting *periodic* but not harmonic wave given by Eq. (13.47) rises and falls in a time interval $t_b = 1/(\nu_1 - \nu_2)$ or at a frequency of $\nu_b = (\nu_1 - \nu_2)$. *The periodic variations of the intensity of the wave resulting from the superposition of two waves of different frequencies is known as the phenomenon of beats.* One maximum of intensity followed by a minimum is technically called a *beat*. Hence the time interval between successive beats is $t_b = 1/(\nu_1 - \nu_2)$ and is called the *beat period*. The *beat frequency*, i.e. the number of beats produced per second is $\nu_b = \nu_1 - \nu_2$. Thus

beat frequency = difference between the frequencies of interfering waves

Figure 13.17 shows how the intensity (i.e. R^2) of the resulting wave varies with time.

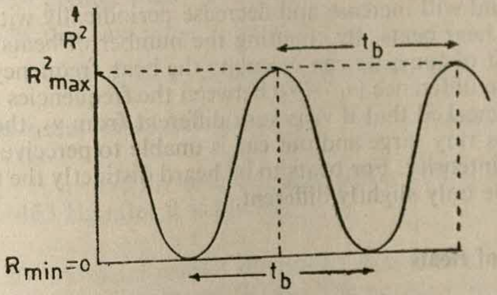


Fig. 13.17 Intensity fluctuation in beats

In Fig. 13.18 we have plotted the displacement-time curves corresponding to Eq. (13.44) (Fig. 13.18a) and Eq. (13.45) (Fig. 13.18b) and

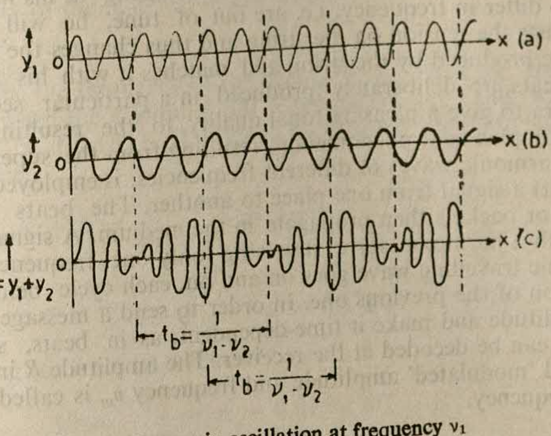


Fig. 13.18 (a) Harmonic oscillation at frequency ν_1
(b) Harmonic oscillation at frequency ν_2
(c) Superposition of (a) and (b) is a non-harmonic periodic oscillation of period $t_b = 1/(\nu_1 - \nu_2)$.

their superposition given by Eq. (13.47) in Fig. 13.18c. Notice that Figs 13.18a and b are harmonic oscillations but their superposition given in Fig. 13.18c is periodic and not harmonic.

Experimental Demonstration of Beats in Sound

Although the phenomenon of beats extends to all types of waves, we will demonstrate it for the case of sound waves. Place two tuning forks of the same frequency on a resonance box and sound them. A continuous sound will be heard. The intensity of the sound does not increase or decrease with time. Now stick a little wax to the prong of one of them so as to reduce its frequency. The frequency of sound waves emitted by the two forks will now be different. The intensity of the resulting sound will increase and decrease periodically with time. We will actually hear beats. By counting the number of beats heard in a given amount of time, we can calculate the beat frequency and hence determine the difference ($\nu_1 - \nu_2$) between the frequencies of the forks. It may be remarked that if ν_1 is very different from ν_2, the beat frequency becomes very large and our ear is unable to perceive such a rapid variation of intensity. For beats to be heard distinctly the two frequencies should be only slightly different.

Applications of Beats

The phenomenon of beats is of great practical importance. Beats can be used to determine the small difference between the frequencies of two sources of sound. Musicians often make use of beats in tuning their instruments. A piano-tuner uses beats to see whether his standard tuning fork has the same frequency as the string of his instrument. If the two differ in frequency, i.e. are out of tune, he will hear beats. He adjusts the tension on the string and thus changes the frequency of the note produced by the string and matches it with his fork. Sometimes beats are deliberately produced in a particular section of an orchestra to give a pleasing tonal quality to the resulting sound. A more complex beat phenomenon, resulting from the superposition of many harmonic waves of different frequencies, is employed to send (or transmit) a signal from one place to another. The beats called wave groups or packets then propagate in the medium. A signal cannot be sent with a harmonic wave involving only one frequency because a harmonic travelling wave goes on and on, each cycle being an exact repetition of the previous one. In order to send a message we change its amplitude and make it time-dependent, as in beats, so that this change can be decoded at the receiver. The amplitude R in Eq. (13.47) is called 'modulated' amplitude and frequency v_m is called its modulation frequency.

EXAMPLE 13.19 Two tuning forks A and B produce 10 beats per second when sounded together. On loading the fork A slightly it was observed that 15 beats are heard in one second. If the frequency of the fork B is 480 Hz, calculate the frequency of the fork A (a) before loading and (b) after loading.

Solution Let ν_1 be the frequency of A and ν_2 that of fork B. There are two possibilities (a) $\nu_1 > \nu_2$ and (b) $\nu_1 < \nu_2$.

(a) $\nu_1 > \nu_2 : \nu_2 = 480$ Hz, $\nu_b = 10$ Hz. Now, $\nu_b = \nu_1 - \nu_2$

$$\therefore \qquad \nu_1 = \nu_2 + \nu_b = 480 + 10 = 490 \text{ Hz}$$

On loading fork *A* its frequency decreases. In other words, ν_1 decreases.

$\therefore$ $\nu_1 - \nu_2 = \nu_b$ should also decrease. But ν_b increases to 15 beats/s. Hence, ν_1 is not greater than ν_2.

(b) $\nu_1 < \nu_2$: In this case $\nu_2 - \nu_1 = \nu_b$.

$$\therefore \qquad \nu_1 = \nu_2 - \nu_b = 480 - 10 = 470 \text{ Hz}$$

On loading ν_1 decreases, therefore $(\nu_2 - \nu_1) = \nu_b$ increases which is what is observed.

Therefore, $\nu_1 = 470$ Hz before it is loaded, and $\nu_1 = \nu_2 - \nu_b = 480 - 15 = 465$ Hz after it is loaded.

EXAMPLE 13.20 A metal wire of diameter 1 mm is held on two knife edges separated by a distance of 50 cm. The tension in the wire is 100 N. The wire vibrating with its fundamental frequency and a vibrating tuning fork together produce 5 beats per second. The tension in the wire is then reduced to 81 N. When the two are excited, beats are heard at the same rate. Calculate (a) the frequency of the fork and (b) the density of the material of the wire.

Solution

(a) Let *N* be the frequency of the tuning fork. Then,the frequency of the wire, when the tension is 100 N will be $(N + 5)$ and when the tension is 81 N, it is $(N, -5)$; since in each case 5 beats are heard per second. Hence

$$N + 5 = \frac{1}{2L}\sqrt{\frac{T_1}{m}} = \frac{1}{2 \times 0.5}\sqrt{\frac{100}{m}} = \frac{10}{\sqrt{m}} \qquad \text{(i)}$$

and

$$N - 5 = \frac{1}{2L}\sqrt{\frac{T_2}{m}} = \frac{1}{2 \times 0.5}\sqrt{\frac{81}{m}} = \frac{9}{\sqrt{m}} \qquad \text{(ii)}$$

Subtracting (ii) from (i) we have

$$10 = \frac{1}{\sqrt{m}} \text{ or m} = 0.01 \text{ kg m}^{-1}$$

Using this value of m in (i) or (ii) gives $N = 95$ Hz.

(b) Now

$$m = \pi r^2 \rho = \frac{\pi d^2 \rho}{4}$$

where $d = 1$ mm $= 1 \times 10^{-4}$ m and ρ is the density of the wire. Thus

$$\rho = \frac{4m}{\pi d^2} = \frac{4 \times 0.01}{3.142 \times (10^{-4})^2} = 12731 \text{ kg m}^{-3}$$

13.17 THE DOPPLER EFFECT

When the source emitting waves, and the observer receiving them, are in relative motion with respect to the medium in which the waves propagate, the frequency of the waves received by the observer is different from the frequency of the vibrating source. This phenomenon was first noticed by C.J. Doppler in 1842 in sound waves and is known as the *Doppler effect.* This phenomenon of the apparent change in the observed frequency of the wave, when there is a relative motion between the source and the observer is exhibited by all types of waves, including light waves which do not require a medium for their propagation. We shall discuss the Doppler effect in sound waves.

The Doppler effect in sound waves may be readily observed by a person at a railway platform. When a through train, blowing its whistle, approaches the observer, the pitch of the whistle appears to rise and it suddenly appears to drop when the engine has passed by. For waves which require a medium for propagation, the apparent change in the frequency depends on which of the three—the source, the observer or the medium is moving. Let us now compute this change in frequency for the following cases.

Source in Uniform Motion—Observer and Medium Stationary

To fix our idea let us take the source to be a tuning fork of frequency ν. It produces sound (compressional) waves of frequency ν, which

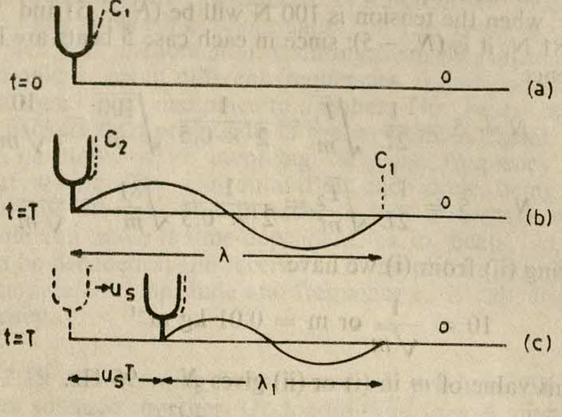


Fig. 13.19

travel through the medium with a velocity $v = \sqrt{\frac{B}{\rho}}$ where B is the modulus of elasticity and ρ, the density of the medium. Let u_s be the velocity with which the source, i.e. the fork, moves bodily towards or away from a stationary observer stationed at O. As shown in Figure 19.19, the wavelength of the waves in the medium, depends on the magnitude and direction of u_s. In Fig. 13.19 we consider the source approaching the observer. Let, at an instant of time $t = 0$, (Fig. 13.19 a) the prongs of the fork move outwards producing a compression C_1 in the medium close to the prong. This compression will travel in the medium with the velocity v of the wave of compression. After one full period, i.e. at $t = T$, where $T (= 1/\nu)$ is the period of vibration of the fork, the second compression C_2 is produced because at $t = T$, the prongs will once again move outwards. If the fork is at rest, the distance λ (Fig. 13.19b) between two consecutive compressions (which is the distance travelled by the compression in one period) is, by definition, the wavelength of the wave in the medium.

If, however, the fork is moving with velocity u_s (Fig. 13.19c) it travels a distance $u_s T$ in time T, towards the observer. At this position the fork produces the second compression C_2, the distance between two consecutive compressions having been reduced to λ_1, which is the wavelength of the wave when the fork is in motion. Obviously λ and λ_1 are related as

$$\lambda_1 = \lambda - u_s T$$

Now, since wave velocity v is fixed (being determined by the properties of the medium, namely, B and ρ), the frequency of the wave of wavelength λ_1 is given by

$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{\lambda - u_s T} = \frac{v}{\frac{v}{\nu} - \frac{u_s}{\nu}}$$

or

$$\nu_1 = \frac{v}{\left(1 - \frac{u_s}{v}\right)} \tag{13.49}$$

Hence, the frequency of the waves received by the observer is ν_1, which is greater than ν. If the source is moving away from a stationary observer, the apparent wavelength $\lambda_1' = \lambda + u_s T$, in which case, the apparent frequency as observed by the observer is given by (since u_s changes sign)

$$\nu_1' = \frac{\nu}{\left(1 + \frac{u_s}{v}\right)} \tag{13.50}$$

In this case, $\nu_1' < \nu$. The cause of the change in frequency is the increase or decrease of the wavelength in the medium brought about by the motion of the source.

Note: A special situation occurs in the case of supersonic sources which move with velocities greater than that of sound. The relations (13.49) and (13.50) do not hold because then in a given time the source advances more than the wave. The resultant wave motion is a conical wave called a shock wave which is a sudden and violent sound we hear when a supersonic jet plane flies by.

Observer in Uniform Motion—Source and Medium Stationary

The effect is different when the source is stationary and the observer moves through the medium. Let the observer move away from a source with a uniform velocity u_0. If ν is the frequency of the vibrating source it emits ν waves in one second. If the observer is at rest, he receives these waves in one second. Therefore, the frequency of wave he hears is ν. If he now starts running away from the source, he will miss the waves which occupy the distance he travels. In one second he moves a distance of u_0 away from the source. Therefore, the number of waves missed by him in one second = number of waves contained in the distance u_0 he moves in one second = u_0/λ, since, in a distance of wavelength λ, there is exactly one wave, by definition. Thus, out of ν waves, he misses u_0/λ waves in one second.

Therefore, the number of waves received by him in one second = $\nu - u_0/\lambda$ or since $\lambda = v/\nu$, the frequency ν_2 of sound heard by him is given by

$$\nu_2 = \nu - \frac{\nu u_0}{v} = \nu\left(1 - \frac{u_0}{v}\right) \tag{13.51}$$

On the other hand, if the observer were approaching the source, he would receive u_0/λ additional waves in each second and would hear a sound of frequency ν_2' given by

$$\nu_2' = \nu\left(1 + \frac{u_0}{v}\right) \tag{13.52}$$

Figure 13.20 illustrates how the abserver hears a sound of higher frequency when he moves towards it. Notice that the apparent change in the frequency of sound heard by the observer results from his inter-

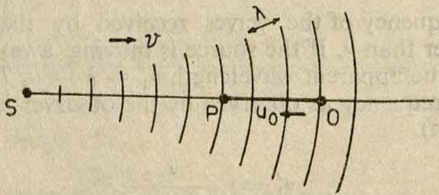


Fig. 13.20 Spherical wave from a point source S travelling to the right. Imagine the observer to stay at O for one second and then move suddenly to P. He now receives more waves than if he had stayed at O and hence hears a higher frequency.

cepting more waves (when he approaches) or fewer waves (when he recedes) each second.

Source and Observer both in Uniform Motion—Medium Stationary

We shall compute the apparent frequency in the following possible situations

Source Approaching a Receding Observer If the source approaches a stationary observer with a velocity u_s, the wavelength of the waves reduces to $\lambda_1 = \lambda - u_s T$ and the frequency of these waves is ν_1 given by Eq. (13.49). The observer receives ν_1 waves in one second. If the observer is receding with a velocity u_0, he will miss u_0/λ_1 waves each second. In one second the number of waves received by him will be given by

$$\nu_3 = \nu_1 - \frac{u_0}{\lambda} = \nu_1\left(1 - \frac{u_0}{v}\right)$$

Using Eq. 13.49, the expression for the apparent frequency becomes

$$\nu_3 = \nu \frac{(v - u_0)}{(v - u_s)} \tag{13.53}$$

If $u_0 = u_s$, $\nu_3 = \nu$. since then there is no relative motion between the source and the observer.

Both the Source and Observer Receding from Each Other The apparent frequency ν_3', in this case, is obtained by reversing the sign of u_s in Eq. (13.53).

$$\nu_3' = \nu \frac{(v - u_0)}{(v + u_s)} \tag{13.54}$$

Both the Source and Observer Approaching Each Other Reversing the sign of u_0 in Eq (13.53) the apparent frequency in this case is

$$\nu_4 = \nu \frac{(v + u_0)}{(v - u_s)} \tag{13.55}$$

Observer Approaching a Receding Source In this case the signs of u_0 and u_s in Eq. (13.53) are reversed and the apparent frequency ν_4', is given by

$$\nu_4' = \nu \frac{(v + u_0)}{(v + u_s)} \tag{13.56}$$

Notice that $\nu_4' = \nu$, if $u_0 = u_s$.

Effect of the Motion of the Medium The velocity of material waves is affected by the motion of the medium. If the medium is moving with a velocity u_m in the direction of propagation of sound, the effective velocity of sound is increased from v to $(v + u_m)$. In the relations (13.49) to (13.56) v is replaced by $(v + u_m)$. On the other hand, if the medium is moving in the opposite direction, v is replaced by $(v - u_m)$.

Example 13.21 A through train blowing a whistle of frequency 700 Hz is passing a railway station with a speed of 72 km/hr. Calculate the frequency of the sound heard by a person standing on the platform (a) when the train is approaching and (b) when it has departed. Velocity of sound = 340 m s^{-1}.

Solution

$$u_s = 72 \text{ km/hr} = 20 \text{ m s}^{-1}$$
$$v = 340 \text{ m s}^{-1}$$
$$\nu = 700 \text{ Hz}$$

(a) *Train approaching*

$$\nu_1 = \frac{\nu}{1 - \frac{u_s}{v}} = \frac{700}{1 - \frac{20}{340}} = 743.75 \text{ Hz}$$

(b) *Train departing*

$$\nu_1' = \frac{\nu}{1 + \frac{u_s}{v}} = \frac{700}{1 + \frac{20}{340}} = 661.1 \text{ Hz}$$

Example 13.22 A car with a velocity of 108 km/hr is approaching a stationary train which is sounding a whistle of frequency 800 Hz. Calculate the frequency of the sound heard by a man in the car.

(a) while it is approaching the engine; and
(b) after it has passed by.

Velocity of sound = 350 m s^{-1}

Solution

$$\nu = 800 \text{ Hz}$$
$$u_0 = 108 \text{ km/hr} = 30 \text{ m s}^{-1}$$
$$v = 350 \text{ m s}^{-1}$$

(a) *Observer approaching*

$$\nu_2' = \nu\left(1 + \frac{u_0}{v}\right) = 800\left(1 + \frac{30}{350}\right) = 868.6 \text{ Hz}$$

(b) *Observer receding*

$$\nu_2 = \nu\left(1 - \frac{u_0}{v}\right) = 800\left(1 - \frac{30}{350}\right) = 731.4 \text{ Hz}$$

Example 13.23 A railway engine and a car are moving on parallel tracks with speeds of 72 km/hr and 36 km/hr respectively. The engine is continuously sounding a whistle of frequency 600 Hz. If the velocity of sound is 340 m s^{-1}, calculate the frequency of sound heard by the driver of the car when

(a) the car and the engine are moving in opposite directions and approaching each other;

(b) the two are moving in opposite directions but going away from each other;

(c) the two are moving in the same direction, the car being ahead of the engine; and

(d) the two are moving in the same direction, the car being behind the engine.

Solution

$$\nu = 600 \text{ Hz}$$

$$v = 340 \text{ m s}^{-1}$$

$$u_0 = 36 \text{ km/hr} = 10 \text{ m s}^{-1}$$

$$u_s = 72 \text{ km/hr} = 20 \text{ m s}^{-1}$$

(a) *Both approaching*: We use Eq. (13.55)

$$\nu_4 = 600\left(\frac{340 + 10}{340 - 20}\right) = 656.25 \text{ Hz}$$

(b) *Both receding*: We use Eq. (13.54)

$$\nu_3' = 600\left(\frac{340 - 10}{340 + 20}\right) = 550 \text{ Hz}$$

(c) *Source approaching and observer receding*: We use Eq. (13.53)

$$\nu_3 = 600\left(\frac{340 - 10}{340 - 20}\right) = 618.75 \text{ Hz}$$

(d) *Observer approaching and source receding*: The appropriate relation is Eq. (13.56)

$$\nu_4' = 600\left(\frac{340 + 10}{340 + 20}\right) = 583.3 \text{ Hz}$$

Notice that in cases (a) and (b) the relative speed ($= 72 + 36 = 108$ km/hr) is the same, yet the apparent frequencies are different. In cases (c) and (d) the relative speed $= 72 - 36 = 36$ km/hr. The apparent frequencies are, however, different. The apparent frequency depends on the absolute velocities of the source and the observer are not on their relative velocity.

EXAMPLE 13.24 An ambulance blowing a siren of frequency 700 Hz is travelling slowly towards a vertical reflecting wall with a speed of 2 m s^{-1}. Calculate the number of beats heard in one second by the driver of the ambulance. Velocity of sound $= 350$ m s^{-1}.

Solution The driver will hear two sounds, one coming directly from the siren and the other reflected at the wall or coming from the acoustic image of the car. ν = frequency of direct sound = 700 Hz. The reflected sound can be imagined to be coming from the mirror image (shown dotted in Fig. 13.21). The observer (driver) is approaching this image-source which is also approaching him with the same speed. Hence, the frequency of sound heard by him is given by $(u_0 = u_s)$

$$v' = \nu \frac{v + u_0}{\nu - u_s} = 700 \times \frac{350 + 2}{350 - 2} = 708 \text{ Hz}$$

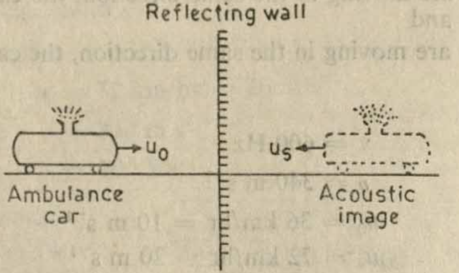


Fig. 13.21

Number of beats heard per second $= \nu' - \nu = 708 - 700 = 8$

$\therefore$ Beat frequency $= 8$ Hz

EXAMPLE 13.25 Two persons A and B, each carrying a source of sound of frequency 600 Hz, are standing a few metres apart in a quiet field. Calculate the beats heard by each when A moves towards B with a velocity 3 m s^{-1}. Velocity of sound = 340 m s^{-1}.

Solution

Beats heard by the person A

The person A hears the sound of his own source whose frequency $\nu =$ 600 Hz and the sound of the source carried by B, towards whom he is moving. The apparent frequency of this sound is given by

$$\nu' = \nu\left(1 + \frac{u}{v}\right) = 600\left(1 + \frac{3}{340}\right) = 605.29 \text{ Hz}$$

Beat frequency apparent to A $= 605.29 - 600 = 5.29$ Hz, i.e. A hears 529 beats in 100 seconds.

Beats heard by the person B

The person B hears the sound of his own source of frequency = 600 Hz and the sound of the source carried by A, which is approaching him with a velocity of 3 m s^{-1}. The apparent frequency of this sound is given by

$$v'' = \frac{\nu}{1 - \frac{u}{v}} = \frac{600}{1 - \frac{3}{340}} = 605.34 \text{ Hz}$$

Beat frequency apparent to B $= 605.34 - 600 = 5.34$ Hz, i.e. B hears 534 beats in 100 seconds.

Notice again that, although the relative velocity is the same, the apparent frequency is different for A and B.

13.18 MUSICAL SOUND AND NOISE

Sounds are usually divided into two classes, *musical sound* and *noise*. A sound which produces a pleasing effect on the ear is called a musical sound such as the sound produced by a violin, guitar or flute. On the other hand, a sound not pleasing to the ear is called noise, such as the sound of a fire cracker or that produced when two metal pieces are clapped together.

Generally speaking, bodies which vibrate with regular frequency and with no abrupt changes of amplitude produce musical sound. Musical sounds are produced by periodic vibrations. The wave functions of musical sounds are periodic functions. We know that any periodic function can be expressed as a superposition of a large number of harmonic functions. Thus a musical sound is a combined effect of a superposition of harmonic functions. The harmonic component with the lowest frequency is the fundamental note and that with twice the frequency of the fundamental the second harmonic, and so on.

On the othar hand, bodies which vibrate with irregular frequency, with abrupt changes in amplitude, produce noise. Their wave functions have no definite regularity.

Characteristics of a Musical Sound

There are three characteristics which distinguish one musical sound from another. They are (1) pitch, (2) loudness (or intensity) and (3) quality (or timber).

Pitch Pitch is that characteristic of sound which enables us to distinguish between a high, acute or shrill note from a low, grave or flat note. *Pitch depends upon the frequency of the note*. Sound with a greater frequency has a higher pitch than that with a smaller frequency.

It must be clearly understood that *pitch is not the same thing as frequency*. Frequency is a measurable physical quantity but pitch is a sensation (in the consciousness of a human being) which corresponds to and depends upon frequency.

Loudness Like pitch, *loudness* of a sound is a sensation which is related to a physically measurable quantity called *intensity*. The intensity of a sound refers to the amount of energy carried by the sound wave and *is numarically proportional to the square of the amplitude of the wave*. Thus the greater the amplitude of sound, the greater is its intensity and the louder it seems. Thus *loudness is that characteristic which distinguishes a loud sound from a faint one, both having the same pitch.*

Quality The quality of a musical sound is that characteristic which distinguishes between two sounds of the same pitch and loudness, but *emitted by different instruments.* When a note of a particular pitch is played on a piano and then on a violin, there is a marked difference in the overall sound. The characteristic quality of sound from a clarinet is different from that from a piano which in turn is different from that from a violin.

The quality of these sounds is different because of their harmonic content. Each instrument emits a sound which has a periodic waveform consisting of a fundamental note and some of its higher harmonics. *The quality of a sound depends upon the number of higher harmonics of the fundamental.* We know that an open pipe has all the harmonics but a closed pipe has only odd harmonics. It is for this reason that they emit sounds of different quality. The presence of a large number of harmonics enriches the tonal quality of sound. Thus *the quality of a musical sound is governed by the number of harmonics present in it along with the fundamental note.*

13.19 MUSICAL SCALE

A series of notes whose fundamental frequencies have definite ratios and which produce a pleasing effect on the ear when sounded in succession constitute a musical scale. The simplest musical scale called the *diatonic scale* has eight notes comprising an *octave*. The frequency ratio of the eighth and the first note is 2 : 1. Conventionally the fundamental frequency of the first note is taken to be 256 Hz and that of the last 512 Hz. Table 13.2 gives the frequencies of the intermediate notes with their Western and Indian names.

Table 13.2 The musical scale

Symbal	*Western name*	*Indian name*	*Fre-quency* (Hz)	*Fre-qnency ratio*
C	Do	Sa	256	
				9/8
D	Re	Re	288	
				10/9
E	Mi	Ga	320	
				16/15
F	Fa	Ma	$341\frac{1}{3}$	
				9/8
G	Sol	Pa	384	
				10/9
A	La	Dha	$426\frac{2}{3}$	
				9/8
B	Ti	Ni	480	
				16/15
C_1	Do	Sa	512	

The note C of the lowest frequency is called the *key note*. The key note is taken to be of frequency 256 Hz, i.e. 2^8 Hz. Musical instruments, such as the piano or harmonium, have many octaves above and below the one given in the table above.

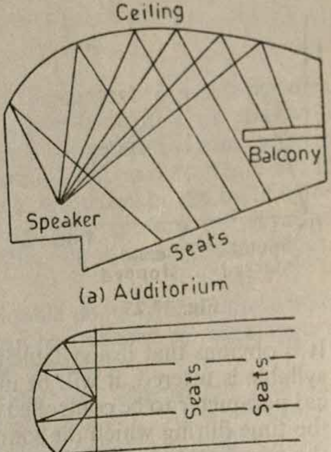
13.20 ACOUSTICS OF BUILDINGS

Acoustics of buildings (or architectural acoustics) deals with the design and construction of buildings (like concert halls, auditoria, etc.) and the behaviour of sound waves in the closed spaces of buildings. In the following we will discuss the various factors the designer of a building should bear in mind so as to achieve the desired quality, clarity and level of sound intensity at each point of the hall. For good rendering of speech or music the following factors must be considered in the design of the room or auditorium.

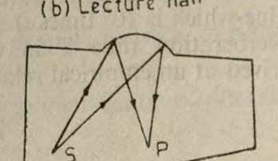
Loudness

We know that the intensity of a sound wave travelling in open space decreases with distance. Also, the sound energy not propagating in the direction of the listener is irretrievably lost. In closed halls, an adequate level of intensity (which is a measure of loudness) can be achieved by a proper use of the reflecting properties of the walls and the ceiling. The reflected waves reinforce the direct sound to give uniform loudness of sound in the hall. Figure 13.22a illustrates how a curved ceiling helps in the reinforcement of sound on the balcony and at the back of the hall. A concave ceiling is useful for concert halls. For lecture halls, where the position of the speaker is fixed, the use of a paraboloidal reflector at the back of the speaker is most adequate. The use of such a reflecting curved surface to give a uniform parallel beam of sound is illustrated in Fig. 13.22b. The speaker is at the focus of the reflecting surface.

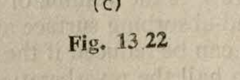
The designer has to be extremely careful in deciding the curvature of the ceiling. A faulty curved surface may lead to a focussing effect of sound, as shown in Fig. 13.22c, which may



(a) Auditorium



(b) Lecture hall



(c)

Fig. 13.22

result in an unduly high intensity of the resulting sound at a particular place P in the hall.

Echoes

In big buildings the echo (or reflected sound) from a distant wall reaches the listener much later than the direct sound. If the interval of time between the arrival of the direct sound and the echo is more than 1/15 of a second, the listener will get a distinct impression of the echo. To correct this defect, such walls or other offending surfaces must be covered with some material which absorbs sound. Porous asbestos tiles have a high absorption coefficient for sounds in the audible frequency range.

Reverberation

Reverberation is the most important factor to be controlled in the acoustics of a building. If a transient sound (i.e. of short duration) is produced in open space, it is received by the listener as such. But if such a sound is produced in a hall, it is found to persist even after the source has ceased to vibrate. This persistence of sound is due to multiple reflections from the ceiling, walls, etc. The persistence of sound in an enclosure, after the source has stopped producing it is called *reverberation*. Professor W.C. Sabine in 1895 studied the problem of reverberation in considerable detail. He measured the intensity of sound in a hall and found that the intensity varied with time. Figure 13.23 shows how the intensity grows and decays with time. It is obvious that if a syllable persists for a time during which the next syllable is uttered, it will be unintelligible. The most important physical parameter to be controlled is the *reverberation time* which is roughly the time during which the sound persists in the hall. More precisely, the reverberation time is the time taken by sound intensity to fall to the minimum audible value (audibility threshold) from an initial value which is 10^6 times the audibility threshold. Sabine measured the reverberation time T in various rooms of different sizes and thus arrived at an empirical relation

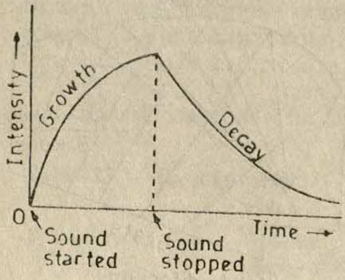


Fig. 13.23

$$T = \frac{0.053\,V}{A}$$

where V is the volume of the room in cubic feet and A is the total sound-absorbing surface area in square feet. Thus, the reverberation time can be reduced if the total absorbing area A is increased. For a given hall the reverberation time can be controlled by inserting or removing sound-absorbing material at the walls.

Reverberation time T is the most important parameter to be considered in the design of an auditorium. If T is too large, there is excessive reverberation resulting in confusion owing to overlapping of successive sounds. On the other hand, if T is too short, the loudness becomes inadequate because the reflections have been eliminated or reduced to a minimum. Hence, for good acoustics some amount of reverberation is essential. This optimum reverberation is determined by the use to which the building is put. For speech the optimum reverberation time varies from 0.5 s in small rooms to about 7.5 s in large halls. For music it can vary from 1.0 s to about 2.5 s. Reverberation helps to enrich the tonal quality of music.

Echelon Effect

If an enclosure has a flight of steps with equal width, the sound waves reflected from them will consist of echoes with regular phase difference, which produce a musical note which will be heard along with the direct sound. This is called the *echelon effect.* It can be remedied by either having steps of unequal widths or by covering them with sound-absorbing material.

Extraneous Noise

All unwanted sounds constitute noise in the context of acoustics of a building. Noise may be of two types: (i) *air-borne* and (ii) *structure-borne*. Air-borne noise includes all sounds originating from sources outside the building and entering it through air, and structure-borne noise comprises sounds originating in the structure of the building itself. This noise may be the noise of machinery, motors or elevators and may be communicated through walls, floor, steel girders, pipes, etc.

Air-borne noise is reduced by using double doors and windows with heavy curtains. The noise of footsteps can be reduced by making use of carpets. The walls should be rigid and broad.

Structure-borne noise is difficult to eliminate. Typewriters and exhaust fans should be mounted on absorbent padding. Machines should be lubricated. Sounds that enter the building through sound-conducting structures can be avoided by breaking the continuity and inserting layers of adsorbents like cork.

In short, the factors resposnible for the proper acoustics of a building are:

1. Adequate level of loudness and uniform distribution of intensity
2. Absence of echoes
3. Optimum reverberation
4. Minimization of echelon effect
5. Minimization of extraneous noise

13.21 DISPERSION

Dispersion is a very important phenomenon that is exhibited by several types of waves. *When the velocity of propagation of a wave travelling in a medium depends on the wavelength (or the frequency) we say that there is dispersion.* It is not true for the velocity of transverse waves on a stretched string, which is governed by the relation $v = \sqrt{\frac{T}{m}}$, where T is the tension with which the string is stretched and m is the mass of unit length of the string. Since wavelength does not appear in this expression, a given string under a given tension will carry waves of all wavelengths with the same speed v. Since $v = \nu\lambda$, the velocity of the waves is also independent of the frequency ν. In other words, transverse waves on a string are nondispersive.

Sound waves are also nondispersive, since the velocity of sound in a medium depends only on the elastic properties and density of the medium. High and low frequency sounds travel with the same speed in a given medium. This is a blessing in disguise. Imagine the state of affairs if while listening to an orchestra, the sounds of different frequencies emitted by different musical instruments were to reach our ear at different times and at different speeds!

Water waves in a ripple tank are dispersive. The velocity of these waves is given by $v = \left(\frac{2\pi\sigma}{\rho\lambda}\right)^{1/2}$, where σ is the surface tension, ρ the density of water and λ is the wavelength of the waves. The waves of longer wavelengths have slower speed. If the frequency of the vibrator in a ripple tank is increased, the velocity of water waves also increases. It is for this reason that a gentle breeze produces waves (or ripples) of longer wavelength and slower speed in a lake.

We are familiar with the dispersion of light waves. We have seen it occur when while light passes through a prism. White light breaks up into its constituent colours because velocity of the red waves of light is greater than that of the violent waves. Each colour has its characteristic wavelength. Waves of different wavelengths travels in glass with different speeds. In a vacuum they all travel with same speed. Light waves do not disperse in a vacuum.

SUPPLEMENT

Periodic Waves

In this chapter we have mainly dealt with harmonic waves, i.e. the waves which are described by wave functions which are given by the sine or cosine of $(vt \pm x)$. Harmonic waves are not the only ones that exist. We often come across waves which are periodic but not harmonic. We have encountered one such situation while discussing beats in Sec. 13.16. We have seen that two harmonic waves of different frequencies [shown in Figs 13.18 (a) and (b)] superpose to give a wave [shown in Fig. 13.18(c)] which is periodic (with period t_b called the beat period) but not harmonic. Notice that the displacements in Fig. 13.18c are not described by a sine or cosine function.

Periodic waves are produced by the superposition of a number of harmonic waves of different frequencies. Musical sounds (see Sec. 13.18) are also periodic. A musical sound consists of a fundamental note and a number of higher harmonics whose frequencies are simple multiples of the fundamental frequency. Thus each harmonic repeats itself after the period of the fundamental. This means that their superposition is periodic having the period of the fundamental. This is the key idea of the Fourier analysis of periodic waves. It is a mathematical technique of breaking down a periodic wave into a linear superposition of its constituent harmonic components. Let us consider an example which shows how three harmonic waves of frequencies ν, 3ν and 5ν produce a periodic wave of frequency ν. Let the three harmonic waves be given by

$$y_1 = \sin 2\pi\,(t - x)$$

$$y_2 = \tfrac{1}{3} \sin 6\pi\,(t - x)$$

$$y_3 = \frac{1}{5} \sin 10\pi\,(t - x)$$

The frequency of the first is $\nu = 1$ Hz. The resultant wave travelling in the medium will be a periodic wave given by

$$\begin{aligned} y &= y_1 + y_2 + y_3 \\ &= \sin 2\pi\,(t - x) + \frac{1}{3} \sin 6\pi\,(t - x) \\ &\quad + \frac{1}{5} \sin 10\pi\,(t - x) \end{aligned} \tag{13.57}$$

The frequency of y is 1 Hz, the frequeney of the fundamental. The displacement y given by Eq. (13.57) can be plotted for all values of t and x. Figure 13.24 shows the graph of y against x for $t = 0$ and 0.3 s.

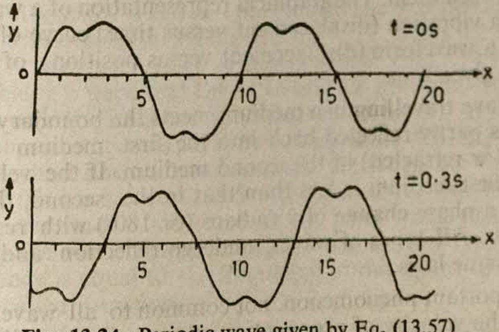


Fig. 13.24 Periodic wave given by Eq. (13.57) at $t = 0$ s and $t = 0.3$ s.

From these figures we observe that although the appearance of the wave is more complicated than its harmonic components, yet the resultant wave is periodic with the period of the fundamental.

SUMMARY

The key word in wave motion is 'disturbance'. A wave is the propagation of the disturbance. The nature of the disturbance may be mechanical (as in waves in strings and other material media including fluids) or electrical (as in light waves) or even thermal (as in waves of heat transfer). If a disturbance can exist only in a material medium, the particles of the medium must be coupled by some force so that the disturbance can propagate. The wave velocity is the speed with which the wave is propagated and is determined by the properties of the medium. The wavelength of a wave is the shortest distance between two particles having the same phase of vibration. Wavelength λ and wave velocity v are related as $v = \nu\lambda$, where ν is the frequency of the source generating the waves.

A transverse wave is a wave in which the particles vibrate perpendicular to the direction of propagation. In a longitudinal wave the particles vibrate along the direction of propagation. Waves on strings are transverse and sound waves in a medium are longitudinal. The various alternative mathematical representations of a harmonic wave are

$$\begin{aligned} y(x, t) &= A \sin\left\{\frac{2\pi}{T}\left(t \mp \frac{x}{v}\right)\right\} \\ &= A \sin\left\{\frac{2\pi}{\lambda} vt \mp x)\right\} \\ &= A \sin(\omega t \mp kx) \end{aligned}$$

where the $(-)$ sign refers to the wave travelling in the $+x$ direction and the $(+)$ sign to the wave travelling in the $-x$ direction.

The feature distinguishing a wave from a vibration is that a wave consists of many vibrations of many particles with definite phase relationships between them. The graphical representation of a wave involves two curves, a vibration (displacement versus time) curve of any given particle and a waveform (displacement versus position) of the whole wave at any given time.

When a wave travelling in a medium meets the boundary of another medium, it is partly reflected back into the first medium and partly transmitted (or refracted) in the second medium. If the velocity of the wave in the first medium is less than that in the second, the reflected wave suffers a phase change of π radians (or 180°) with respect to the incident wave. All types of waves undergo reflection and refraction obeying the same laws.

A very important phenomenon, not common to all wave, is dispersion. When the velocity of propagation of a wave in a medium depends on its wavelength (or frequency) we say that there is dispersion. Transverse waves on strings and longitudinal (sound) waves in a medium are not dispersive. Water waves in a ripple tank and light waves in a material medium are examples of waves that disperse.

The term 'interference' is used to describe the physical effects of superposition of two or more waves travelling through the same region of space. The resultant motion of a particle can be found by the superposition principle which states that the resultant displacement is equal to the algebraic sum of the individual displacements imparted to it by the individual waves.

Two waves of unequal frequencies, travelling in the same direction, interfere to give rise to the phenomenon of beats. The intensity of the wave resulting from their superposition varies periodically with time. The intensity rises to a maximum value and falls to a minimum value at regular intervals of time. One maximum of intensity followed by a minimum is technically called a beat. The beat frequency is equal to the difference of the frequencies of individual waves.

When two waves, having the same frequency, wavelength and speed, travel in opposite directions through a medium, the resultant effect is a stationary (or standing) wave. Certain points in the stationary wave, called nodes, remain permanently at rest, and certain other points, called antinodes, vibrate with maximum amplitude. The distance between two consecutive nodes (or antinodes) is half the wavelength of either wave. The most striking feature of a standing wave is that energy is not transported along the medium. The wave keeps 'standing' as it were.

A vibrating body has many possible modes of vibration which are nothing but stationary waves on it. When a disturbance is created at any point in a body, a wave travels along it. This wave gets reflected at the boundaries of the body. The interference between the incident and the reflected waves gives rise to standing waves on it. The body of a string, fixed at both ends, can vibrate in 1, 2, 3, 4, . . . segments, with a node at each end. The lowest frequency of vibration is called the fundamental frequency and the higher frequencies are called overtones. An overtone whose frequency is an integral multiple of the fundamental frequency is called a harmonic. The frequencies of various models of vibration of a string are 1, 2, 3, 4, . . . times the fundamental frequency. The body of a gas in a pipe can also vibrate, the closed end of the pipe being a node and the open end an antinode. If the pipe is open at both ends the frequencies of the various modes of the vibration of the air column in it are 1, 2, 3, 4, . . . times its fundamental frequency. If it is open at one end and closed at the other, the frequencies of the modes of vibrations are 1, 3, 5, 7, . . . times the fundamental frequency. The amplitude of vibration in any given mode can be increased by linking the body, in some way, to an external periodic force whose frequency is equal to the frequency of the given mode of vibration. This is called resonance.

The phenomenon of apparent change in the frequency of waves, when there is a relative motion between the source and the observer is known as the Doppler effect. This effect is exhibited by all types of waves. The observed frequency is greater than the true frequency if either the source or the observer, or both, approach each other. On the other hand, the apparent frequency is lowered if the source and the observer are moving away from each other. In sound waves the Doppler effect is not symmetric, i.e. although the relative velocity is the same, the apparent change in the frequency depends on whether it is the observer or the source that moves.

Acoustics of buildings is a branch of acoustics which deals with the design of a building from the point of view of its acoustics. The various factors that the designer has to bear in mind so as to achieve the desired clarity, quality and intensity of sound at each point in a building are (i) uniform distribution and adequate level of intensity, (ii) optimum reverberation, (iii) absence of echoes, (iv) minimization of echelon effects, and (v) minimization of extraneous noise.

EXERCISES

1. What do you understand by wave motion? Distinguish between a vibration and a wave.
2. Give two examples of a disturbance which can propagate as wave motion. What are transverse and longitudinal waves?
3. Define wavelength and wave velocity. Deduce a relation between them.
4. Starting with equation of simple harmonic disturbance, obtain the wave equation.
5. Name the parameters that characterize a harmonic wave. Define each of them.
6. A harmonic wave has period T and wavelength λ. How far does it advance in time T? How long does it take to travel a distance of λ? How are λ and T related?
7. Show by dimensional considerations that the velocity of a transverse wave along a string is given by $v = K\sqrt{\frac{F}{m}}$, where F is the tension, m is the mass per unit length of the string and K is a dimensionless constant. How is K evaluated?
8. The velocity of a longitudinal wave in a metallic rod is given by $v = \sqrt{\frac{Y}{\rho}}$, where Y is Young's modulus and ρ is the density of the metal. Check this relation for dimensional consistency.
9. Write down the expression for the velocity v in Question 7 in terms of the diameter and density of the string. How is the velocity of the wave modified if (a) the tension is doubled and (b) the diameter is doubled?
10. Obtain Newton's formula for the velocity of sound in a gas.
11. Is wave motion in a gas an adiabatic process? If so, why?
12. Show that the velocity of sound in air increases by about 0.61 m s^{-1} for each degree Celsius rise in temperature.

13. Two strings A and B, of the same material, are joined at C. The cross-sectional area of B is greater than that of A. Discuss the mechanism of reflection and transmission at the boundary C of a pulse travelling in the string A. What can you say about the phase of the reflected and the transmitted pulse in relation to that of the incident pulse?
14. What will happen if the pulse mentioned in Exercise 13 were travelling in the string B?
15. State the superposition principle. Is it always applicable?
16. Show graphically how the superposition of two harmonic waves of equal amplitude, equal phase and slightly different frequencies, causes the phenomenon of beats.
17. Discuss the formation of beats and deduce the expression for the beat frequency in terms of the frequencies of the constituent waves.
18. Explain the difference between interference and beats. How will you experimentally demonstrate the phenomenon of beats in sound?
19. What is a stationary wave? How is it produced? How does it differ from a travelling wave?
20. Discuss graphically the formation of stationary waves. What are nodes and antinodes?
21. 'Stationary waves cannot transport energy but travelling wavss can'. Comment.
22. What is the Doppler effect? Give an example from everyday life that establishes the existence of this effect.
23. Deduce the expression for the frequency of sound as heard by an observer.
    (a) when the source of sound is approaching him with a uniform speed and he is at rest; and
    (b) when he is moving towards a stationary source of sound with the same speed.
24. Show that the apparent frequencies in cases (a) and (b) of Exercise 23 are nearly equal if the velocity of the source or observer is much smaller than the velocity of sound.
25. Deduce the expression for the frequency of sound as heard by an observer
    (a) when he is at rest and the source of sound is receding from it with a uniform velocity; and
    (b) when the source is at rest and he is receding from it with the same velocity.
26. Show that the frequencies in cases (a) and (b) of Exercise 25 reduce to each other if the velocity of the source or observer is much smaller than that of sound.
27. The asymmetry in the Doppler effect in sound is a consequence of the fact that the sound waves require a medium for propagation. Comment.
28. What is meant by the term reverberation? Define reverberation time.
29. Define reverberation time for an auditorium? What causes reverberation and how is it controlled?
30. Write an essay on the acoustics of buildings.
31. Distinguish between a musical sound and noise.

32. What do you understand by a musical scale?
33. Distinguish clearly between periodic and harmonic waves.

PROBLEMS

1. Find the amplitude, time period, speed and wavelength of the waves given by the following equations. All quantities are in SI units.
   (a) $y(x, t) = 0.01 \sin 2\pi(3t - 2x)$
   (b) $y(x, t) = 0.02 \sin(2t + 3x)$
   (c) $y(x, t) = 0.03 \cos(vt + x)$
   (d) $y(x, t) = 0.05 \cos(6\pi t - \pi x)$
2. Write down the equations (using the sine function) of a harmonic wave travelling in the $-x$ direction and having the following characteristics:
   (a) Amplitude 0.05 m, speed 2 m s^{-1}, frequency 4 Hz
   (b) Amplitude 0.02 m, frequency 800 Hz, wavelength 0.3 m
   (c) Amplitude 0.01 m, speed 10 m s^{-1}, wavelength 0.2 m
3. Two strings A and B, of equal thickness, are made of the same material. The length of A is half that of B while the tension in A is twice that in B. Compare the velocities of the transverse waves on them.
4. Two strings A and B are of the same length and are made of the same material. The cross-sectional area of A is half that of B while the tension in A is twice that in B. Compare the velocities of the transverse waves on them.
5. A string 25 m long has a mass of 0.5 kg. A 10 m long specimen of the string is stretched with force equal to the weight of 5 kg. How long will the transverse wave take to travel the length of the string?
6. The intensity of the sound of a barking dog is 10^{-3} Wm^{-2}. If the frequency is 1000 Hz, calculate the amplitude of the sound waves in air at standard conditions.
7. Calculate the amplitude of the sound wave of Prob. 6 if the air temperature were 35 °C.
8. It is desired to tune the string of a sonometer with a tuning fork of frequency 512 Hz. When vibrated together they produce 10 beats per second. (a) By what percentage should the length of the string be altered to achieve tuning, the tension remaining the same? (b) By what percentage should the tension in the string be altered to achieve tuning, the length of the string remaining the same?
9. A string under a tension of 100 N is emitting its fundamental note. While tuning with a tuning fork, 5 beats per second are heard. When the tension is increased to 121 N, again 5 beats per second are heard. Determine the frequency of the fork.
10. Two identical pipes are sounding notes of frequency 500 Hz. The air temperature in both is 25 °C. The temperature of the air in one pipe is increased to 30 °C. How many beats are heard per second when both the pipes are blown? Velocity of sound at 0 °C = 330 m s^{-1}.

11. Calculate the percentage change in the frequency of the fundamental note emitted by a closed pipe of length 1.0 m if the temperature of the air is changed from 15 to 30 °C. Velocity of sound at 0 °C = 330 m s^{-1}.

12. Two tuning forks A and B produce 4 beats per second when vibrated together. If a prong of fork A is filed, the beats reduce to 2 per second. If the frequency of fork B is 480 Hz, what is the frequency of fork A (a) before and (b) after it is filed?

13. A sonometer wire carries a metal block at the other end. When the block is in air, the resonant length with a tuning fork is found to be 90 cm. When the block is wholly immersed in water, the resonant length with the same tuning fork decreases to 80 cm. Find the relative density of the block.

14. Two tuning forks A and B produce 10 beats per second when sounded together. When each fork is sounded above the water column in a resonance tube apparatus, their respective resonance lengths of the air column are found to be 16 and 16.5 cm. Determine the frequencies of forks A and B.

15. A pipe of length 1.5 m closed at one end is filled with a gas and resonates in its fundamental mode with a tuning fork. Another pipe of the same length but open at both ends is filled with air and resonates in its fundamental mode with the same tuning fork. Calculate the velocity of sound at 0 °C in the gas, given that the velocity of sound in air is 360 m s^{-1} at 30 °C where the experiment is performed.

16. Two tuning forks A and B produce 8 beats per second when sounded together. A gas column 37.5 cm long in a pipe closed at one end resonates in its fundamental mode with fork A, whereas a column of length 38.5 cm of the same gas in a similar pipe is required for a similar resonance with fork B. Calculate the frequencies of the two forks.

17. A tube of a certain diameter and length 48 cm is open at both ends. Its fundamental frequency of resonance is found to be 320 Hz. The velocity of sound in air is 320 m s^{-1}. Estimate the diameter of the tube. One end of the tube is now closed. Calculate the lowest frequency of resonance for the tube.

18. A steel wire of length 1 m, mass 0.1 kg and uniform cross-sectional area 10^{-6} m^2 is rigidly fixed at both ends. The temperature of the wire is lowered by 20 °C. If transverse waves are set up by plucking the wire in the middle, calculate the frequency of the fundamental mode of vibration. Coefficient of linear expansion of steel = 1.21×10^{-5} (°C)$^{-1}$. Young's modulus of steel = 2×10^{11} Nm^{-2}.

19. The sound of a horn in a car is recorded by a stationary tape recorder as the car approaches it and also as it recedes from it. The ratio of the two apparent frequencies is found to be 1.2. If the speed of sound is 340 m s^{-1}. determine the speed of the car.

20. At what speed is a car approaching a stationary observer if he hears the music of the car radio with a frequency 10% higher than it actually is? The speed of the sound is 350 m s^{-1}.

21. At what speed must the driver of a car approach a stationary siren so that the frequency of the sound heard by him is 10% higher than it actually is? Speed of sound = 340 m s^{-1}.

22. A railway engine and a car are moving on parallel tracks with speeds of 30 m s^{-1} and 15 m s^{-1} respectively. The engine is continuously sounding a whistle of frequency 1000 Hz. Calculate the frequency of sound heard by the driver of the car when (a) the car and the engine move in opposite directions and approach each other (b) they move in opposite directions but go away from each other (c) they move in the same direction, the car being ahead of the engine and (d) they move in the same direction, the car being behind the engine.

21. A source of sound of frequency 256 Hz moves rapidly towards a wall with a velocity of 5 m s^{-1}. How many beats per second will be heard if sound travels at a speed of 330 m s^{-1}?

Answers to Problems

CHAPTER 3

1. (a) C is the fastest and A is the slowest, (b) No, (c) 6 km from origin, (d) 6 km (e) 7 km/h (f) −2 km/h; **2.** 2.19 s, 11.63 m s^{-1}; **3.** (i) 14.8 m s^{-1}, (ii) 20.4 m, (iii) 31.7 m s^{-1}; **4.** (i) 4.8 m s^{-1}, (ii) 40.4 m, (iii) 31.7 m s^{-1}; **5.** (i) 98 m, (ii) 46.02 m s^{-1}; **6.** 11s, 115 m; **7.** 2.3 s, No; **8.** 6.53 s; **9.** 22.5 m; **10.** 15 s; **12.** 0.01 s; **14.** $2V \times (V' + V'')/(V' + V'' + 2V)$; **15.** 10 m s^{-1}, + 3.75 m, 5 m, −3.75 m; **16.** $v_1 = 9.8$ m s^{-1}, $v_7 = 8.23$ m s^{-1}; and **17.** (30 m, 20 m s^{-1}, −30 m s^{-2}), (0.67 s, 36.67 m), yes.

CHAPTER 4

1. 230.9 km h^{-1}, 115.5 km h^{-1}; **2.** 15.8 m; **3.** Roughly 18° with the vertical, towards the south; **4.** 2 s, 5.7 m s^{-1}; **5.** (i) 19.6 m, (ii) 400 m, (iii) 19.6 m s^{-1}; **6.** $\frac{\pi}{5}$ s; **7.** 25 N; **8.** 46.2 m, 122.6 m, 37.8 m; **9.** 2.1×10^4 m, 8.3×10^4 m, both = 7.5×10^4 m; **10.** 20 s, 527.4 m s^{-1} at 21° 48′ with vertical, 0.2 km; **11.** 516.2 m, 386.4 m s^{-1} at 4° below horizontal; **12.** 5.4 m; **13.** 11.6 m s^{-1} (approx.); **14.** $y = h - \frac{gx^2}{2v^2}$, parabola opposite to the direction of motion of the train; **15.** 52° 14′, 1.6 N, 3.16 m s^{-1}; and **16.** $h = \frac{5}{4} D$.

CHAPTER 5

1. 2.0×10^7 N; **2.** (i) 85.7 kg, (ii) 35 kg, (iii) 70 kg; **3.** 0.25 (approx.); **4.** 12 N, 2 N; **5.** 1.79×10^4 N; **6.** $\frac{m_2 g}{(m_1 + m_2)}$, $\frac{m_1 m_2 g}{(m_1 + m_2)}$; **7.** $\frac{2\, m_1 m_2\, g}{(m_1 + m_2)}$, $\frac{(m_2 - m_1)\, g}{(m_1 + m_2)}$; **8.** 50 m; **9.** 3000 N, 2.5 cm; **10.** 0.49 m s^{-1}; **11.** 20.66 m s^{-1}, 13.07 m s^{-1}, 4.77 s, 2.67 s; **12.** 0.6 N, 30.62 N, 0.0196; and **13.** 10.99 N, 30.59 N.

CHAPTER 6

1. 1/4 or 25%; **2.** 294 W; **3.** (i) 1960 J, (ii) 1176 J, (iii) 1960 J; **4.** 4 : 1; **5.** 10^4 W, 1980 N; **6.** 37°; **7.** $\frac{2}{3}$ m s^{-1}; **8.** 3.34×10^{-27} kg; **9.** 2×10^6 m s^{-1}, -1.2×10^6 m s^{-1}; **10.** 3 m s^{-1}; **11.** $v_p = 5.85$ m s^{-1}, $v_Q = 2.07$ m s^{-1}; **12.** 2.2×10^{-3}; **13.** 1000 J; and **14.** 22 kW.

CHAPTER 7

1. $x = \frac{L}{2}$, $y = \frac{L}{3}$; **2.** 47 km h^{-1}; **3.** 588 N, yes, $\theta = 27°$; **4.** 36° (approx.) **5.** (a) 2.5 kg m^2, (b) 5 kg m^2, (c) 5 kg m^2; **6.** 80 rad s^{-2}, 8 m s^{-2}, **7.** (a) 44.4 rad s^{-2}, (b) 4.44 m s^{-2}, (c) 11.1 N; **8.** 25 m s^{-1}; **9.** (a) 76 kg m^2 s^{-1} along $-z$ direction, (b) 28 Nm in $+z$ direction, (c) 28 kg m^2 s^{-2}; **10.** 2.6×10^{29} J; **11.** 6.75×10^{12} rad s^{-1}; **12.** (a) 0.5 rad s^{-1}, (b) 2.5 rad s^{-1}, (c) 4, (d) 312.5 J; **13.** 11.2 cm; **14.** 6.43 m; **15.** 0.62 rad s^{-1}; **16.** 25 Nm along $+z$ direction; **17.** 2.8×10^{-2} J; and **18.** 1 : 1.225.

CHAPTER 8

1. 7.35×10^{22} kg; **2.** 4.0×10^{23} N, 1.28×10^{4} m s^{-1}; **3.** (i) 1.04×10^{4} m s^{-1} (ii) 1.89×10^{3} m s^{-1}, (iii) 6.17×10^{5} m s^{-1}; **4.** 6.53×10^{-5} m s^{-1}; **5.** zero, (approx.); **6.** 3.594×10^{4} km; **7.** (a) 1.12×10^{4} m s^{-1}, (b) 9.145×10^{3} m s^{-1}; **8.** 889 N; **9.** 7.1×10^{10} J; **10.** 6×10^{24} kg **11.** $g/4$; and **12.** 3.0 m, 6 : 1A.

CHAPTER 9

1. 3.6 Å; **2.** 4×10^{-4}; **3.** 6×10^{-4}; **4.** 3×10^{-10} m; **5.** (i) repulsive, (ii) attractive, (iii) attractive, the magnitude of force is greatest at $R = 0.5$ Å and (iv) Zero; **6.** 21 J; **7.** 4.75 eV; **8.** 4.07×10^{4} J mol^{-1}; **9.** 9.1×10^{-20} J, and **10.** 0.2 kg.

CHAPTER 10

1. 67 N; **2.** 1.2×10^{8} Nm^{-2}, 29.6 N; **3.** 2.9 km; **4.** 100 N, 150 N, 6 mm; **5.** 1002 kg m^{-3}; **6.** 4.9 m s^{-2}; **7.** 180 m; **8.** 612 N; **9.** 1.13 cm; **10.** 2.23×10^{8} Nm^{-2}; **11.** 1.25×10^{-3}, 0.75×10^{-3}, 471 N; **12.** 9.69×10^{4} N; **13.** 9.8 N, 8.575 N; **14.** 25 m^2; **15.** 670 kg m^{-3}, 740 kg m^{-3}; **16.** (a) 5 cm, (b) 490 Pa; **17.** (a) 0.84 kg, (b) 9.8×10^{2} Pa; **18.** (a) 950 kg m^{-3}, (b) 1190 kg m^{-3}; **19.** 21.8 Ns m^{-2}; **20.** 1.41×10^{-2} N; **21.** 0.1 s; **22.** 20 m s^{-1}; **23.** (i) 4 m, (ii) 4 m below the water surface; **24.** 1.1×10^{-5} J; **25.** 3.0×10^{3} N m^{-2}; **26.** 2.56×10^{3} N m^{-2}; **27.** 7.0×10^{-2}; **28.** 75 cm of Hg; **29.** 5 cm; **30.** 1.26×10^{-5} J; **31.** 8.8×10^{-5} J, 125; **32.** -2×10^{6} Pas^{-1}; **33.** (i) 6.5 cm, (ii) 5.4 cm; **34.** 4.61×10^{2} m s^{-1}; **35.** 1.93×10^{3} m s^{-1} **36.** 100 K; **37.** 2768 K; **38.** 4.84×10^{3} m s^{-1}; **39.** 6.06×13^{-20} J; **40.** 68.25 K; and **41.** (i) 1.28×10^{-2} kg, (ii) 1.37×10^{3} m s^{-1}, (iii) 1.64×10^{3} m s^{-1}.

CHAPTER 11

1. 10.4 cm; **2.** 3.36 cm^3; **3.** 8.3×10^{-4} $(°C)^{-1}$; **4.** 25.5 °C; **5.** 22.3 °C; **6.** Zero, 6.25 g of water and 3.75 g of ice; **7.** 114.3 °C; **8.** 52 J s^{-1}; **9.** (a) 93.3 °C, (b) 1.1 J s^{-1}; **10.** 5.34×10^{3} W; **11.** 6.2×10^{-8} Wm^{-2} K^{-4}; **12.** 95.5 Js^{-1}, 213.3 °C; and **13.** 5770 K.

CHAPTER 12

1. 13.6 cm s^{-1}; **2.** 2.83 cm, 8.89 cm s^{-1}, 28 cm s^{-2}; **3.** 2.5 s, 4.53 cm s^{-1}; **4.** 5 cm, 0.4 s, 53.1°; **5.** 5 cm, 4 rad s^{-1}; **6.** 0.5 s, 1.95×10^{-3} J;

7. (a) 0.1 m, (b) 0.7 s; **8.** 5 Hz, approx.; **9.** $T = 2\pi\sqrt{\frac{M}{\rho Ag}}$; **10.** No, the frequency in the second case is $\sqrt{2}$ times that in the first; **11.** $3\,k, \frac{3}{2}\ k$; **12.** The clock slows down by 43.2 seconds per day; **13.** 0.2%; **14.** 83.7 kg wt, 36.3 kg wt; **15.** (a) 2.5 cm, (b) 1.25 cm; **16.** $t_1 : t_2 = \pi : 2\sqrt{2}$; **17.** 3.535 cm, 11.5 cm s^{-1}, 34.9 cm s^{-2}, **18.** 5 cm; **19.** $\frac{\pi}{10}$ or 0.314 s; **20.** 3.14 s; **21.** Angular frequency $= \sqrt{\frac{k}{m_2}}$, amplitude $= \frac{m_1 g}{k}$; **22.** $T = 2\pi\sqrt{\frac{mL}{YA}}$; **23.** (i) simple harmonic, $T = 2\pi/\omega$, (ii) periodic but not simple harmonic, $T = 2\pi/\omega$, (iii) simple harmonic, $T = \pi/\omega$, (iv) periodic but not simple harmonic, $T = 2\pi/\omega$, (v) Non-periodic, (vi) Non-periodic.

CHAPTER 13

1. (a) 0.001 m, 1/3 s, 3 Hz, 1.5 m s^{-1}, 0.5 m, (b) 0.02 m, 3.14 s, 0.32 Hz, 0.67 m s^{-1}, 2.09 m, (c) 0.03 m, $2\pi/v$ s, $v/2\pi$ Hz, v m s^{-1}, 2π m, (d) 0.05 m, 1/3 s, 3 Hz, 6 m s^{-1}, 2 m; **2.** (a) $y = 0.05 \sin 4\pi\,(2t - x)$, (b) $y = 0.02 \sin 2\pi\,(800\,t - x/3)$ (c) $y = 0.01 \sin 10\pi\,(10\,t - x)$ **3.** $\sqrt{2} : 1$; **4.** 2 : 1; **5.** 0.2 s; **6.** 3.44×10^{-7} m; **7.** 3.33×10^{-7} m; **8.** (a) 1.95%, (b) 4.02% or 3.8%; **9.** 105 Hz; **10.** 4; **11.** 2.63%; **12.** 476 Hz, 478 Hz; **13.** 4.76; **14.** 330 Hz, 320 Hz; **15.** 683.4 m s^{-1}; **16.** 300 Hz, 308 Hz; **17.** 3.33 cm, 163.3 Hz; **18.** 11 Hz; **19.** 30.9 m s^{-1}; **20.** 31.8 m s^{-1}; **21.** 34 m s^{-1}; **22.** (a) 1112.5 Hz, (b) 881.6 Hz, (c) 1046.9 Hz, (d) 960.5 Hz; and **23.** Zero, 7.7 or 7.9 beats per second depending on the position of the observer.

Tata McGraw-Hill's
PHYSICS
for Class XI

The book is written strictly according to the latest CBSE syllabus to be implemented in 1989-90.

Salient Features

- ☐ Each concept in a topic is explained in a comprehensive manner with a number of illustrations.
- ☐ Stress has been laid on practical applications throughout the book.
- ☐ Each concept is followed by a number of solved examples and solved numericals.
- ☐ Exercises include descriptive type and short-answer type questions followed by a number of numerical problems.
- ☐ Answers to numerical problems are provided at the end of the book.

Dr N K Bajaj has been teaching physics at St Stephen's College, University of Delhi, for the past three decades. Between 1969 and 1971 he was a member of the physics faculty at the University of Maryland, USA. He has been a resource person in physics at various workshops organised by NCERT for the improvement of teaching and evaluation in Physics in schools.

Other Books of Interest

Kapoor: *Chemistry for Class XI*

Varadakutty: *Mathematics for Class XI*

Sankaran: *Biology for Class XI*

Tata McGraw-Hill Publishing Company Limited
4/12 Asaf Ali Road, New Delhi 110 002

0-07-460043-5